tively, and calculated a lower limit for t of 10 I^{129} half-lives. From the systematics of nuclear abundance data, he thought that Xe^{129} did not have an abnormally high abundance compared to the other Xe isotopes.

Aten,[7] however, has pointed out that in all cases, where an odd isotope occurs between two even ones, the odd one is always less abundant than the sum of the abundances of the other two. The only exception is Xe^{129} whose abundance is 26.23 percent while Xe^{128} and Xe^{130} have abundances of 1.90 percent and 4.07 percent, respectively. Therefore, most of the Xe^{129} present in the atmosphere today originated by decay of I^{129} after the formation of the earth just as most of the A^{40} originated from decay of K^{40}.

It was suggested by Dr. M. Goldhaber of this Laboratory that t probably could be calculated more accurately by comparing the I^{127}/K^{40} abundance ratio with the Xe^{129}/A^{40} abundance ratio. In this way the terrestrial abundance data given by Goldschmidt[8] can be applied directly; and as a first approximation it can be assumed that the atmosphere contains the same fraction of the earth's Xe^{129} that it does of the earth's A^{40}. The following equation was used:

$$(I^{127}e^{-\lambda t}/K^{40})_T=(Xe^{129}/A^{40}).$$

I^{127} refers to an abundance[8] of 0.24 atom per 10^6 atoms of silicon; λ is the disintegration constant of I^{129}; K^{40} refers to the number of atoms (per 10^6 atoms of Si) which have decayed to A^{40} in time T, the age of the earth, 3.35×10^9 years.[9] Its present abundance was taken as 0.0119 percent of the total potassium abundance of 44,200 atoms[8] per 10^6 Si atoms; its half-life -1.27×10^9 years[10] with 12 percent decaying to A^{40}. The A^{40} in the equation refers to its abundance in the atmosphere by volume, 0.93 percent; and Xe^{129} refers to the radiogenic Xe^{129} which is assumed to be *20 percent* of the total Xe abundance in the atmosphere, 8×10^{-6} percent Xe by volume.[11] This leaves 6.23 percent Xe^{129} which is not radiogenic. Thus, the rule of Aten is approximately satisfied, and the rule which states that two odd isotopes of an element have roughly the same abundance[5,6] is not seriously violated either (abundance of Xe^{131} is 21.2 percent).

Solving the equation for t gives 15.4 half-lives of I^{129} or 2.7×10^8 years for the time between the formation of the elements and the formation of the earth. Thus, the age of the elements $(T+t)$ is 3.6×10^9 years. This calculation of t is very insensitive to the accuracy of the assumptions and approximations that were used. For example, if the amount taken for radiogenic Xe^{129} is wrong by a factor of 10 the error in t is only 22 percent.

[7] A. H. W. Aten, Jr., Phys. Rev. **73**, 1206 (1948).
[8] V. M. Goldschmidt, Geochem. Verteilungsgesetze IX, Videnskapsakademien, Oslo (1937).
[9] A. Holmes, Nature **159**, 127 (1947).
[10] G. A. Sawyer and M. L. Wiedenbeck, Phys. Rev. **79**, 480 (1950).
[11] G. Damköhler, Z. Elektrochem. **41**, 74 (1935).

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Conservation of Angular Momentum in the Statistical Theory of Nuclear Reactions

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The conservation of total angular momentum J, z-component of angular momentum m, and parity p are considered in connection with the statistical theory of nuclear reactions. Results are obtained for the angular distribution of a single group of outgoing particles and for the cross section as a function of the spin of the residual nucleus. The energy and angular distributions are also considered for the case in which many groups are observed simultaneously. The density of nuclear states as a function of spin affects results appreciably.

INTRODUCTION

WHEN a particle of energy less than 50 Mev strikes a nucleus it is usually pictured as forming a compound nucleus.[1] In this process the incident energy is distributed among the many particles of the nucleus with the result that the incident particle can no longer be distinguished from the others. The outgoing product particles of the collision are the products of the decay of the compound nucleus. The state of the compound nucleus and, consequently, its decay products are often said to be independent of the way in which the compound nucleus is produced. This statement must be qualified, however, by the condition that any compound nuclear state must have the same total angular momentum J, z-component of total angular momentum m, and parity p as the initial state.[2] The conservation of J and p, as has been noted a number of times,[3] results in selection rules which forbid transitions between certain compound nuclear states and particular initial or final states. Little mention, however, has been made of the conservation of m, which may be said to have as its

[1] N. Bohr, Nature **137**, 344 (1936).
[2] It is assumed here and throughout that the initial state is resolved into a sum of products of spin and orbital functions which in turn can be expressed as a sum of states with particular values of J, m, and p.
[3] For example, H. A. Bethe, Revs. Modern Phys. **9**, 108 (1937).

consequence that the compound nucleus "remembers" the axis of incidence of the initial particles. In the present paper these conservation principles are applied to nuclear reactions in which the compound nucleus has so large a density of levels that it may be treated by the statistical, or continuum theory.[4,5] In particular, the angular distribution of such reactions is discussed.

