A Collective Description of Electron Interactions. I. Magnetic Interactions*

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A new approach to the treatment of the interactions in a collection of electrons is developed, which we call the collective description. The collective description is based on the organized behavior produced by the interactions in an electron gas of high density; this organized behavior results in oscillations of the system as a whole, the so-called "plasma oscillations." The collective description, in contrast to the usual individual particle description, describes in a natural way the long-range correlations in electron positions brought about by their mutual interaction. In this paper we confine our attention to the magnetic interactions between the electrons; the coulomb interactions will be discussed in a subsequent paper.

The transition from the usual single-particle description to the collective description of the electron motion in terms of organized oscillations is obtained by a suitable canonical transformation. The complete hamiltonian for a collection of charges interacting with the transverse electromagnetic field is re-expressed as a sum of three terms. One involves the collective field coordinates and expresses the degree of excitation of organized oscillations. The others represent the kinetic energy of the electrons and the residual particle interaction, which is not describable in terms of the organized oscillations, and corresponds to a screened interparticle force of short range.

Both a classical and a quantum-mechanical treatment are given, and the criteria for the validity of the collective description are discussed.

I. INTRODUCTION

BECAUSE of the long range of the coulomb force, the interactions in a collection of electrons involve many particles simultaneously. However, the usual description of these electron interactions is based on a free-particle approximation. In a metal, for instance, the motion of a given electron is assumed to be independent of the motion of all the other electrons in first approximation. The effect of the other electrons on this electron is then represented by a smeared-out potential, which can be determined by using the self-consistent field methods of Hartree and Fock. This means that the effects of the correlations in the positions of the electrons brought about by the long range of the coulomb force are almost entirely neglected; we can therefore expect that in any problem in which the electron-electron interactions are important, as, for example, the calculation of the cohesive energy or the electronic contribution to the specific heats at low temperatures, the free-particle model may not be adequate. In order to obtain a better mathematical treatment of this problem, we have adopted a new approach based on a collective description of the motion. This collective description is most appropriate for systems of high particle density.

It is well known that an electron gas of high density[1] can undergo organized oscillations resembling sound waves.[2,3] These oscillations, the so-called "plasma oscillations," represent the effects of the long-range correlation of electron positions brought about by coulomb interactions. A description in terms of these organized oscillations therefore provides a natural way of treating the long-range electron interactions, and leads to greater insight into the dynamical behavior of the electron gas than is afforded by the free-particle approximation. Thus, it may be expected that such a collective description will make possible a better understanding of the interactions between the electrons in a metal.

In a treatment of these organized oscillations,[3] one considers a particular fourier component of the average field, proportional to

$$\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)].$$

For small amplitudes (which are of interest to us here), the linear approximation is valid, and an arbitrary field can therefore be expanded as a sum of such trigonometric terms. In response to this oscillating field, each electron undergoes a small corresponding trigonemetric change in its velocity and in its contribution to the mean charge density. Owing to the long range of the coulomb force, the mean field at each point can become quite large as a result of the cumulative effects of small contributions arising from each particle. The condition for sustained oscillations is that the field arising from the response of the particles must be consistent with the field producing this response. This requirement leads to a dispersion relation connecting ω and k. For longitudinal waves the approximate dispersion relation, good for long wavelengths, is[3]

$$\omega^2=(4\pi n_0e^2/m)+3k^2\kappa T/m, \qquad (1)$$

where n_0 is the electron density, T the temperature, and κ is Boltzmann's constant. For infinite wavelength,

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[1] The electron gas must be neutralized by an approximately equal density of positive charge. In practice the positive charge can usually be regarded as immobile relative to the electrons, and for the most applications can also be regarded as smeared out uniformly throughout the system.

[2] L. Tonks and I. Langmuir, Phys. Rev. 33, 195 (1929).

[3] D. Bohm and E. P. Gross, Phys. Rev. 75, 1851 and 1864 (1949); Paper A discusses the origin of mediumlike behavior; Paper B deals with the excitation and damping of oscillations.

this reduces to the well-known plasma frequency,

$$\omega_p^2 = 4\pi n_0 e^2/m.$$

There are, however, certain limitations on the collective description of the electron gas in terms of organized longitudinal oscillations which arise from the fact that these oscillations cannot be sustained for wavelengths shorter than a critical distance known as the Debye length,

$$\lambda_D = (\kappa T/4\pi n_0 e^2)^{\frac{1}{2}} \cong \bar{V}/\omega_p. \quad (2)$$

This length is of the order of the distance traveled during the period of an oscillation by a particle moving with the mean thermal speed, $\bar{V}$, and thus might reasonably be expected to constitute a limitation on organized oscillation. It may be shown, furthermore, that whenever there is a static field in the electron gas, either externally imposed or arising from a lack of charge neutrality, the electrons redistribute themselves in such a way as to screen out the field within a distance of the order of a Debye length.[4]

Screening and organized oscillation are different but related manifestations of the collective behavior of the electron gas, brought about by cumulative responses of the particles to the average force; and the Debye length determines the smallest distance for which this collective behavior is significant. Thus, one effectively obtains a separation between the long-range part of the force, which is best described collectively, and the short-range part, which is best described in terms of the coordinates of the individual particles. The higher the density, the shorter the distances at which the collective description applies, and the more useful this description becomes. For a metal, the Debye length is of the order of 10^{-8} cm; and therefore, the collective description is applicable practically down to interparticle distances.

Organized transverse oscillations of an electron gas are also possible. These oscillations are electromagnetic waves strongly modified by the fields arising from the collective particle response. Such oscillations are obtained in the transmission of radio waves through a highly ionized medium, such as the heaviside layer. The organized transverse oscillations can be given a treatment similar to that of the longitudinal oscillations.[2,5] The dispersion relation for these waves is

$$\omega^2 \cong \omega_p^2 + c^2 k^2.$$

The minimum wavelength for which the collective behavior is important is, in this case,

$$\lambda_C \cong c/\omega_p.$$

This distance is clearly much longer than the Debye length.

[4] P. Debye and E. Huckel, Physik Z. **24**, 185 (1923). This phenomenon was first studied in connection with highly ionized electrolytes.

[5] For a treatment along the lines of Bohm and Gross, reference 3, see D. Pines, PhD. thesis, Princeton University (1950).

In order to obtain a collective description of the electrons in a metal we must use a quantum-mechanical treatment, since the electron gas is highly degenerate. Previous treatments of the organized oscillations of an electron gas have been comparatively unsystematic in that hamiltonian methods were not employed, so that the results were not extensible to quantum theory. In our treatment of the collective description hamiltonian methods will be employed throughout.

Let us consider the hamiltonian for a collection of charges interacting with the electromagnetic field, the particles and the field being described by appropriate canonical coordinates. This hamiltonian may be represented schematically as

$$H_0 = H_{\text{part}} + H_{\text{inter}} + H_{\text{field}} \quad (3)$$

where H_{part} represents the kinetic energy of the electrons, H_{inter} represents the interaction between the electrons and the electromagnetic field, and H_{field} represents the energy contained in the electromagnetic field.

Our program is to find a canonical transformation to a new set of variables which will provide a collective description of the system leading to results classically equivalent to those obtained in the noncanonical treatment. Thus, we shall require that the new field variables oscillate independently of the new particle variables with the characteristic frequency of organized oscillation. When we do this we find that the hamiltonian in the collective description can be represented schematically as

$$H^{(1)} = H^{(1)}{}_{\text{part}} + H_{\text{osc}} + H_{\text{part int}}, \quad (4)$$

where $H^{(1)}{}_{\text{part}}$ corresponds to the kinetic energy in these new coordinates and H_{osc} is a sum of harmonic oscillator terms with frequencies given by the dispersion relation for organized oscillations. $H_{\text{part int}}$ then corresponds to a screened force between particles, which is large only for distances shorter than the appropriate minimum distance associated with organized oscillations. Thus, we obtain explicitly in hamiltonian form the effective separation between long range collective interactions, described here in terms of organized oscillations, and the short-range interactions between individual particles.[6]

In our treatment, certain approximations must be made, which are discussed in detail in Sec. II, in connection with the collective approximation. These approximations reflect the fact that while the effect of the average field on an individual particle is small, these cumulative small contributions from each particle

[6] In the case of longitudinal oscillations, this model for electron interactions provides a physical basis for the hitherto empirical use of a screened coulomb force to represent correlation effects. Furthermore, it predicts a screening radius which produces agreement with the experimental results for both the electronic contribution to the specific heat of a metal, and the width of the tail of the soft x-ray emission curve for sodium. D. Bohm and D. Pines, Phys. Rev. **80**, 903 (1950).

to the average field may produce a large change in these fields relative to what they would be in a medium of low charge density. Thus, perturbation theory is applied to the solution for the motion of each particle in the average field of all the others, but cannot be applied in similar fashion for the solution of the field equations of motion. On the other hand, because of the high density of particles, we may assume that only the response of a particle which is in phase with the field producing it will be important and that other responses which depend on the position of the particle can be neglected.

