and

$$P=\pm\frac{\{J(J+1)-K(K\mp1)\}\{J(J+1)-(K\mp1)(K\mp2)\}}{2(K\mp1)\{(1-\xi)C_v-B_v\}}\times\left(\frac{B_e^2a}{\omega_s}\right)^2$$

for K other than ±1 and $l=\pm1$, the upper sign being taken for $l=+1$ and the lower for $l=-1$. This holds for K different from ±1 only when P is small. The frequencies of the transitions are then

$$\nu=2B_v(J+1)-4D_J(J+1)^3-2D_{JK}(J+1)K^2+4(2D_J+D_{JK})(J+1)Kl\xi+\Delta P(J,K,l),$$

where

$$\Delta P=\pm2(J+1)B_e^2a/\omega_s \quad\text{for}\quad K=\pm1,\quad l=\pm1$$

$$\Delta P=\pm\frac{(J+1)\{(J+1)^2-(K\mp1)^2\}}{(K\mp1)\{(1-\xi)C_v-B_v\}}(B_e^2a/\omega_s)^2,$$

for K other than ±1 and $l=\pm1$. As before, the upper sign is taken for $l=+1$ and the lower for $l=-1$.

The assignments of the excited vibrational lines, the observed frequencies, and the calculated positions of the lines are given in Table III. The parameters used to give the calculated line positions are: $B_v=2883.46$ Mc/sec, $D_J=0.2$ kc/sec, $D_{JK}=7.0$ kc/sec, $\xi=1.5$, $B_e^2a/\omega_s=1.81$ Mc/sec, and $(B_e^2a/\omega_s)^2/\{(1-\xi)C_v-B_v\}=7.0$ kc/sec. The parameters listed are those which gives the best fit to the data. Although the upper limit for ξ is $+1$ from theoretical considerations, it is not possible to fit the observed data using values of $\xi\leq+1$. From the value of B in the ground state and this excited state, $\alpha_{10}=-6.51$ Mc/sec.

We would like to thank Mr. Charles Greenhow for suggesting the method of solution of the secular equations for the excited vibrational states.

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The Effective Range of Nuclear Forces. Effect of the Potential Shape

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Corrections to the theory of the effective range, which take account of the shape of the neutron-proton potential, are discussed. The following quantities are calculated for a Yukawa, exponential, and square well potential: (*a*) various triplet ranges compatible with the value obtained for $\rho_t(0,-\epsilon)$ from the latest experiments, $(1.72\pm0.035)\times10^{-13}$ cm. (*b*) The singlet effective range r_{0s} from neutron-proton scattering cross sections at energies up to 6 Mev. (*c*) r_{0s} from neutron absorption cross sections by hydrogen. (*d*) Photoelectric and photomagnetic disintegration cross sections for the deuteron for various γ-rays.

It is shown that a comparison of the values obtained for r_{0s} from (*b*) and (*c*) could, with a slight increase in experimental accuracy, give an estimate of the potential shape. Present, very tentative, indications are for a short-tailed potential and a value for r_{0s} of about $(2.7\pm0.5)\times10^{-13}$ cm.

I. INTRODUCTION

In a previous paper[1] (quoted as **B**) Bethe developed formulas for nuclear scattering using the theory of the effective range. In a second paper[2] (quoted as **BL**) Bethe and Longmire applied the effective range theory to the photodisintegration of the deuteron. Throughout this paper we use, wherever possible, the same notation as in these references. In these two papers and in a paper[3] (quoted as **BJ**) by Blatt and Jackson it was shown that the effect of the shape of the nuclear potential on the various quantities which can be calculated from experiments is small; these effects were masked completely by the experimental inaccuracies in the results available at that time. In the meantime, many of the relevant experimental determinations have been repeated with greatly increased accuracy; notably the measurements of the deuteron binding energy, the coherent neutron-proton scattering amplitude, and the neutron-proton scattering cross section for neutron energies up to 5 Mev. It therefore seemed worthwhile to calculate the deviations from the simple formulas obtained on the effective range theory (quoted as *ERT*) for the various potential shapes.

In this paper we derive formulas for the evaluation of the effective singlet range r_{0s} for a neutron-proton potential of Yukawa, exponential-well, and square-well shape from two independent experimental measurements: (i) neutron-proton scattering cross sections for neutron energies up to 5 Mev. (ii) Cross section for the capture of slow neutrons by protons. We also derive expressions for the photomagnetic and photoelectric disintegration cross sections of the deuteron, for γ-rays of "classical" energies, for the potential shapes mentioned above.