GENERAL FORMULATION

A collision is considered of particle a with spin s and particle α with spin i in which both spins are unpolarized. The general procedure is to resolve the initial state into states characterized by the set of quantum numbers $(ljJmp)$, where l is the orbital angular momentum, and j is the vector sum of i and s. The parity p is actually not an independent quantum number but is determined by the value of l. It is noted separately, however, since it is conserved and l is not. If only a single compound nuclear state is involved, a fairly simple equation can be written for the angular distribution of the product particles. Even for this case, however, the equation contains a number of constants about which theory tells us very little. These constants describe the spin-orbit coupling of the compound state relative to the initial particle states and to the final particle states. Several attempts have been made to determine the constants in this equation (or in the corresponding equation for two or three compound states) for particular reactions.[6]

In the statistical theory it is assumed that the transition from the initial state of particles a and α to the compound state, and the decay of the compound state to particles b and β can be treated as independent processes. This may be interpreted as meaning that if a large number of compound levels are considered, the phase relations between the matrix elements of the transition to the compound state, and those from the compound state, are random.[3,4] Consequently, one may define $\sigma(Jmp,\, a\alpha ljm_j\theta)$ = cross section for capture of a by α forming a compound nucleus with quantum numbers (Jmp) starting with a plane wave making an angle θ with the z-axis, in a spin state characterized by the quantum numbers (jm_j), and considering only contributions from orbital angular momentum l. This may be written[7]

$$\sigma(Jmp,\, a\alpha ljm_j\theta)=|A(lm-m_j\theta)|^2 \times(ljJm|ljm-m_jm_j)^2\sigma_{al}\xi(Jp,\, a\alpha lj). \quad (1)$$

The first factor in Eq. (1) is proportional to the square of a coefficient in the expansion of the incident plane wave function $\psi_\theta(\Theta)$ in terms of spherical harmonics $Y_l^{m_l}(\Theta)$,[8] where $m_l=m-m_j$. Since angles measured relative to the axis of incidence are given by $(\Theta-\theta)$,

$$\psi_\theta(\Theta)=\mathrm{const}P_l^0(\Theta-\theta).$$

By the addition theorem for legendre polynomials, it follows that

$$|A(lm_l\theta)|\equiv|\mathrm{const}\int\psi_\theta(\Theta)Y_l^{m_l}(\Theta)\,d\Omega| =\mathrm{const}P_l^{m_l}(\theta). \quad (1a)$$

Equation (1) has been written so that this first factor equals unity (times δ_{m_l0}) for the usual choice of θ equals zero. This will follow from Eq. (1a) if the proportionality constant is set equal to unity.

The second factor in Eq. (1) represents the usual coefficients for combining angular momenta.[9] The third factor,

$$\sigma_{al}=(2l+1)\pi\lambda\!\!\!^{-}{}_a^2P_{al}, \quad (1b)$$

is the product of the "target area" for particles of angular momentum l and the penetration factor P_{al} for the combined angular momentum and coulomb barriers. The last factor plays the role of a "sticking probability," which will be assumed later to be independent of the various quantum numbers.

Similarly,[10] $\Gamma(Jmp,\, b\beta\mathbf{ljm}_j\boldsymbol{\theta})d\boldsymbol{\Omega}/4\pi$ = width (averaged over a large number of compound levels all at essentially the same energy) for decay of a compound nucleus with quantum numbers (Jmp) to form particles b and β moving along a line making an angle $\boldsymbol{\theta}$ with the z-axis, within a solid angle $d\boldsymbol{\Omega}$, and in a spin state characterized by the quantum numbers $(\mathbf{jm}_j)$, considering only orbital angular momentum $\mathbf{l}$.

Applying the law of detailed balancing,

$$\Gamma(Jmp,\, b\beta\mathbf{ljm}_j\boldsymbol{\theta})=D^{Jp}\frac{\sigma(Jmp,\, b\beta\mathbf{ljm}_j\boldsymbol{\theta})}{2\pi^2\lambda\!\!\!^{-}{}_b^2} =\frac{D^{Jp}}{2\pi}(\mathbf{lj}Jm|\mathbf{lj}m-\mathbf{m}_j\mathbf{m}_j)^2(2\mathbf{l}+1) \times P_{b\mathrm{l}}\xi(Jp,\, b\beta\mathbf{lj})|A(\mathbf{l}m-\mathbf{m}_j\boldsymbol{\theta})|^2, \quad (2)$$

[4] This theory as used here is described by V. F. Weisskopf, Lecture Series in Nuclear Physics (U. S. Govt. Printing Office, December, 1947), p. 80.

[5] J. M. Blatt and V. F. Weisskopf, "Theory of nuclear reactions," Technical Report No. 42 (Massachusetts Institute of Technology, 1950) (unpublished).

[6] $Li^7(p,\alpha)\alpha$: C. Critchfield and E. Teller, Phys. Rev. **60**, 10 (1941); D. R. Inglis, Phys. Rev. **74**, 21 (1948); $Li^6(d,\alpha)\alpha$: R. Resnick and D. R. Inglis, Phys. Rev. **76**, 1318 (1949); $Li^7(p,n)$: G. Breit and I. Bloch, Phys. Rev. **74**, 397 (1948); $F^{19}(p,\alpha)$: E. Gerjuoy, Phys. Rev. **58**, 503 (1940); C. Y. Chao, Phys. Rev. **80**, 1035 (1950).

[7] A rigorous derivation may be given following the method of reference 5. The only assumption is that there is no *elastic* scattering with a change of orbital angular momentum. This assumption follows from the statistical, or continuum theory in which the only appreciable elastic nuclear scattering is due to the surface of the nucleus. Equation (10.16) of reference 5 may be seen to be a sum over l of Eq. (1) above for the case $\theta=0°$ by replacing $\eta_{ll'}$ by $(1-\xi P_{al})^{\frac{1}{2}}\delta_{ll'}$. Incidentally, this derivation shows that, with the assumption mentioned, it is possible to isolate contributions to the cross section from different l values.

[8] The argument Θ here stands for both polar and azimuthal angles. On the other hand φ is simply a polar angle since the results are independent of φ as long as polarization of spins is not considered. The functions $P_l^{m_l}(\theta)$ then are associated legendre polynomials normalized to $2/(2l+1)$.