We confine our attention in this paper to the collective description of the interactions between electrons brought about through the medium of the transverse electromagnetic field. These magnetic interactions are weaker than the corresponding coulomb interactions by a factor of approximately v^2/c^2 and, consequently, are not usually of great physical interest. However, the canonical treatment of the transverse field is more straightforward mathematically than that of the longitudinal field; and since we would like to illustrate clearly the techniques and approximations involved in our methods, we therefore investigate first the role of the organized transverse oscillations in a description of electron interactions. In Sec. II, we give a classical treatment and discuss in detail the collective approximation and the effective residual interparticle force. In Sec. III we give the analogous quantum-mechanical treatment of the collective description. The longitudinal oscillations will be treated in a subsequent paper.

II. THE CLASSICAL CANONICAL TRANSFORMATION TO THE COLLECTIVE DESCRIPTION

(A) Generating Function for the Transformation

The hamiltonian for a collection of charges interacting with a transverse electromagnetic field may be written

$$H=\sum_i[\mathbf{p}_i+e\mathbf{A}(\mathbf{x}_i)/c]^2/2m+\int([E^2(\mathbf{x})+H^2(\mathbf{x})]/8\pi)d\mathbf{x}. \quad (5)$$

We expand $\mathbf{A}(\mathbf{x})$ in a fourier series in a cube of volume L^3, and impose periodic boundary conditions. Thus,

$$\mathbf{A}(\mathbf{x})=\sum_{k\mu}(4\pi c^2/L^3)^{\frac{1}{2}}q_{k\mu}\boldsymbol{\varepsilon}_{k\mu}\exp(i\mathbf{k}\cdot\mathbf{x}), \quad (6)$$

where $\boldsymbol{\varepsilon}_{k\mu}$ is a unit polarization vector, and for this transverse field, μ takes on values 1 and 2, representing the two possible values of polarization perpendicular to the direction of propagation. The wave number vector $\mathbf{k}$ is assumed to take on both plus and minus values, i.e., $k_x=2\pi n_x/L$, where $n_x=-\infty\cdots+\infty$, etc. To ensure that $\mathbf{A}(\mathbf{x})$ is real, we take

$$\boldsymbol{\varepsilon}_{k\mu}=\boldsymbol{\varepsilon}_{-k\mu},\quad q_{k\mu}=(q_{-k\mu})^*. \quad (7)$$

Then, following the usual treatments,[7] it can be shown that our hamiltonian (5) becomes:

$$H=\sum_i(p_i^2/2m)+\sum_{ik\mu}(4\pi e^2/L^3)^{\frac{1}{2}}[(\mathbf{p}_i\cdot\boldsymbol{\varepsilon}_{k\mu})/m]q_{k\mu}\exp(i\mathbf{k}\cdot\mathbf{x}_i)+\sum_{ikl\mu\nu}(2\pi e^2/mL^3)q_{k\mu}q_{l\nu}\boldsymbol{\varepsilon}_{k\mu}\cdot\boldsymbol{\varepsilon}_{l\nu}\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i]+\tfrac{1}{2}\sum_{k\mu}(p_{k\mu}p_{-k\mu}+c^2k^2q_{k\mu}q_{-k\mu}). \quad (8)$$

Here $p_{k\mu}$ is the canonical conjugate to $q_{k\mu}$, and we also have

$$p_{k\mu}=(p_{-k\mu})^*. \quad (9)$$

We find it convenient, for reasons which will become clear later, to split up the term

$$(2\pi e^2/mL^3)\sum_{ikl\mu\nu}q_{k\mu}q_{l\nu}\boldsymbol{\varepsilon}_{k\mu}\cdot\boldsymbol{\varepsilon}_{l\nu}\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i]$$

in those terms for which $\mathbf{l}=-\mathbf{k}$ and $\mathbf{l}\neq-\mathbf{k}$. When this is done, our hamiltonian corresponds to the schematic hamiltonian of the previous section, with

$$H_{\text{part}}=\sum_i p_i^2/2m, \quad (10a)$$

$$H_{\text{inter}}=\sum_{ik\mu}(4\pi e^2/m^2L^3)^{\frac{1}{2}}(\mathbf{p}_i\cdot\boldsymbol{\varepsilon}_{k\mu})q_{k\mu}\exp(i\mathbf{k}\cdot\mathbf{x}_i)+(2\pi e^2/mL^3)\sum_{\substack{kli\mu\nu\\ l\neq-k}}q_{k\mu}q_{l\nu}\boldsymbol{\varepsilon}_{k\mu}\cdot\boldsymbol{\varepsilon}_{l\nu}\times\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i], \quad (10b)$$

$$H_{\text{field}}=\tfrac{1}{2}\sum_{k\mu}[p_{k\mu}p_{-k\mu}+(c^2k^2+\omega_p^2)q_{k\mu}q_{-k\mu}]. \quad (10c)$$

We will now show that a canonical transformation to the new coordinates $(X_i, P_i; Q_k, \Pi_k)$, which is generated by $S(x_i, P_i; q_k, \Pi_k)$ as given below, constitutes the desired transformation to the collective description. This generating function is

$$S(x_i, P_i; q_k, \Pi_k)=\sum_i\mathbf{x}_i\cdot\mathbf{P}_i+\sum_{k\mu}q_{k\mu}\Pi_{k\mu}+F(x_i, P_i; q_k, \Pi_k), \quad (11)$$

where

$$F(x_i, P_i; q_k, \Pi_k)=\sum_{k\mu}\xi_{k\mu}(\mathbf{P}_i)q_{k\mu}\Pi_{k\mu}-(4\pi e^2/m^2L^3)^{\frac{1}{2}}i\times\sum_{l\nu j}(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{l\nu})\frac{[(\mathbf{l}\cdot\mathbf{P}_j/m)q_{l\nu}+i\Pi_{-l\nu}]}{\omega^2-(\mathbf{l}\cdot\mathbf{P}_j)^2/m^2}\exp i\mathbf{l}\cdot\mathbf{x}_j. \quad (12)$$

$\xi_{k\mu}(\mathbf{P}_i)$ depends only on the particle momenta and is of the order v^2/c^2; it will be determined in the course of working out the consequences of this transformation (see Eq. (25)); ω represents the frequency of the organized transverse field oscillations.[8] With this gener-

[7] See, for instance, G. Wentzel, *The Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), Chapter IV.