[1] H. A. Bethe, Phys. Rev. **76**, 38 (1949), to be referred to as **B**.
[2] H. A. Bethe and C. Longmire, Phys. Rev. **77**, 647 (1950), to be referred to as **BL**.
[3] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949), to be referred to as **BJ**.

There are now at least three different accurate determinations of the deuteron binding energy ϵ which agree fairly well with each other. The corrected value obtained by Bell and Elliot,[4] by direct energy measurement of the γ-rays from the n–p capture reaction, is

$$\epsilon=(2.230\pm0.007)\text{ Mev.} \quad (1)$$

A direct determination of the threshold energy for photodisintegration by Mobley and Laubenstein,[5] using γ-rays obtained from an electrostatic generator, gives

$$\epsilon=(2.226\pm0.003)\text{ Mev.} \quad (2)$$

The accurate value for the neutron–proton mass difference obtained by Taschek *et al.*[6] using the $H^3(p,n)He^3$ reaction agrees fairly well now with the values from other nuclear cycles.[7] We take

$$n-H=(0.782\pm0.002)\text{ Mev.} \quad (3a)$$

Using the mass spectrographic result of Roberts and Nier[8]

$$2H-D=(1.442\pm0.005)\text{ Mev,} \quad (3b)$$

we obtain a third, accurate although indirect, value for the deuteron binding energy

$$\epsilon=(2.224\pm0.006)\text{ Mev.} \quad (3c)$$

The calculations throughout this paper are based on a value of

$$\epsilon=2.227(1\pm0.0015)\text{ Mev} \quad (4)$$

(although 2.226 would probably be a better value now).

For the coherent scattering amplitude f, we use the accurate value obtained recently by Hughes *et al.*,[9] using the reflection of thermal neutrons from a liquid hydrocarbon mirror,

$$f=2(\tfrac{3}{4}a_t+\tfrac{1}{4}a_s)=-3.76(1\pm0.008)\times10^{-13}\text{ cm.} \quad (5)$$

For the scattering cross section σ_{free} of slow neutrons by free protons, we use the same value as that used in **B**, obtained by Melkonian,[10]

$$\sigma_{\text{free}}=\pi(3a_t^2+a_s^2)=20.36(1\pm0.005)\text{ barns} \quad (6)$$

The values chosen above give for the radius of the deuteron

$$1/\gamma=4.314(1\pm0.0008)\times10^{-13}\text{ cm,} \quad (7)$$

and for the two scattering lengths,

$$a_t=5.388(1\pm0.0045)\times10^{-13}\text{ cm,} \quad (8)$$

$$a_s=-23.68(1\pm0.0025)\times10^{-13}\text{ cm.} \quad (9)$$

[4] R. E. Bell and L. G. Elliot, Phys. Rev. **79**, 282 (1950).
[5] R. C. Mobley and R. A. Laubenstein, Phys. Rev. **80**, 309 (1950).
[6] Taschek, Argo, Hemmendinger, and Jarvis, Phys. Rev. **76**, 325 (1940).
[7] Hornyak, Lauritsen, Morrison, and Fowler, Revs. Modern Phys. **22**, 291 (1950).
[8] T. R. Roberts and A. O. Nier, Phys. Rev. **77**, 746 (1950).
[9] Hughes, Burgy, and Ringo, Phys. Rev. **77**, 291 (1950); Phys. Rev. **79**, 227 (1950); and private communication.
[10] E. Melkonian, Phys. Rev. **76**, 1744 (1949).

TABLE I. $\rho_t(E_1, E_2)$ and b_t in 10^{-13} cm for three potential shapes; $\rho_t\equiv\rho_t(0,-\epsilon)$ being 1.720×10^{-13} cm.