[9] Condon and Shortley, *Theory of Atomic Spectra* (The Macmillan Company, New York, 1935), p. 73. The tables given there have been extended by Dr. David Falkoff, who kindly sent me a copy of this work.

[10] Values after collision are indicated by bold-face type, except in subscripts where they can be distinguished by the fact that they are not in italic type.

where D^{Jp} is the average spacing of compound nuclear levels with quantum numbers (Jp). This differs from the usual formulation[11] only in the designation of the direction $\boldsymbol{\theta}$ and the quantum numbers m and $\mathbf{m}_j$. To obtain the usual form, one may define

$$\Gamma_b=\int\sum_{m\mathbf{m}_j}\Gamma(Jmp,\, b\beta\mathbf{ljm}_j\boldsymbol{\theta})d\boldsymbol{\Omega}/4\pi,$$
$$\sigma_b=\sum_{m\mathbf{m}_j}\sigma(Jmp,\, b\beta\mathbf{ljm}_j0). \tag{2a}$$

It then follows from Eq. (2) that

$$\Gamma_b=(D^{Jp}\sigma_b/2\pi^2\lambda\!\!\!^{-}{}_b{}^2). \tag{2b}$$

However, Γ_b and σ_b are of doubtful significance because they are summed over m independently, whereas m is actually conserved throughout the reaction. If one omits the summation over m in Eq. (2a), thus defining $\Gamma_b(m)$ and $\sigma_b(m)$, one finds a relation less simple than Eq. (2b):

$$\Gamma_b(m)=\frac{D^{Jp}\sigma_b(m)}{2\pi^2\lambda\!\!\!^{-}{}_b{}^2(2\mathbf{l}+1)(\mathbf{lj}Jm|\mathbf{lj}0m)^2}.$$

The differential cross section of the reaction is given by

$$\sigma(a\alpha,\, b\beta\boldsymbol{\theta})=\sum_j\sum_{\mathbf{m}_j=-\mathbf{j}}^{\mathbf{j}}\sum_{\mathbf{l}=J-\mathbf{j}}^{J+\mathbf{j}}{}'\sum_{l=0}^{\infty}\sum_{j=|i-s|}^{i+s}\sum_{m=-j}^{j}\sum_{J=|l-j|}^{l+j}\frac{\sigma(Jmp,\, a\alpha ljm0)}{(2i+1)(2s+1)}\times\frac{\Gamma(Jmp,\, b\beta\mathbf{ljm}_j\boldsymbol{\theta})}{\int\sum_{c\gamma\mathbf{l}'\mathbf{j}'\mathbf{m}_j'}\Gamma(Jmp,\, c\gamma\mathbf{l}'\mathbf{j}'\mathbf{m}_j'\boldsymbol{\theta}')d\boldsymbol{\Omega}'/4\pi}. \tag{3}$$

Here the first factor represents the formation of the compound nucleus (the incident θ equals zero) and the second represents the width for outgoing particles b and β at angle $\boldsymbol{\theta}$ divided by the total width for all possible decay products. The sums over m and j represent an average over the initial polarization of the spins[12] which averaging also requires the division by $(2i+1)(2s+1)$. The sum over initial orbital angular momenta l automatically gives the sum over parity. The sum over $\mathbf{l}$ is restricted to even or odd values, depending on the parity p and the intrinsic parities of b and β; this restriction is indicated by the prime on the summation sign. The sum over $(\mathbf{jm}_j)$ represents an integration over the outgoing spin space, which is required since the final polarization is not measured. If a single residual state is assumed, $\mathbf{j}$ ranges from $|\mathbf{i}-\mathbf{s}|$ to $(\mathbf{i}+\mathbf{s})$.

A fundamental assumption involved in the derivation of Eq. (3) is the possibility of neglecting interference terms between outgoing waves differing as to the quantum numbers $(Jmp\mathbf{ljm}_j)$. This is justified as follows:

(1) There cannot possibly be any interference from terms corresponding to different values of m because the incident spins are unpolarized and $m=m_j$. Similarly, in summing over the outgoing spins, any interference terms between states having different values of $\mathbf{j}$ or $\mathbf{m}_j$ would drop out.

(2) The neglect of interference terms between outgoing waves arising from compound states, with different values of J or p, follows from the assumption that the reaction can be divided into two independent parts, the formation and the decay of the compound nucleus. One way of stating this is to say that these interference terms cancel out when averaging over many compound states for each value of J and p because the outgoing waves have random phases.

(3) Interference terms between different outgoing orbital angular momenta $\mathbf{l}$ arising from the same compound state do not enter σ (see reference 7), and consequently, by the law of detailed balancing, will not enter Γ.

The main concern here is in the distribution of outgoing particles as to energy, angle, etc. Thus the denominator of the second term in Eq. (3) is of interest only insofar as it depends on the quantum numbers (Jmp). Substituting from Eqs. (1a) and (2) into this denominator and making use of the identity

$$\sum_{\mathbf{m}_j'}(\mathbf{l}'\mathbf{j}'Jm|\mathbf{l}'\mathbf{j}'m-\mathbf{m}_j'\mathbf{m}_j')^2=1, \tag{4}$$

the denominator is equal to

$$\frac{D^{Jp}}{2\pi}\sum_{c\gamma}\sum_{\mathbf{l}'}{}'\sum_{\mathbf{j}'=|J-\mathbf{l}'|}^{J+\mathbf{l}'}P_{c\mathbf{l}'}\xi(Jp,\, c\gamma\mathbf{l}'\mathbf{j}')=(D_{Jp}/2\pi)\langle\xi(Jp,\, c\gamma)\rangle F(J), \tag{5}$$

where

$$F(J)\equiv\sum_{c\gamma}\sum_{\mathbf{l}'}{}'\sum_{\mathbf{j}'=J-\mathbf{l}'}^{J+\mathbf{l}'}P_{c\mathbf{l}'}.$$

It is assumed that ξ is independent of $\mathbf{l}'$ and $\mathbf{j}'$, and some kind of average over c and γ is factored out of the summation. The function $F(J)$ (times D) represents the dependence on J of the yield of the final product of interest plus all competing possibilities.