[8] It will be seen that ω depends on the momentum P_i. However, this dependence is of the second order (in v/c), and the corresponding changes in the transformation equations may be neglected.

ating function we obtain the following equations relating the old and new coordinates:

$$p_{i\lambda}=\frac{\partial S}{\partial x_{i\lambda}}=P_{i\lambda}+\left(\frac{4\pi e^2}{m^2L^3}\right)^{\frac{1}{2}}\sum_{l\nu}\left\{\frac{\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu}}{\omega^2-(\mathbf{l}\cdot\mathbf{P}_i)^2/m^2}\times\{(\mathbf{l}\cdot\mathbf{P}_i/m)q_{l\nu}+i\Pi_{l\nu}\}l_\lambda\exp(i\mathbf{l}\cdot\mathbf{x}_i)\right\}, \quad (13)$$

$$\begin{aligned}X_{i\lambda}=\partial S/\partial P_{i\lambda}=x_{i\lambda}+(4\pi e^2/m^2L^3)^{\frac{1}{2}}\\ \times\sum_{l\nu}\{[i(\epsilon_{l\nu})_\lambda/[\omega^2-(\mathbf{l}\cdot\mathbf{P}_i)^2/m^2]]\\ \times[[(\mathbf{l}\cdot\mathbf{P}_i)/m]q_{l\nu}+i\Pi_{-l\nu}]\\ +[(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu})l_\lambda/[\omega^2-(\mathbf{l}\cdot\mathbf{P}_i)^2/m^2]]\\ \times[[\omega^2+(\mathbf{l}\cdot\mathbf{P}_i/m)^2]q_{l\nu}+2i(\mathbf{l}\cdot\mathbf{P}_i/m)\Pi_{-l\nu}]\}\\ \times\exp(i\mathbf{l}\cdot\mathbf{x}_i)+\sum_{k\mu}\Pi_{k\mu}q_{k\mu}(\partial\xi_{k\mu}/\partial P_{i\lambda}),\end{aligned} \quad (14)$$

$$\begin{aligned}p_{k\mu}=\partial S/\partial q_{k\mu}=\Pi_{k\mu}(1+\xi_{k\mu})\\ -(4\pi e^2/m^2L^3)^{\frac{1}{2}}i\sum_j\{(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{k}\cdot\mathbf{P}_j)/\\ m[\omega^2-(\mathbf{k}\cdot\mathbf{P}_j/m)^2]\}\exp(i\mathbf{k}\cdot\mathbf{x}_j),\end{aligned} \quad (15)$$

$$\begin{aligned}Q_{-k\mu}=\partial S/\partial\Pi_{-k\mu}=q_{-k\mu}(1+\xi_{k\mu})-(4\pi e^2/m^2L^3)^{\frac{1}{2}}\\ \times\sum_j\{(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})/[\omega^2-(\mathbf{k}\cdot\mathbf{P}_j/m)^2]\}\exp(i\mathbf{k}\cdot\mathbf{x}_j).\end{aligned} \quad (16)$$

We see that there is a rather complicated interrelationship between the old and the new coordinates defined by these equations. We will not be able to solve these equations *exactly* to obtain the old coordinates in terms of the new coordinates or vice versa. We could obtain approximate expressions for the old coordinates in terms of the new, and then determine what form the hamiltonian will take in terms of the new variables. However, we feel that the nature of the collective approximation will be revealed more clearly by a somewhat different procedure. We will substitute the "mixed" expressions for p_i, $p_{k\mu}$, and $q_{k\mu}$ [(13), (15), and (16)], into our old hamiltonian and thus obtain a hamiltonian in terms of both old and new coordinates. However, this hamiltonian can be simplified considerably with the aid of the collective approximation, and after achieving this simplification, we then express the hamiltonian entirely in terms of the new variables.

B. Nature of the Collective Approximation

The approximations we find it necessary to make, in order to apply the collective description to the electron gas, have been grouped by us under the general heading of the collective approximation. The collective approximation involves the following requirements:

(1) The short-range electron-ion and electron-electron collisions are neglected. This assumes that we are dealing with an electron gas in which the mean free time for such collisions is considerably longer than the period of an organized oscillation. Actually, such collisions tend to disrupt the organized oscillation of the system as a whole; but if they are not too frequent, they lead only to a small damping of the oscillations.[9] The long free path needed for the validity of this approximation actually occurs because of the screening of the long-range part of the force, which leaves only the short-range interactions to be accounted for in terms of collisions.

(2) The organized oscillations are assumed to be of sufficiently small amplitude that each particle suffers only a small perturbation in its straight line motion due to the combined fields of all the other particles. Thus, we will neglect quadratic field terms in the electron or field equations of motion and apply perturbation theory to the particle motions. This is the customary linear approximation, appropriate for small oscillations.

(3) We distinguish between two kinds of response of the electrons to a wave. One of these is in phase with the wave, so that the phase difference between the particle response and the wave producing it is independent of the position of the particle. This is the response which contributes to the organized behavior of the system. The other response has a phase difference with the wave producing it which depends on the position of the particle. Because of the general random location of the particles, this second response tends to average out to zero when we consider a large number of electrons, and we shall neglect the contributions arising from this. This procedure we call the "random phase approximation."

(4) We shall assume the smallness of $(\mathbf{k}\cdot\mathbf{v})/\omega$. In our case, $\omega/k\gtrsim c$, so that the smallness of $(\mathbf{k}\cdot\mathbf{v})/\omega$ follows from the smallness of v/c. For longitudinal oscillations, however, we can have small $(\mathbf{k}\cdot\mathbf{v})/\omega$ only for wavelengths appreciably longer than the Debye length, λ_D. This follows because λ_D is essentially $\bar{V}/\omega$, where $\bar{V}$ is a suitable mean speed. Thus, the collective description is applicable only for long enough wavelengths. This result agrees with the general conclusion cited previously that both statically and dynamically a dense ion gas exhibits organized behavior only for wavelengths longer than λ_D or λ_C.

The collective approximation is similar to a complete perturbation theory treatment in that perturbation theory is applied to the particle motion. The difference between the two approaches lies in the assumptions made regarding the fields produced by the particles. For instance, in Eqs. (15), and (16) we see that the old and new field coordinates and momenta differ by a series of terms summed over all particle coordinates. A complete perturbation theoretical treatment would require that this sum be small, but we do not make this assumption here. Instead we note that the small modifications of the net field arising from each particle may add up to a large change of the average field.

The principal advantage of the collective approximation is that it does not require a small cumulative response to the fields. Thus, in this approximation we shall retain terms in the hamiltonian proportional to the square of the vector potential, whereas in a perturbation treatment these would be discarded, since they are formally of second order in the perturbing potential. As we shall see, in the case of high particle density, these can bring about a significant modification in the behavior of the electromagnetic field.

[9] This damping is discussed in detail in Bohm and Gross, reference 3.

C. Results of the Transformation; the Hamiltonian for the Collective Description

Let us now see what effect the transformation has on the various terms in our hamiltonian. Using Eq. (13) we have

$$H_{\text{part}}=\sum_i P_i^2/2m+(4\pi e^2/m^2L^3)^{\frac{1}{2}}\sum_{il\nu}[(\mathbf{l}\cdot\mathbf{P}_i)/m]$$
$$\times[\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu}]\frac{\{[(\mathbf{l}\cdot\mathbf{P}_i)/m]q_{l\nu}+i\Pi_{-l\nu}\}}{\omega^2-[(\mathbf{l}\cdot\mathbf{P}_i)/m]^2}\exp(i\mathbf{l}\cdot\mathbf{x}_i)$$
$$+\frac{2\pi e^2}{mL^3}\sum_{kl\mu\nu i}\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu})(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{k}\cdot\mathbf{l})/m}{[\omega^2-(\mathbf{l}\cdot\mathbf{P}_i/m)^2][\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2]}$$
$$\times\{(\mathbf{l}\cdot\mathbf{P}_i/m)q_{l\nu}+i\Pi_{-l\nu}\}\{((\mathbf{k}\cdot\mathbf{P}_i)/m)q_{k\mu}+i\Pi_{-k\mu}\}$$
$$\times\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i].\quad(17)$$

We consider the terms for which $\mathbf{l}=-\mathbf{k}$, and $\mathbf{l}\neq-\mathbf{k}$, separately in the quadratic field terms in (17). When $\mathbf{l}=-\mathbf{k}$, this quadratic term reduces to

$$\frac{2\pi e^2}{mL^3}\sum_{k\mu\nu i}\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})^2k^2}{m[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2]}$$
$$\{(\mathbf{k}\cdot\mathbf{P}_i/m)^2q_{k\mu}q_{-k\mu}+\Pi_{k\mu}\Pi_{-k\mu}\}\quad(18)$$

provided we assume an isotropic distribution of the $\mathbf{P}_i$. Using Eq. (13) we may write the terms arising from H_{inter} as

$$H_{\text{inter}}=(4\pi e^2/m^2L^3)^{\frac{1}{2}}\sum_{ik\mu}q_{k\mu}(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})\exp(i\mathbf{k}\cdot\mathbf{x}_i)$$
$$+\frac{4\pi e^2}{m^2L^3}\sum_{\substack{ikl\mu\nu\\ l\neq-k}}\left\{\frac{(\mathbf{l}\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})}{\omega^2-(\mathbf{l}\cdot\mathbf{P}_i/m)^2}\right.$$
$$\times[(\mathbf{l}\cdot\mathbf{P}_i/m)q_{l\nu}q_{-k\mu}+i\Pi_{-l\nu}q_{k\mu}]$$
$$\left.+\tfrac{1}{2}(\boldsymbol{\varepsilon}_{k\mu}\cdot\boldsymbol{\varepsilon}_{l\nu})q_{k\mu}q_{l\nu}\right\}\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i],\quad(19)$$

where we have split up the terms for which $\mathbf{l}=-\mathbf{k}$, and $\mathbf{l}\neq-\mathbf{k}$, and applied the transversality condition $\sum_\mu\mathbf{k}\cdot\boldsymbol{\varepsilon}_{k\mu}=0$. It is convenient to group the terms arising from the application of (15) and (16) to H_{field} as follows:

$$H_{\text{field}}=\sum_{k\mu}\{(\Pi_{k\mu}+\partial F/\partial q_{k\mu})(\Pi_{-k\mu}+\partial F/\partial q_{-k\mu})$$
$$+\omega^2[Q_{k\mu}Q_{-k\mu}-2q_{k\mu}(\partial F/\partial\Pi_{-k\mu})$$
$$-(\partial F/\partial\Pi_{k\mu})(\partial F/\partial\Pi_{-k\mu})]\}$$
$$+\tfrac{1}{2}\sum_{k\mu}(c^2k^2+\omega_p^2-\omega^2)q_{k\mu}q_{-k\mu}.\quad(20)$$

This particular combination is taken because F is a function of q_k. We then obtain

$$H_{\text{field}}=\tfrac{1}{2}\sum_{k\mu}\{\Pi_{k\mu}\Pi_{-k\mu}(1+2\xi_{k\mu})$$
$$+\omega^2(Q_{k\mu}Q_{-k\mu}-2\xi_{k\mu}q_{k\mu}q_{-k\mu})+(\omega_p^2+c^2k^2-\omega^2)q_{k\mu}q_{-k\mu}\}$$
$$-\left(\frac{4\pi e^2}{m^2L^3}\right)^{\frac{1}{2}}\sum_{jk\mu}\frac{\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu}(1+\xi_{k\mu})}{\omega^2-(\mathbf{k}\cdot\mathbf{P}_j/m)^2}$$
$$\times\{-i(\mathbf{k}\cdot\mathbf{P}_j/m)\Pi_{-k\mu}+\omega^2q_{k\mu}\}\exp(i\mathbf{k}\cdot\mathbf{x}_j)-(2\pi e^2/m^2L^3)$$
$$\times\sum_{ijk\mu}\frac{\{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)(\mathbf{k}\cdot\mathbf{P}_j/m)]\}}{[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2][\omega^2-(\mathbf{k}\cdot\mathbf{P}_j/m)^2]}$$
$$\times\exp[i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)].\quad(21)$$

In (21) we have neglected terms involving $\xi_{k\mu}^2$, since these are of order v^4/c^4, and for the particles with which we will be concerned $v^2/c^2\ll1$.

The sum of the expressions (17), (19), and (21) will give us our hamiltonian (8) expressed in terms of a mixture of the old and new coordinates. When we add these expressions, we find that those terms which are linear in the field coordinates are reduced to

$$-(4\pi e^2/m^2L^3)^{\frac{1}{2}}\sum_{jk\mu}\xi_{k\mu}(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})$$
$$\times\frac{\{\omega^2q_{k\mu}+i(\mathbf{k}\cdot\mathbf{P}_j/m)\Pi_{-k\mu}\}}{\omega^2-(\mathbf{k}\cdot\mathbf{P}_j/m)^2}\exp(i\mathbf{k}\cdot\mathbf{x}_j).\quad(22)$$

But this term is of order v^2/c^2 smaller than our original H_{inter}, Eq. (10b). We could devise a further canonical transformation which would eliminate terms of this order too, but this would only introduce corrections of order v^4/c^4 in the interparticle force. Since we will confine ourselves to the terms of lowest order (v^2/c^2) in the interaction between particles, we can neglect this small term (22) entirely.

Let us now investigate the quadratic field terms resulting from the sum of (17), (19), and (21). These fall into two categories, which are given in (23) and (24) below:

$$\frac{2\pi e^2}{mL^3}\sum_{\substack{ikl\mu\nu\\ k\neq-l}}\left\{\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu})(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})\mathbf{l}\cdot\mathbf{k}[(\mathbf{l}\cdot\mathbf{P}_i/m)(\mathbf{k}\cdot\mathbf{P}_i/m)q_{l\nu}q_{k\mu}-\Pi_{k\mu}\Pi_{l\nu}+2i(\mathbf{k}\cdot\mathbf{P}_i/m)q_{k\mu}\Pi_{-l\nu}]}{m^2[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2][\omega^2-(\mathbf{l}\cdot\mathbf{P}_i/m)^2]}\right.$$
$$\left.+\frac{2(\mathbf{l}\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu})}{m[\omega^2-(\mathbf{l}\cdot\mathbf{P}_i/m)^2]}\left[\left(\frac{\mathbf{l}\cdot\mathbf{P}_i}{m}\right)q_{l\nu}q_{-k\mu}+i\Pi_{-l\nu}q_{k\mu}\right]+q_{k\mu}q_{l\nu}(\boldsymbol{\varepsilon}_{k\mu}\cdot\boldsymbol{\varepsilon}_{l\nu})\right\}\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i],\quad(23)$$

$$\frac{2\pi e^2}{mL^3}\sum_{k\mu i}\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})^2k^2}{m^2[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2]}\left\{\left(\frac{\mathbf{k}\cdot\mathbf{P}_i}{m}\right)^2 q_{k\mu}q_{-k\mu}+\Pi_{k\mu}\Pi_{-k\mu}\right\}$$
$$+\tfrac{1}{2}\sum_{k\mu}\{(c^2k^2+\omega_p^2-\omega^2)q_{k\mu}q_{-k\mu}+2\xi_{k\mu}(\Pi_{k\mu}\Pi_{-k\mu}-\omega^2q_{k\mu}q_{-k\mu})\}+\tfrac{1}{2}\sum_{k\mu}(\Pi_{k\mu}\Pi_{-k\mu}+\omega^2Q_{k\mu}Q_{-k\mu}). \quad (24)$$

The terms in (23) are all quadratic in the field variables, and in addition have a phase factor $\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i]$, the argument of which never vanishes. In order to discuss these terms, we transform from the old coordinates x_i to the new coordinates X_i, according to Eq. (14). We will then have two types of terms; those in which x_i is replaced by X_i, and those which arise from the difference between x_i and X_i. Now the X_i, because $H_{\text{part int}}$ is a screened short-range interaction, behave to a good degree of approximation like the coordinates of a free particle. We may then assume that these are distributed at random, as in a perfect gas. Since we have a very large number of particles, the nonvanishing argument of the phase factor, $\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{X}_i]$, will cause the contribution of those terms in which it is present to average out to zero. This is essentially the random-phase approximation.

The effects of the correlations in the x_i are represented in the difference between x_i and X_i. However, the terms arising from this difference multiply quadratic field terms; and it may be shown that the resultant products either make nonlinear contributions to the equations of motion, which we neglect in the linear approximation, or average out to zero because they contain a phase factor with nonvanishing argument.[5] Thus, in general, terms which are quadratic in the field variables and contain a phase factor with nonvanishing argument may be neglected in the collective approximation.