The summation over outgoing particles c in Eq. (5) may usually be limited to neutrons, because the penetration factor for outgoing neutrons is much greater than for charged particles. The sum over residual nuclei γ requires a knowledge of the density of residual nuclear states as a function of energy, spin, and parity. Here it is assumed that the density is independent of parity and that the dependence on spin and energy may be separated, giving

$$\rho(E_c,\,\mathbf{i})dE_c=\rho_c(\mathbf{i})W(E_c)dE_c \tag{6}$$

[11] Reference 4, p. 87.
[12] G. Breit and B. T. Darling, Phys. Rev. 71, 402 (1947).

= number of residual nuclear states corresponding to an outgoing neutron energy between E_c and E_c+dE_c and residual nuclear spin $\mathbf{i}$. (The $(2\mathbf{i}+1)$ degenerate states corresponding to different m_{i} values are counted as one state.) The number of final states with a total spin $\mathbf{j}$ may be expressed by a similar equation by introducing

$$\rho_c(\mathbf{j})=\tfrac{1}{2}\{\rho_c(\mathbf{i}=\mathbf{j}+\tfrac{1}{2})+\rho_c(\mathbf{i}=\mathbf{j}-\tfrac{1}{2})\}. \tag{6a}$$

The function $F(J)$ then becomes

$$F(J)=\int\sum_{\mathbf{l}'}\sum_{\mathbf{j}'=|J-\mathbf{l}'|}^{J+\mathbf{l}'}\rho_c(\mathbf{j}')P_{c\mathbf{l}'}(E_c)W(E_c)dE_c. \tag{7}$$

The main contribution to Eq. (7) becomes from low values of $\mathbf{l}'$ corresponding to low energies E_c for which the density $W(E_c)$ is large; as a consequence, only values of $\mathbf{j}'$ close to $\mathbf{j}'=J$ contribute to the sum over $\mathbf{j}'$. Approximating $\rho(\mathbf{j}')$ as a linear function of $\mathbf{j}'$ over this interval

$$F(J)=\int[\sum_{\mathbf{l}'=0}^{J}\rho_c(J)P_{c\mathbf{l}'}(E_c)(2\mathbf{l}'+1)$$
$$+\sum_{\mathbf{l}'=J+1}^{\infty}\rho_c(\mathbf{l}')P_{c\mathbf{l}'}(E_c)(2J+1)]W(E_c)dE_c. \tag{7a}$$

The first term in the brackets is the more important, except for J less than 2 or 3. If it is assumed that[13]

$$\rho_c(J)=\mathrm{const}(2J+1), \tag{8}$$

at least for the values of J for which the second term in Eq. (7a) is important, the two terms become identical in form and

$$F(J)\sim\rho_c(J).$$

Making use of Eqs. (1) through (3) and (5), one finds for the outgoing distribution

$$\sigma(a\alpha,\,b\beta\boldsymbol{\theta})=\frac{\pi\lambda\!\!\!^{-}{}_a^2}{(2i+1)(2s+1)}$$
$$\times\sum_{l=0}^{\infty}\sum_{j=|i-s|}^{i+s}\sum_{m=-j}^{j}\sum_{J=|l-j|}^{l+j}(ljJm|lj0m)^2(2l+1)$$
$$\times P_{al}\xi(Jp,\,a\alpha)\sum_{\mathbf{j}=|\mathbf{i}-\mathbf{s}|}^{\mathbf{i}+\mathbf{s}}\sum_{\mathbf{l}=|J-\mathbf{j}|}^{J+\mathbf{j}}{}'\sum_{\mathbf{m}_j=-\mathbf{j}}^{\mathbf{j}}\frac{\xi(Jp,\,b\beta)}{\langle\xi(Jp,\,c\gamma)\rangle}$$
$$\times(2\mathbf{l}+1)(\mathbf{l}\mathbf{j}Jm|\mathbf{l}\mathbf{j}m-\mathbf{m}_j\mathbf{m}_j)^2\frac{P_{b\mathbf{l}}}{F(J)}|P_{\mathbf{l}}^{m-\mathbf{m}_j}(\boldsymbol{\theta})|^2. \tag{9}$$

ANGULAR DISTRIBUTION

Equation (9) gives the angular distribution of a single outgoing group of particles b resulting from the collision of particles a and α, assuming statistical theory can be applied to the compound nucleus. Of course, statistical theory is not being applied to the residual nucleus, since only a single residual nuclear state β is considered. A very simple case is one in which the spins (i, s, $\mathbf{i}$, and $\mathbf{s}$) are zero, giving for the angular distribution

$$\sigma(a\alpha,\,b\beta\boldsymbol{\theta})\sim\sum_{l=0}^{\infty}\frac{(2l+1)^2}{F(l)}P_{al}P_{bl}|P_l^0(\boldsymbol{\theta})|^2, \tag{10}$$

assuming that the ξ factors are independent of l. The amount of anisotropy is determined by the maximum value of l effective in the sum which depends on the ability of the incident and outgoing particles to overcome the angular momentum barrier. It should be emphasized that only the factor $\{(2l+1)^2/F(l)\}$ in Eq. (10) is dependent on the detailed assumptions of the present formulation; the general features of the equation, including the anisotropy, follow directly from conservation of orbital angular momentum. If there is only one possible final state so that the sum over c and γ in Eq. (3) does not appear, the angular distribution is for the case of no spins

$$\sigma(a\alpha,\,b\beta\boldsymbol{\theta})\sim\sum_{l=0}^{\infty}(2l+1)^2P_{al}|P_l^0(\boldsymbol{\theta})|^2. \tag{10a}$$