When we consider the terms in (24), we see that if we take

$$\xi_{k\mu}=-(2\pi e^2/mL^3)\sum_i k^2(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu}/m)^2/[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2], \quad (25)$$

then the quadratic terms in $\Pi_{k\mu}\Pi_{-k\mu}$, reduce to $\frac{1}{2}\sum_{k\mu}\Pi_{k\mu}\Pi_{-k\mu}$. We similarly see that with this choice of $\xi_{k\mu}$, the quadratic terms in $q_{k\mu}q_{-k\mu}$ become

$$\tfrac{1}{2}\sum_{k\mu}q_{k\mu}q_{-k\mu}\left\{c^2k^2+\omega_p^2+(4\pi e^2/mL^3)\times\sum_i\frac{\{k^2(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu}/m)^2[\omega^2+(\mathbf{k}\cdot\mathbf{P}_i/m)^2]\}}{[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2]^2}-\omega^2\right\}; \quad (26)$$

but, if we take

$$\omega^2=c^2k^2+\omega_p^2+(4\pi e^2/mL^3)\times\sum_i\frac{\{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu}/m)^2k^2[\omega^2+(\mathbf{k}\cdot\mathbf{P}_i/m)^2]\}}{[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2]^2}, \quad (27)$$

then this term vanishes. This choice of ω^2 is just the dispersion relation for transverse plasma oscillations.[2,10] Thus we see that we have reduced the field terms represented in (24) to

$$\tfrac{1}{2}\sum_{k\mu}(\Pi_{k\mu}\Pi_{-k\mu}+\omega^2Q_{k\mu}Q_{-k\mu}). \quad (28)$$

The remaining terms are given by

$$-\frac{2\pi e^2}{m^2L^3}\sum_{ijk\mu}\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)(\mathbf{k}\cdot\mathbf{P}_j/m)]}{[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2][\omega^2-(\mathbf{k}\cdot\mathbf{P}_j/m)^2]}\times\exp[i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)]. \quad (29)$$

We now wish to express this term entirely in terms of our new coordinates. One can show, using the linear approximation and the random-phase approximation, that the lowest order (in v/c) field term resulting from substituting our expression for x_i, (14) into (29), is

$$-\left(\frac{4\pi e^2}{m^2L^3}\right)^{\frac{1}{2}}\sum_{ik\mu}\frac{\xi_{k\mu}\omega^2(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})q_{k\mu}}{\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2}\exp(i\mathbf{k}\cdot\mathbf{X}_i). \quad (30)$$

This term, which is of the same sort as (22), may be neglected for the same reason—that its inclusion will lead to fourth-order terms in v/c in $H_{\text{part int}}$.[11] Thus, we obtain, to this order of approximation, only the above term (29) with x_i replaced by X_i.

Our transformed hamiltonian, expressed entirely in terms of the new coordinates, the "collective" variables, thus takes the form:

$$H^{(1)}=\sum_i(P_i^2/2m)+\tfrac{1}{2}\sum_{k\mu}\Pi_{k\mu}\Pi_{-k\mu}+\omega^2Q_{k\mu}Q_{-k\mu}-(2\pi e^2/m^2L^3)\times\sum_{ij\mu}\frac{\{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)(\mathbf{k}\cdot\mathbf{P}_j/m)]\}}{[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2][\omega^2-(\mathbf{k}\cdot\mathbf{P}_j/m)^2]}\times\exp[i\cdot\mathbf{k}(\mathbf{X}_i-\mathbf{X}_j)]. \quad (31)$$

This is just the hamiltonian we sought, as discussed in Sec. I, with

$$H^{(1)}{}_{\text{part}}=\sum_i P_i^2/2m, \quad (32a)$$

[10] This differs from the dispersion relation quoted by Langmuir in the terms of order v^2/c^2 which he neglected. It may be shown, using methods similar to those of Langmuir, that the above dispersion relation is obtained when these terms are not neglected —see D. Pines, reference 5.

[11] The term in Eq. (14) involving $\partial\xi/\partial P_i$ leads to quadratic field terms multiplied by phase factors which do not vanish. According to the general properties of the collective approximation this term can be neglected.

$$H_{\mathrm{osc}}=\tfrac{1}{2}\sum_{k\mu}\Pi_{k\mu}\Pi_{-k\mu}+\omega^2 Q_{k\mu}Q_{-k\mu},\tag{32b}$$

$$H_{\mathrm{part\ int}}=-(2\pi e^2/m^2L^3)\times\sum_{ij\mu}\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)(\mathbf{k}\cdot\mathbf{P}_j/m)]}{[\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2][\omega^2-(\mathbf{k}\cdot\mathbf{P}_j/m)^2]}\times\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)].\tag{32c}$$

We see that the new field and particle variables no longer interact. The new field coordinates carry out oscillations of frequency ω given by the transverse field dispersion relation, Eq. (27), and thus correctly describe the organized transverse oscillations. The new particle coordinates act like those of a free particle, to the extent that we are able to neglect $H_{\mathrm{part\ int}}$, which represents an effective residual particle interaction.

D. Effective Residual Particle Interaction

Let us now investigate $H_{\mathrm{part\ int}}$ in some detail. It effectively corresponds to that part of the magnetic interaction between electrons which is not describable in terms of the organized transverse oscillations. Since $\omega^2 \geqslant c^2k^2$, we see that this term will be at least of order v^2/c^2 smaller in magnitude than the corresponding coulomb interaction between the electrons, which is given by

$$(2\pi e^2/L^3)\sum_{ijk}(1/k^2)\exp[i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)].$$

Hence, we shall consider only the lowest order (in v/c) terms represented in (32c), neglecting those terms of order v^4/c^4 as compared with the coulomb term. The lowest order terms may be written:

$$H_{\mathrm{part\ int}}\cong-(2\pi e^2/m^2L^3)\times\sum_{ijk\mu}\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})}{\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2}\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)].\tag{33}$$

Now to this order of approximation, we may write $\omega^2=c^2k^2+\omega_p{}^2$, so that we have

$$H_{\mathrm{part\ int}}=-\frac{2\pi e^2}{m^2L^3}\sum_{ijk\mu}\left\{\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})}{c^2k^2+\omega_p{}^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2}\right\}\times\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)].\tag{34}$$

We now compare the above expression with the analogous term in the hamiltonian from which may be derived the Biot-Savart law for the magnetic interaction between particles. This may be shown to be

$$H_{\mathrm{magn\ int}}=-(2\pi e^2/m^2L^3)\times\sum_{ijk\mu}\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})}{c^2k^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2}\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)].\tag{35}$$

Because of the presence of $\omega_p{}^2$ in the denominator in Eq. (34), we see that when the ion density is large, the interaction is greatly reduced for small values of k, or to put it another way, screened out for sufficiently large interparticle distances. To see this in more detail, let us evaluate $H_{\mathrm{part\ int}}$ in terms of the interparticle distances. First, we carry out the sum over polarization directions, obtaining

$$H_{\mathrm{part\ int}}=-\frac{2\pi e^2}{m^2L^3}\sum_{ijk}\frac{\{P_i\cdot P_j-(\mathbf{k}\cdot\mathbf{P}_i)(\mathbf{k}\cdot\mathbf{P}_j)/k^2\}}{c^2k^2+\omega_p{}^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2}\times\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)].\tag{36}$$

For simplicity we consider only the first term in $H_{\mathrm{part\ int}}$ and we neglect $(\mathbf{k}\cdot\mathbf{P}_i/m)^2$ in the denominator, since it will always be order v^2/c^2 smaller than c^2k^2. We replace the sum over k by an integral, multiplying by the density in k space, $(L/2\pi)^3$. We thus have:

$$H^{(1)}{}_{\mathrm{part\ int}}=-[2\pi e^2/m^2(2\pi)^3]\sum_{ij}\mathbf{P}_i\cdot\mathbf{P}_j\int dk\times\{[1/\omega_p{}^2+c^2k^2]\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)]\}.\tag{37}$$

The integral over k yields just

$$2\pi^2[\exp\{-[\omega_p/c]|\mathbf{X}_i-\mathbf{X}_j|\}]/|\mathbf{X}_i-\mathbf{X}_j|,$$

so we have

$$H^{(1)}{}_{\mathrm{part\ int}}=-(e^2/2)\sum_{ij}(\mathbf{P}_i\cdot\mathbf{P}_j/m^2)\times[\exp\{-(\omega_p/c)|\mathbf{X}_i-\mathbf{X}_j|\}]/|\mathbf{X}_i-\mathbf{X}_j|.\tag{38}$$

This leads to a screened Biot-Savart law of interaction between the particles, with the interaction being screened out at distances $\sim c/\omega_p$. For a metal, where $n_0\sim 10^{22}$, this screening distance is $\sim 3\times 10^{-5}$ cm.