The main features of the angular distribution (Eq. (9)) are (a) symmetry about 90° and usually (b) peaks in the forward and backwards directions. The former follows from the fact that interference terms between outgoing waves of different parity are assumed to cancel out. The latter is expected because the initial zero value for m_l should show itself in a large intensity for the outgoing wave with $\mathbf{m}_{\mathrm{l}}$ equal to zero, which wave is peaked in the forward direction. While this is certainly the case when no spins are present (Eq. (10)), it is not always true when large nuclear spins are involved.

To get a measure of the anisotropy it is convenient to obtain, from Eq. (9), expressions proportional to the average cross section

$$\sigma_{\mathrm{Av}}\equiv(1/4\pi)\int\sigma d\Omega,$$

and the cross section for $\boldsymbol{\theta}=0°$. Once again ignoring the dependence of the ξ factors on J

$$\sigma_{\mathrm{Av}}(a\alpha,\,b\beta)\sim\sum_{l=0}^{\infty}\sum_{j=|i-s|}^{i+s}\sum_{J=|l-j|}^{l+j}\frac{P_{al}(2J+1)}{F(J)}$$
$$\times\sum_{\mathbf{j}=|\mathbf{i}-\mathbf{s}|}^{\mathbf{i}+\mathbf{s}}\sum_{\mathbf{l}=|J-\mathbf{j}|}^{J+\mathbf{j}}{}'P_{b\mathbf{l}}, \tag{11a}$$

$$\sigma(a\alpha,\,b\beta 0)\sim\sum_{l=0}^{\infty}\sum_{j=|i-s|}^{i+s}\sum_{m=-j}^{j}\sum_{J=|l-j|}^{l+j}\frac{P_{al}(2l+1)}{F(J)}$$
$$\times(ljJm|lj0m)^2\sum_{\mathbf{j}=|\mathbf{i}-\mathbf{s}|}^{\mathbf{i}+\mathbf{s}}\sum_{\mathbf{l}=|J-\mathbf{j}|}^{J+\mathbf{j}}{}'$$
$$\times(\mathbf{l}\mathbf{j}Jm|\mathbf{l}\mathbf{j}0m)^2(2\mathbf{l}+1)P_{b\mathbf{l}}. \tag{11b}$$

[13] This assumption is not unreasonable (see Appendix B).

TABLE I. Ratio of forward yield $\sigma(0)$ to average cross section σ_{Av} as a function of residual nuclear spin $\mathbf{i}$ and maximum effective outgoing orbital angular momentum $\mathbf{l}_{max}$. Initial spins are zero.

$\mathbf{i}$	$\mathbf{l}_{max}$	Case A (Yes or No)	Case B (No)	Case B (Yes)
1/2	0	1	1	1
	1	1.33	1.37	1.50
	2	1.80	2.00	2.10
	3	2.28	2.66	2.74
	4	2.78	3.32	3.38
	5	3.27	3.98	4.04
3/2	0	1	1	1
	1	1	1.18	1
	2	1.12	1.37	1.20
	3	1.33	1.62	1.47
	4	1.56	1.90	1.78
	5	1.80	2.20	2.09

The anisotropy will be particularly sensitive to l_{max} and $\mathbf{l}_{max}$, the maximum values of l and $\mathbf{l}$, respectively, effective in the sums; in fact, the lower of these sets an absolute bound on the degree of anisotropy of any reaction.[14] It will also be sensitive to the values of $\mathbf{i}$ and $\mathbf{s}$, since the larger these are the less directly the conservation of m is reflected in the outgoing orbital functions.

Calculations have been made of this anisotropy under the following assumptions:

(1) i and s are equal to zero.

(2) $\mathbf{s}=\frac{1}{2}$.

(3) As an approximation the penetration factors P_{al} and $P_{b\mathbf{l}}$ are assumed equal to unity for $l\leqslant l_{max}$, and $\mathbf{l}\leqslant \mathbf{l}_{max}$, respectively, and equal to zero otherwise.

(4) The incident energy is sufficiently large that $l_{max}\geqslant \mathbf{l}_{max}+\mathbf{i}+\mathbf{s}$.

(5) Two forms of $F(J)$ are considered:

Case A. $F(J)$ is proportional to $2J+1$, which corresponds to Eq. (8).

Case B. $F(J)$ is constant for all values of J greater than zero; $F(0)$ equals $\frac{1}{2}F(1)$. An analysis of Eq. (7) indicates that this corresponds approximately to $\rho_c(\mathbf{j})$ constant for reasonable choices of $W(E_c)$.