III. QUANTUM-MECHANICAL TRANSFORMATION TO THE COLLECTIVE DESCRIPTION

A. Generating Function for the Transformation

We now carry out a quantum-mechanical treatment of the preceding classical material. We shall see that the quantum-mechanical calculation yields essentially the same results as the classical treatment in the preceding section.

We find it convenient to work in the following representation. We expand in terms of the creation and annihilation operators for the transverse field, $a_{k\mu}$ and $(a_{k\mu})^*$,

$$\mathbf{A}(\mathbf{x})=\sum_{k\mu}(2\pi\hbar c^2/\omega L^3)^{\frac{1}{2}}[a_{k\mu}\exp(i\mathbf{k}\cdot\mathbf{x})+a_{k\mu}{}^*\exp(-i\mathbf{k}\cdot\mathbf{x})]\boldsymbol{\varepsilon}_{k\mu}.\tag{39}$$

These are connected with $p_{k\mu}$ and $q_{k\mu}$ by the following relations:

$$\begin{aligned}a_{k\mu}&=(\omega/2\hbar)^{\frac{1}{2}}[q_{k\mu}+i(p_{-k\mu}/\omega)],\\ a_{k\mu}{}^*&=(\omega/2\hbar)^{\frac{1}{2}}[q_{k\mu}-i(p_{-k\mu}/\omega)],\\ q_{k\mu}&=(\hbar/2\omega)^{\frac{1}{2}}[a_{k\mu}+a_{-k\mu}{}^*],\\ p_{k\mu}&=-i(\hbar\omega/2)^{\frac{1}{2}}[a_{k\mu}-a_{-k\mu}{}^*].\end{aligned}\tag{40}$$

Using these, one can show that the classical hamiltonian (8), which is equally valid in the quantum-mechanical case, leads to the following hamiltonian[12] in this representation:

$$H=\sum_i(p_i^2/2m)+(e/m)\sum_{k\mu i}(2\pi\hbar/\omega L^3)^{\frac{1}{2}}(\mathbf{p}_i\cdot\boldsymbol{\varepsilon}_{k\mu})$$
$$\times(a_{k\mu}+a_{-k\mu}^*)\exp(i\mathbf{k}\cdot\mathbf{x}_i)+(2\pi e^2/mL^3)$$
$$\times\sum_{\substack{kl\mu\nu i\\ l\neq -k}}(\hbar/\omega)(\boldsymbol{\varepsilon}_{k\mu}\cdot\boldsymbol{\varepsilon}_{l\nu})(a_{k\mu}+a_{-k\mu}^*)(a_{l\nu}+a_{-l\nu}^*)$$
$$\times\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i]+\tfrac{1}{2}\sum_{k\mu}\hbar\omega(a_{k\mu}a_{k\mu}^*+a_{k\mu}^*a_{k\mu})$$
$$+\tfrac{1}{4}\sum_{k\mu}[\hbar(\omega_p^2+c^2k^2-\omega^2)/\omega][a_{k\mu}a_{k\mu}^*+a_{k\mu}^*a_{k\mu}$$
$$+a_{-k\mu}a_{k\mu}+a_{-k\mu}^*a_{k\mu}^*].\quad(41)$$

We may adopt the schematic notation of Sec. I, with

$$H_{\text{part}}=\sum_i(p_i^2/2m),\quad(42\text{a})$$

$$H_{\text{inter}}=(e/m)\sum_{k\mu}(2\pi\hbar/\omega L^3)^{\frac{1}{2}}(a_{k\mu}+a_{-k\mu}^*)(\mathbf{p}_i\cdot\boldsymbol{\varepsilon}_{k\mu})$$
$$\times\exp(i\mathbf{k}\cdot\mathbf{x}_i)+(4\pi e^2/mL^3)\sum_{\substack{kl\mu\nu i\\ l\neq -k}}(\hbar/2\omega)(\boldsymbol{\varepsilon}_{k\mu}\cdot\boldsymbol{\varepsilon}_{l\nu})$$
$$\times(a_{k\mu}+a_{-k\mu}^*)(a_{l\nu}+a_{-l\nu}^*)\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i],\quad(42\text{b})$$

$$H_{\text{field}}=\tfrac{1}{2}\sum_{k\mu}\hbar\omega(a_{k\mu}a_{k\mu}^*+a_{k\mu}^*a_{k\mu})$$
$$+\tfrac{1}{4}\sum_{k\mu}[\hbar(\omega_p^2+c^2k^2-\omega^2)/\omega][a_{k\mu}a_{k\mu}^*+a_{k\mu}^*a_{k\mu}$$
$$+a_{-k\mu}a_{k\mu}+a_{k\mu}^*a_{-k\mu}^*].\quad(42\text{c})$$

The order of factors in (42b) is not essential because, due to the transversality condition, $\mathbf{p}_i\cdot\boldsymbol{\varepsilon}_{k\mu}$ commutes with $\exp(i\mathbf{k}\cdot\mathbf{x}_i)$. We are working in the Heisenberg representation and our operators satisfy the usual commutation rules:

$$\begin{aligned}[][a_{k\mu}^*,a_{l\nu}]&=\delta_{kl}\delta_{\mu\nu},\\ [a_{k\mu}^*,a_{l\nu}^*]&=[a_{k\mu},a_{l\nu}]=0,\\ [p_{i\lambda},x_{j\mu}]&=(\hbar/i)\delta_{ij}\delta_{\lambda\mu},\\ [p_{i\lambda},p_{j\mu}]&=[x_{i\lambda},x_{j\mu}]=0.\end{aligned}\quad(43)$$

Just as we did in the classical case, we now seek to find a canonical transformation to the collective description, in which the field coordinates describe the organized transverse oscillations. The transformation theory most suited to our purpose is briefly as follows.[13] We define new operators $(\mathbf{X}_i,\mathbf{P}_i)$ which possess the same eigenvalues, and satisfy the same commutation rules as the $(\mathbf{x}_i,\mathbf{p}_i)$ by

$$\begin{aligned}\mathbf{x}_i&=\{\exp(-iS/\hbar)\}\mathbf{X}_i\{\exp(iS/\hbar)\},\\ \mathbf{p}_i&=\{\exp(-iS/\hbar)\}\mathbf{P}_i\{\exp(iS/\hbar)\},\end{aligned}\quad(44\text{a})$$

and similarly

$$a_{k\mu}=\{\exp(-iS/\hbar)\}A_{k\mu}\{\exp(iS/\hbar)\}.\quad(44\text{b})$$

We may consider (44a) and (44b) as operator equations, and take S to be a function of the new operators $(\mathbf{X}_i,\mathbf{P}_i,A_{k\mu},A_{k\mu}^*)$ only. S will be the quantum-mechanical analog of our classical generating function. The relationship between the old and the new hamiltonians may similarly be viewed as an operator equation:

$$H=\{\exp(-iS/\hbar)\}\mathcal{H}\{\exp(iS/\hbar)\}\quad(44\text{c})$$

where $\mathcal{H}$ represents the hamiltonian expressed in the new coordinates.

The desired canonical transformation to the "collective" representation is then defined by the following generating function

$$S=-(ei/m)\sum_{il\nu}(2\pi\hbar/\omega L^3)^{\frac{1}{2}}$$
$$\times\{[(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu})A_{l\nu}/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+\hbar l^2/2m)]\exp(i\mathbf{l}\cdot\mathbf{X}_i)$$
$$-[\exp(-i\mathbf{l}\cdot\mathbf{X}_i)][A_{l\nu}^*\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu}]/$$
$$(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+\hbar l^2/2m)\}.\quad(45)$$

B. Consequences of the Transformation

In working out the consequences of this transformation, we shall expand the exponentials, $\exp(\pm iS/\hbar)$, so that (44c) may be written:

$$H=\mathcal{H}-(i/\hbar)[S,\mathcal{H}]-(1/2\hbar^2)[S,[S,\mathcal{H}]]+(i/6\hbar^3)[S,[S,[S,\mathcal{H}]]]+\cdots.\quad(46)$$

We will classify terms according to the power of S they contain—i.e., $[S,\mathcal{H}]$ is the first-order commutator of S and $\mathcal{H}$. We shall see that the terms arising from $[S,[S,[S,\mathcal{H}]]]$, and higher order commutators can be neglected if we restrict our attention to the lowest order terms in v/c.

The following relationships are useful in applying (46):

$$p_{i\lambda}=P_{i\lambda}-(i/\hbar)[S,P_{i\lambda}]+\cdots$$
$$=P_{i\lambda}+(e/m)\sum_{l\nu}(2\pi\hbar/\omega L^3)^{\frac{1}{2}}l_\lambda$$
$$\times\{[(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu})A_{l\nu}/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+\hbar l^2/2m)]$$
$$\times[\exp(i\mathbf{l}\cdot\mathbf{X}_i)]+[\exp(-i\mathbf{l}\cdot\mathbf{X}_i)]$$
$$[A_{l\nu}^*\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu}/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+\hbar l^2/2m)]\},\quad(47)$$

[12] This hamiltonian differs from the customary one, because we have expanded in terms of an arbitrary oscillation frequency ω, rather than taking $\omega=ck$.

[13] Quantum-mechanical transformation theory is developed in, for instance, P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, London, 1935), second edition.

$$\exp(i\mathbf{k}\cdot\mathbf{x}_i)=\exp(i\mathbf{k}\cdot\mathbf{X}_i)-(i/\hbar)[S,\exp(i\mathbf{k}\cdot\mathbf{X}_i)]+\cdots$$
$$=\exp(i\mathbf{k}\cdot\mathbf{X}_i)-(e/m\hbar)\sum_{l\nu}(2\pi\hbar/\omega L^3)^{\frac{1}{2}}$$
$$\times\{\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu}A_{l\nu}[1/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+\hbar l^2/2m)$$
$$-1/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+(\hbar l^2/2m)-\hbar\mathbf{l}\cdot\mathbf{k}/m)]$$
$$\times\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{X}_i]+[\exp[i(\mathbf{k}-\mathbf{l})\cdot\mathbf{X}_i]]$$
$$\times\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{l\nu}A_{l\nu}^*[1/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+(\hbar l^2/2m)$$
$$-\hbar\mathbf{l}\cdot\mathbf{k}/m)-1/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+\hbar l^2/2m)]$$
$$+\hbar\mathbf{k}\cdot\boldsymbol{\varepsilon}_{l\nu}[(\exp(i\mathbf{k}\cdot\mathbf{X}_i))(A_{l\nu}/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)$$
$$+\hbar l^2/2m))\exp(i\mathbf{l}\cdot\mathbf{X}_i))-(\exp(-i\mathbf{l}\cdot\mathbf{X}_i))$$
$$\times(A_{l\nu}^*/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+\hbar l^2/2m))$$
$$\times(\exp(-i\mathbf{k}\cdot\mathbf{X}_i))]\},\quad(48)$$

$$a_{k\mu}=A_{k\mu}-(i/\hbar)[S,A_{k\mu}]+\cdots$$
$$=A_{k\mu}-(e/m\hbar)\sum_i(2\pi\hbar/\omega L^3)^{\frac{1}{2}}[\exp(-i\mathbf{k}\cdot\mathbf{X}_i)]$$
$$\times[(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})/(\omega-(\mathbf{k}\cdot\mathbf{P}_i/m)+\hbar k^2/2m)].\quad(49)$$

The calculation of the commutators for (47) and (49) is trivial. In obtaining (48) one encounters commutators like

$$[\{1/(\omega-(\mathbf{l}\cdot\mathbf{P}_i/m)+\hbar l^2/2m)\},\{\exp(i\mathbf{k}\cdot\mathbf{X}_i)\}].$$

Such a commutator may be easily calculated, provided one notices that:

$$\{1/[\omega-\mathbf{l}\cdot\mathbf{P}_j/m+\hbar l^2/2m]\}\exp(i\mathbf{k}\cdot\mathbf{X}_i)$$
$$=\exp(i\mathbf{k}\cdot\mathbf{X}_i)\{1/[\omega-\mathbf{l}\cdot\mathbf{P}_j/m+\hbar l^2/2m-\hbar\mathbf{l}\cdot\mathbf{k}\delta_{ij}/m]\}.\quad(50)$$

That (50) is true, may be seen by multiplying both sides by $(\omega-\mathbf{l}\cdot\mathbf{P}_j/m+\hbar l^2/2m)$ and commuting this with $\exp(i\mathbf{k}\cdot\mathbf{X}_i)$ on the right-hand side. Using (50), we see that:

$$[\{1/(\omega-(\mathbf{l}\cdot\mathbf{P}_j/m)+\hbar l^2/2m)\},\exp(i\mathbf{k}\cdot\mathbf{X}_i)]$$
$$=\{\exp(i\mathbf{k}\cdot\mathbf{X}_i)\}\{1/(\omega-(\mathbf{l}\cdot\mathbf{P}_j/m)+(\hbar l^2/2m)-\hbar\delta_{ij}\mathbf{l}\cdot\mathbf{k}/m)-1/(\omega-(\mathbf{l}\cdot\mathbf{P}_j/m)+\hbar l^2/2m)\}.\quad(51)$$

We now consider the first-order terms arising from H_{part} and $\frac{1}{2}\sum_{k\mu}\hbar\omega(a_{k\mu}a_{k\mu}^*+a_{k\mu}^*a_{k\mu})$. These are given by

$$-(i/\hbar)[S,\{\sum_i(P_i^2/2m)+\tfrac{1}{2}\sum_{k\mu}\hbar\omega(A_{k\mu}A_{k\mu}^*+A_{k\mu}^*A_{k\mu})\}]$$
$$=-(e/m)\sum_{k\mu}(2\pi\hbar/\omega L^3)^{\frac{1}{2}}\{(A_{k\mu}+A_{-k\mu}^*)$$
$$\times(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})\exp(i\mathbf{k}\cdot\mathbf{X}_i)\},\quad(52)$$

as may be verified using the commutation rules analogous (43) for the new coordinates. But these terms thus cancel one of the zero-order terms arising from H_{inter}, which we denote by

$$\mathcal{H}^I_{\text{inter}}=(e/m)\sum_{k\mu}(2\pi\hbar/\omega L^3)^{\frac{1}{2}}(A_{k\mu}+A_{-k\mu}^*)$$
$$\times(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})\exp(i\mathbf{k}\cdot\mathbf{X}_i).\quad(53)$$

Thus, there are no interaction terms of this order (in v/c) in the new hamiltonian. The vanishing of the linear field terms in this order is one of the desired properties of the hamiltonian appropriate for the collective description.