Results are given in Table I for various values of $\mathbf{i}$ and $\mathbf{l}_{max}$. "No" or "Yes" indicates that the product of the intrinsic parities of the final particles is the same as, or different from that of the initial particles. For Case A the results are independent of parity change and can be expressed analytically (Appendix A).

$$\sigma_{Av}(a\alpha, b\beta)\sim\tfrac{1}{2}(2\mathbf{i}+1)(2\mathbf{l}_{max}-\mathbf{i}+\tfrac{3}{2}) \quad \text{for} \quad \mathbf{l}_{max}\geqslant\mathbf{i}-\tfrac{1}{2}, \tag{12a}$$

$$\begin{aligned}\sigma_{Av}(a\alpha, b\beta)&\sim(\mathbf{l}_{max}+1)^2 \quad \text{for} \quad \mathbf{l}_{max}\leqslant\mathbf{i}-\tfrac{1}{2},\\ \sigma(a\alpha, b\beta0)&\sim(\mathbf{l}_{max}+1)^2.\end{aligned} \tag{12b}$$

The distribution is isotropic for $\mathbf{i}$ greater than $\mathbf{l}_{max}$, while for $\mathbf{l}_{max}$ considerably greater than $\mathbf{i}$, the anisotropy is approximately

$$\frac{\sigma(0)}{\sigma_{Av}}=\frac{\mathbf{l}_{max}+1}{2\mathbf{i}+1}+\frac{1}{4}. \tag{13}$$

The results are not so simple if the limiting value l_{max} of the entering orbital angular momentum must be considered; that is, assumption (4) is not valid. It is then possible that for some values of the residual nuclear spin the forward yield is less than the average. A particular example is shown in Table II; the results labeled $l_{max}>9$ correspond to Eqs. (12). Except for (4), the assumptions listed above (Case A) have been used.

TABLE II. Ratio of forward yield $\sigma(0)$ to average cross section σ_{Av} for $\mathbf{l}_{max}=3$ as a function of $\mathbf{i}$ and under two assumptions for l_{max}. Initial spins are zero.

$\mathbf{i}$	$l_{max}=4$ (No)	$l_{max}=4$ (Yes)	$l_{max}>9$ (Yes or No)
1/2	2.28	2.28	2.28
3/2	1.15	1.33	1.33
5/2	0.98	0.82	1.07
7/2	0.70	0.88	1
9/2	0.96	0.70	1
11/2	0.80	1.13	1
13/2	1.60	0.86	1

Commonly, a single outgoing group of particles is not observed but rather all outgoing groups within a certain energy range. Equation (9) should then be modified by including a factor $\rho_b(\mathbf{j})W(E_b)dE_b$ (see Eq. (6a)), extending the summation over $\mathbf{j}$ to all values, and integrating over the energy range. It is not clear whether any anisotropy will remain after this summation; the results depend upon the details of the assumptions, particularly, the form of $\rho_b(\mathbf{j})$. Three possibilities have been considered:

Case α. The density $\rho_b(\mathbf{j})$ of residual nuclear states corresponding to the outgoing particles b of interest is proportional to $(2\mathbf{j}+1)$[13] for all values of $\mathbf{j}$ that are possible. This is clearly inconsistent with assumption (4) previously used. Using the others of the previous assumptions (Case A), it is shown in Appendix A that the angular distribution is essentially isotropic. This result is clearly independent of $\mathbf{s}$, l_{max}, and $\mathbf{l}_{max}$, and therefore of assumptions (2) and (3).

Case β. The density $\rho_b(\mathbf{j})$ is constant as a function of $\mathbf{j}$. Using a constant for $\rho_c(\mathbf{j})$ in Eq. (7) gives Case B for $F(J)$ to a good approximation. Using the previous assumptions (except assumption 4), calculations have been made for the case $l_{max}=4$; the resulting values of the anisotropy as measured by $\sigma(0)/\sigma_{Av}$ are 1.14 for $\mathbf{l}_{max}=2$, and 1.23 for $\mathbf{l}_{max}=3$.

Case γ. The density $\rho_b(\mathbf{j})$ is limited to a maximum value j_0, being given by

$$\begin{aligned}\rho_0(\mathbf{j})&=\text{const}(2\mathbf{j}+1) \quad &\mathbf{j}\leqslant j_0,\\ \rho_b(\mathbf{j})&=0 \quad &\mathbf{j}>j_0.\end{aligned} \tag{14}$$

Assumption (4) is now applicable provided $l_{max}\geqslant\mathbf{l}_{max}+j_0$. Using all the previous assumptions (once again Case A), one finds the anisotropy given in Table III.

[14] E. Eisner and R. G. Sachs, Phys. Rev. **72**, 680 (1947); L. Wolfenstein and R. G. Sachs, Phys. Rev. **73**, 528 (1948); C. N. Yang, Phys. Rev. **74**, 764 (1948).

It should be noted that the assumptions used for this case are not completely consistent, since if Eq. (14) were used for $\rho_c(\mathbf{j})$ the resulting $F(J)$ would not be that assumed (Case A). However, it should be noted that the residual nuclei α, corresponding to the outgoing low-energy neutrons c, may have a considerably greater excitation energy than the residual nuclei β corresponding to outgoing particles b, particularly if high-energy particles b are being selected. Thus, it may not be unreasonable to assume a cut-off in $\rho_b(\mathbf{j})$ but not in $\rho_c(\mathbf{j})$.

OTHER RESULTS

Equation (11a) may also be considered as giving the energy distribution of the outgoing particles if the summation over $\mathbf{j}$ is replaced by a sum over residual nuclear states with a density given by Eqs. (6) and (6a).

$$\sigma_{\mathrm{Av}}\sim\sum_{l=0}^{\infty}\sum_{j=|i-s|}^{i+s}\sum_{J=|l-j|}^{l+j}\frac{P_{al}(2J+1)}{F(J)}\times\sum_{\mathbf{l}=0}^{\infty}\sum_{\mathbf{j}=|J-\mathbf{l}|}^{J+\mathbf{l}}P_{b\mathbf{l}}\rho_b(\mathbf{j})W(E_b)dE_b.$$

To simplify this result one may approximate $\rho_b(\mathbf{j})$ as a linear function of $\mathbf{j}$, as was done in deriving Eq. (7a).

$$\sigma_{\mathrm{Av}}\sim\sum_{l=0}^{\infty}\sum_{j=|i-s|}^{i+s}\sum_{J=|l-j|}^{l+j}P_{al}\frac{2J+1}{F(J)}\times\{\rho_b(J)\sum_{\mathbf{l}=0}^{J}P_{b\mathbf{l}}(E_b)(2\mathbf{l}+1)W_b(E_b)+(2J+1)\sum_{\mathbf{l}>J}P_{b\mathbf{l}}(E_b)\rho_b(\mathbf{l})W_b(E_b)\}dE_b. \quad (15)$$

The first term in the brackets has the form usually given for the energy distribution, but the second term shows a dependence of the energy distribution on the density as a function of spin. Indeed, the usual form for the distribution is obtained only if (Eq. (8)) is assumed for $\rho_b(\mathbf{l})$. It should be noted also that the $(2\mathbf{l}+1)$ factor enters from quite different considerations than usual in the present derivation. An alternative simplification follows from assuming no initial spins, initial $l_{\max}$ very large, and $F(J)$ proportional to $2J+1$;

$$\sigma_{\mathrm{Av}}\sim\sum_{\mathbf{l}=0}^{\infty}\{\sum_{\mathbf{j}=0}^{\mathbf{l}}\rho_b(\mathbf{j})(2\mathbf{j}+1)+\sum_{\mathbf{j}>\mathbf{l}}\rho_b(\mathbf{j})(2\mathbf{l}+1)\}P_{b\mathbf{l}}(E_b)W_b(E_b)dE_b.$$

The equations obtained may also be used to determine the dependence of the cross section for a single outgoing group on the spin of the residual nucleus, other things being equal. The cross section tends to be larger for larger residual nuclear spins because the larger spin provides a greater number of modes of decay for the compound nucleus, unless these are forbidden by the angular-momentum barrier. This is clearly seen in Eq. (12a). Thus, for $\mathbf{l}_{\max}=4$, the cross section increases by a factor of almost 3 from $\mathbf{i}=\frac{1}{2}$ to $\mathbf{i}\geqslant 9/2$. On the other hand, the forward yield (Eq. (12b)) is independent of $\mathbf{i}$.

COMPARISON WITH EXPERIMENT

Results of several types of experiments are affected by these considerations:

(1) The angular distribution of a single outgoing group of particles. Such measurements might help to identify the spin of the residual nuclear state; however, it is experimentally difficult to find an isolated group of product particles when the incident energy is large enough to allow the use of statistical theory for the compound nucleus. Of the data available, that of Wyly[15] on the $N^{14}(d, p)N^{15}$ is particularly suggestive. Although only 2- to 3-Mev deuterons were used, the compound nucleus had an excitation of 22 Mev, and the angular distribution varied little with bombarding energy. Two proton groups were observed. One showed forward and backward peaks, approximate symmetry about 90°, and was known to correspond to $\mathbf{i}=\frac{1}{2}$; the second was essentially isotropic and had a much larger yield. This suggested that the second corresponded to a larger value of $\mathbf{i}$. Unfortunately, quantitative calculations are in complete disagreement. The observed anisotropy is much larger than the calculated; furthermore, since the second group has a lower energy, it would not be expected to have a larger yield than the first, even if it corresponded to a larger value of $\mathbf{i}$.

TABLE III. Ratio of forward yield $\sigma(0)$ to average cross section σ_{Av}, averaged over many residual nuclear states with a spin distribution given by Eq. (14).

	$\mathbf{l}_{\max}$			
j_0	2	3	4	5
4	1.05	1.11	1.19	1.30
6	1.03	1.05	1.09	1.19
8	1.02	1.03	1.05	1.08

(2) The angular distribution of all outgoing particles in a certain energy interval. Here the results of the theory are more ambiguous. Cohen has analyzed data on the angular distribution of (α, n) reactions studied with threshold detectors by Allen and others.[16] He finds, for practically all nuclei, a distribution symmetric about 90°, with peaks in the forward and backward directions. The present analysis indicates that this small anisotropy is not at all inconsistent with the statistical theory of nuclear reactions. The results are in qualitative agreement with the assumption of a constant density of residual states as a function of spin (Case β) or a cutoff in the density (Case γ), but are not consistent with a density proportional to $(2\mathbf{i}+1)$ with no cutoff. On the other hand, the large anisotropy, unsymmetrical about

[15] L. D. Wyly, Phys. Rev. 76, 104 (1949).

[16] B. L. Cohen, Ph.D. Thesis (Carnegie Institute of Technology, 1950); Allen, Nechaj, Sun, and Jennings, Phys. Rev. 76, 188 (1949).

90°, observed in (d, n) and (d, p) reactions,[17] is completely inconsistent with statistical theory and definitely requires a different mechanism.

(3) The relative yield of different outgoing particle groups. This might give some information as to the relative spins of different residual nuclear states; however, this effect may be smaller than expected fluctuations.

(4) The energy distribution of outgoing particles. The usual equations are modified significantly only for fairly high outgoing energies.