When we consider the terms arising from the second-order commutators in H_{part} and $\frac{1}{2}\sum_{k\mu}\hbar\omega(a_{k\mu}a_{k\mu}^*+a_{k\mu}^*a_{k\mu})$, we see that they will yield just minus one-half the terms given by the first-order commutator arising from $\mathcal{H}^I_{\text{inter}}$, and that all of the terms arising from the higher order commutators in these two terms will bear a similar simple arithmetical relationship to the terms in the next-lower order commutator in $\mathcal{H}^I_{\text{inter}}$. The first-order commutator of S and $\mathcal{H}^I_{\text{inter}}$ is:

$$-(i/\hbar)[S,\mathcal{H}^I_{\text{inter}}]=-(ei/m\hbar)\sum_{k\mu i}(2\pi\hbar/\omega L^3)^{\frac{1}{2}}$$
$$\times\{[S,(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})](A_{k\mu}+A_{-k\mu}^*)(\exp(i\mathbf{k}\cdot\mathbf{X}_i))$$
$$+(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})[S,(A_{k\mu}+A_{-k\mu}^*)](\exp(i\mathbf{k}\cdot\mathbf{X}_i))$$
$$+(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(A_{k\mu}+A_{-k\mu}^*)[S,(\exp(i\mathbf{k}\cdot\mathbf{X}_i))]\}.\quad(54)$$

It is quite straightforward, but tedious to show that this reduces to

$$-(4\pi e^2/m^2L^3)\sum_{k\mu ij}\{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})/$$
$$[\omega^2-((\mathbf{k}\cdot\mathbf{P}_j/m)+\hbar k^2/2m)^2]\}\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)],\quad(55\text{a})$$
$$+\frac{4\pi e^2}{m^3L^3}\sum_{ik\mu}\frac{(\hbar k^2/2\omega)(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})^2}{[\omega-(\mathbf{k}\cdot\mathbf{P}_i/m)]^2-\hbar^2k^4/4m^2}$$
$$\times\{A_{k\mu}A_{k\mu}^*+A_{k\mu}^*A_{k\mu}+A_{k\mu}A_{-k\mu}+A_{k\mu}^*A_{-k\mu}^*\}.\quad(55\text{b})$$

In reducing (54) to (55a) and (55b) we have neglected a number of terms which are quadratic in field variables and are multiplied by a phase factor with nonvanishing argument, $\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{X}_i]$. This is the same approximation we made in the classical case, when we neglected the corresponding terms (23). The justification used for the classical approximation may be directly applied to this case. Since, as we have noted, the second-order commutators arising from H_{part} and $\frac{1}{2}\sum_{k\mu}\hbar\omega(a_{k\mu}a_{k\mu}^*+a_{k\mu}^*a_{k\mu})$ yield just $-\frac{1}{2}\{-(i/\hbar)[S,\mathcal{H}^I_{\text{inter}}]\}$, then the combined contribution of these terms and the first-order commutator with $\mathcal{H}^I_{\text{inter}}$ to our hamiltonian will be $-(i/2\hbar)[S,\mathcal{H}_{\text{inter}}]$ or one-half the sum of (55a) and (55b).

In addition to the unconsidered higher order commutators in the terms we have already discussed, there are two terms we have not yet investigated. One is a quadratic term in the field variables in H_{inter},

$$(4\pi e^2/mL^3) \sum_{\substack{kl\mu\nu \\ l\neq -k}} (\hbar/2\omega)(\boldsymbol{\varepsilon}_{k\mu}\cdot\boldsymbol{\varepsilon}_{l\nu})(a_{k\mu}+a_{-k\mu}^*) \times(a_{l\nu}+a_{-l\nu}^*)\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i].$$

This term may be neglected because it is quadratic in the field variables and contains a phase factor $\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i]$ with nonvanishing argument. The other term is from H_{field}, and is, to first order,

$$\tfrac{1}{4}\sum_{k\mu}(\omega_p^2+c^2k^2-\omega^2)(\hbar/\omega)\{A_{k\mu}A_{k\mu}^*+A_{k\mu}^*A_{k\mu} +A_{k\mu}A_{-k\mu}+A_{k\mu}^*A_{-k\mu}^*-(i/\hbar)[S,(A_{k\mu}A_{k\mu}^* +A_{k\mu}^*A_{k\mu}+A_{k\mu}A_{-k\mu}+A_{k\mu}^*A_{-k\mu}^*)]\}. \quad (56)$$

On comparing the zero-order terms in (56) with the quadratic field terms resulting from $-(i/2\hbar)[S,\mathcal{H}^I{}_{\text{inter}}]$ obtained from (55), we see that the sum of these will vanish if we take:

$$\omega^2=\omega_p^2+c^2k^2+(4\pi e^2/mL^3)\sum_{ik\mu}(k^2/\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})^2/m^2\} \times\{1/(\omega-\mathbf{k}\cdot\mathbf{P}_i/m)^2-\hbar^2k^4/4m^2\}. \quad (57)$$

This is our quantum-mechanical dispersion relation for the organized transverse oscillations. It is almost identical with the analogous classical dispersion relation (27), and reduces to the latter as $\hbar\rightarrow 0$.

When we combine all of the terms we have considered thus far, and assume that the frequency of the organized oscillations is specified by (57), we obtain

$$\mathcal{H}=\sum_i(P_i^2/2m)+\tfrac{1}{2}\sum_{k\mu}\hbar\omega(A_{k\mu}A_{k\mu}^*+A_{k\mu}^*A_{k\mu}) -\frac{2\pi e^2}{m^2L^3}\sum_{ijk\mu}\left\{\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})}{\omega^2-[(\mathbf{k}\cdot\mathbf{P}_j/m)+\hbar k^2/2m]^2}\right\} \times\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)], \quad (58)$$

where we have used Eqs. (55a), (55b), (56), and (57), and taken one-half of the sum of the terms in (55a) and (55b). This is just our desired hamiltonian for the collective description. It reduces to the classical hamiltonian (31) as $\hbar\rightarrow 0$. $H_{\text{part int}}$ is here given by

$$H_{\text{part int}}=-\frac{2\pi e^2}{m^2L^3}\sum_{ijk\mu}\frac{(\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})(\mathbf{P}_j\cdot\boldsymbol{\varepsilon}_{k\mu})}{\omega^2-[(\mathbf{k}\cdot\mathbf{P}_j/m)+\hbar k^2/2m]^2} \times\exp[i(\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)]. \quad (59)$$

It is essentially the same as our classical result (32c) since the $\hbar k^2/2m$ term is unimportant for most cases of interest.

We may neglect the terms arising from the higher order commutators we have not yet considered, for the higher order commutators in that part of H_{field} given in (56) are of the same character as those resulting from (55b). Thus, we see all of the higher order commutators will be arithmetical fractions of those resulting from $[S,[S,\mathcal{H}^I{}_{\text{inter}}]]$. But as may be seen from (55a) and (55b) the lowest order term arising from $[S,[S,\mathcal{H}^I{}_{\text{inter}}]]$ will be a factor of v^2/c^2 smaller than H_{inter}, and may be neglected, just as (22) was in the classical case, since its inclusion would lead to a fourth or higher power of v/c in $H_{\text{part int}}$.

Thus, in all respects the quantum-mechanical results are essentially the same as those obtained with our classical treatment of the previous sections. We find that we must carry out the same approximations for this case, and the analysis of $H_{\text{part int}}$ will yield results similar to those of Sec. II.

IV. CONCLUSION

In conclusion, we should like to point out that we have verified, both classically and quantum-mechanically, our qualitative picture of the role of organized transverse oscillations in electron interactions. We have seen that, by a suitable canonical transformation to the collective variables, we can show that the effects of magnetic interaction are divided naturally into the two components discussed earlier:

(1) The long-range part, ($\lambda>c/\omega_p$). This is responsible for the long-range organized behavior of the electrons, leading to modified transverse field oscillations. We may interpret these interactions as being redescribed in terms of the coordinates of the modified transverse oscillations.

(2) The short-range part, ($\lambda<c/\omega_p$), given by $H_{\text{part int}}$, which does not contribute to the organized behavior, and represents the residual particle-interaction after the organized behavior of the system has been taken into account.

Furthermore, in those regions in which organized behavior is unimportant (i.e., $\lambda<c/\omega_p$), the new hamiltonian reduces to the appropriate particle hamiltonian, in which the electrons interact according to the Biot-Savart law, and the transverse fields oscillate with frequency $\omega=ck$. For, as we have seen, $H_{\text{part int}}$ describes the Biot-Savart law for short wavelengths, and in this limit our dispersion relation becomes $\omega^2=c^2k^2$.