APPENDIX A

Derivation of Eqs. (12a) and (12b)

Starting with Eq. (11a) and setting the initial spins equal to zero, so that $J=l$, one finds for a single value of j

$$\sigma_{Av}\sim\sum_{J=0}^{\infty}P_{aJ}\frac{2J+1}{F(J)}\sum_{l=|J-j|}^{J+j}{}'P_{bl}.$$

The prime on the summation indicates that

$$(-1)^{l}(-1)^{J}(-1)^{\delta}=1,$$

where δ is zero, if the final intrinsic parity (that is, the product of the intrinsic parities of the final particles) is the same as the initial, and one otherwise. Making assumptions (3), (4), and (5) (Case A), and changing the order of summation,

$$\sigma_{Av}\sim\sum_{l=0}^{l_{max}}\sum_{J=l-j}^{l+j}{}'(1). \qquad (16)$$

For a fixed value of l, the sum over J gives

$$\begin{array}{lll} j+1 & \text{if}\ j\leqslant l & (-1)^{j+\delta}=1,\\ j & \text{if}\ j\leqslant l & (-1)^{j+\delta}=-1,\\ l+1 & \text{if}\ j\geqslant l & (-1)^{j+\delta}=1,\\ l & \text{if}\ j\geqslant l & (-1)^{j+\delta}=-1.\end{array}$$

If these are now summed over j equal to $i-\frac{1}{2}$ and $i+\frac{1}{2}$, the results are clearly independent of δ, and Eq. (16) becomes, for $l_{max}\geqslant(i+\frac{1}{2})$,

$$\sigma_{Av}\sim\sum_{l=i+\frac{1}{2}}^{l_{max}}(2i+1)+\sum_{l=0}^{i-\frac{1}{2}}(2l+1)=\tfrac{1}{2}(2i+1)(2l_{max}-i+\tfrac{3}{2}).$$

If $l_{max}\leqslant(i-\frac{1}{2})$,

$$\sigma_{Av}\sim\sum_{l=0}^{l_{max}}(2l+1)=(l_{max}+1)^2.$$

Similarly, from Eq. (11b), one finds for a single value of j

$$\begin{aligned}\sigma(0)&=\sum_{l=0}^{l_{max}}\sum_{J=|l-j|}^{l+j}{}'(ljJ0|lj00)^2(2l+1)\\ &=(l_{max}+1)^2\quad\text{for}\quad(-1)^{\delta+j}=1\\ &=0\qquad\qquad\ \ \text{for}\quad(-1)^{\delta+j}=-1.\end{aligned}$$

Here use has been made of the normalization condition

$$\sum_{J=|l-j|}^{l+j}(ljJ0|lj00)^2=1$$

plus the fact that this combination coefficient vanishes if $(l+j+J)$ is odd. For the case of $s=\frac{1}{2}$ and given i, there is always one even and one odd possibility for j so that summing over j gives Eq. (12b) independent of i or of δ.

It is worth noting in passing the important effect of parity change for the case $s=0$; that is, the inelastic scattering of fast alpha-particles from a target of spin zero. In this case the forward scattering is zero if $\delta+i$ is odd, whereas there will be a pronounced forward peak if $\delta+i$ is even.

Sum Over Residual Nuclear States (Case α)

Now it is desired to introduce a limit l_{max} on the J summation and to sum over j, using a weighting factor $\rho(j)\sim(2j+1)$. This gives, from Eq. (16), changing the order of summation,

$$\sigma_{Av}\sim\sum_{J=0}^{l_{max}}\sum_{l=0}^{l_{max}}\sum_{j=|J-l|}^{J+l}(2j+1)=\sum_{J=0}^{l_{max}}\sum_{l=0}^{l_{max}}(2J+1)(2l+1).$$

Similarly,

$$\sigma(0)\sim\sum_{J=0}^{l_{max}}\sum_{l=0}^{l_{max}}\sum_{j=|J-l|}^{J+l}(2j+1)(ljJ0|lj00)^2(2l+1)$$

$$=\sum_{J=0}^{l_{max}}\sum_{l=0}^{l_{max}}(2J+1)(2l+1)=\sigma_{Av},$$

where the relationship

$$\sum_{j=|J-l|}^{J+l}(2j+1)(ljJ0|lj00)^2=2J+1$$

has been used.

APPENDIX B

Level Density as a Function of Spin

A very rough discussion will be given here as to the forms to be expected for the function $\rho(i)$, which represents the density of residual nuclear levels as a function of spin i. Suppose a single nucleon carries all the excitation; then it is to be expected that all values of angular momentum up to a fairly large value are approximately equally probable since the energy separating different angular momentum levels is small compared to the excitation energy. The result is quite different if the excitation energy is shared equally by two nucleons. For each nucleon it might be assumed that all values of angular momentum are equally probable, but if the interaction energy between the two nucleons is assumed to be small, the probability of a resultant angular momentum i is proportional to $(2i+1)$. The reason for this is simply that the larger the total angular momentum, the more combinations of two angular momenta one can find having this as a possible resultant. This result can be generalized somewhat. If (a) the different values of angular momentum for the individual nucleons have a probability which may be approximated as a linear function of angular momentum over every interval of length $2I$, and if (b) the probability of an individual nucleon angular momentum greater than I is much larger than that for less than I, then it follows that the probability of a resultant angular momentum i is proportional to $(2i+1)$ for i less than or equal to I. The results hold for more than two nucleons sharing the excitation; however, if many nucleons share the excitation, it is unlikely that the hypotheses are applicable.

For outgoing particles of spin $\frac{1}{2}$ it is convenient to use $\rho(j)$ defined by Eq. (6a) rather than $\rho(i)$. It is immediately evident that if $\rho(i)$ is a linear function of i, such as $(2i+1)$, then $\rho(j)$ is exactly the same function of j.

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[17] See reference 13, also Falk, Creutz, and Seitz, Phys. Rev. **76**, 322 (1949); C. E. Falk, Ph.D. Thesis (Carnegie Institute of Technology, 1950); P. Ammiraju, Phys. Rev. **76**, 1421 (1949); H. Gove, Phys. Rev. **78**, 345 (1950).