	$\rho_t'\equiv\rho_t(-\epsilon,-\epsilon)$	$r_{0t}\equiv\rho_t(0,0)$	$\rho_t(0, 5\text{ Mev})$	b_t
Yukawa	1.787	1.653	1.578	2.99
Exponential	1.737	1.703	1.684	2.40
Square well	1.697	1.743	1.768	2.07

Equation (B, 19) gives for $\rho_t(0,-\epsilon)$ (which we shall simply denote by ρ_t)

$$\rho_t\equiv\rho_t(0,-\epsilon)=(1.720\pm0.035)\times10^{-13}\text{ cm.} \quad (10)$$

The experimental errors in the measurements of σ_{free} and of f now contribute about equally to the stated probable error for ρ_t (and similarly for a_t and a_s); the uncertainty in ϵ contributes a much smaller error for ρ_t. The accuracy of the measurement[9] of f will probably be increased somewhat further in the near future.*

With the help of Eq. (B, 19) we have found a value (10) for ρ_t which is the same for all potential shapes. To determine the effective range $\rho_t(E_1, E_2)$ for other energies, we use the approximate formula [see Eq. (B, 33)]

$$\rho(E_1, E_2)=r_0-2Pr_0^3(k_1^2+k_2^2). \quad (11)$$

The values of P, which depend on the shape and strength of the potential, were taken from the graphs and formulas of **BJ**. In Table I we give the values of r_{0t}, $\rho_t(-\epsilon,-\epsilon)$ (which we shall denote by ρ_t') and of $\rho_t(0, 5\text{ Mev})$ for central triplet potentials of Yukawa well, exponential well, and square well shapes, respectively. The experimental errors of these numbers are practically the same as those of ρ_t and are now *smaller* than the differences between the values for the different potential shapes and also somewhat smaller than the dependence of ρ_t on the energy in the region considered. The effect of potential shape on the values for r_{0s} and for disintegration cross sections, which we shall derive in this paper, is largely due to the fact that ρ_t' or $\rho_t(0, E)$, and not ρ_t, occurs in some relevant formula.

In Table I we also give the values for the intrinsic triplet range b_t for the three different potential shapes considered. For the Yukawa potential the value of b_t corresponds to a value of the meson mass μ_t of

$$\mu_t=(274\pm12)m_e. \quad (12)$$

This is very close to the best experimental value[11] for the mass of the π-meson, $(276\pm6)m_e$, but this excellent agreement is almost certainly fortuitous.

II. NEUTRON-PROTON SCATTERING

The total cross section $\sigma_{\text{tot}}(E)$ for the scattering of neutrons of energy E Mev (wave number k) by sta-

* *Note added in proof:*—A more recent value of $-(3.78\pm0.02)\times10^{-13}$ cm for f would give a value of $(1.704\pm0.030)\times10^{-13}$ cm for ρ_t.

[11] Smith, Barkas, Bishop, Bradner, and Gardner, Phys. Rev. **78**, 86 (1950).

TABLE II. r_{0s} obtained from total n–p scattering cross sections σ_{tot} for a neutron energy of 5.000 Mev.

σ_{tot} (in barns)	r_{0s} (in 10^{-13} cm) for ERT	Yuk.	Exp.	Sq. well	Error in r_{0s}
1.6487	2.00	1.55	1.90	2.11	±0.20
1.6290	2.50	2.11	2.41	2.58	±0.20
1.6086	3.00	2.66	2.92	3.04	±0.20

tionary protons is given by the expression

$$\sigma_{tot}(E)=3\pi\{k^2+[a_t^{-1}-\tfrac{1}{2}\rho_t(0,E)k^2]^2\}^{-1} +\pi\{k^2+[a_s^{2}-\tfrac{1}{2}\rho_s(0,E)k^2]^2\}^{-1}. \quad (13)$$

The three constants ρ_t, a_t, and a_s are now known with sufficient accuracy so that a completely precise experimental value of $\sigma_{tot}(E)$ for E about 3 Mev or more would yield values of $\rho_s(0, E)$ uncertain by only about $(\pm 0.2)\times 10^{-13}$ cm for any given potential shape. In Table II we give values of r_{0s}, obtained by assuming certain values of σ_{tot} for E equal to 5 Mev and using Eqs. (11) and (13). It is seen that the different potential shapes yield values for r_{0s} which differ, for any given value of σ_{tot}, by considerably more than the uncertainty $(\pm 0.2)\times 10^{-13}$ cm which arises from the constants ρ_t, a_t, and a_s. We also give r_{0s} as calculated by means of the simpler formulas of the effective range theory (denoted by ERT) which involve putting P in Eq. (11) equal to zero both for the singlet and triplet states.

Williams *et al.*[12] have recently measured $\sigma_{tot}(E)$ for various energies E between 0 and 5 Mev. In Table III we give values of r_{0s} derived from these values of $\sigma_{tot}(E)$. The probable errors for r_{0s} quoted in Table II are now due almost exclusively to experimental errors in these "high energy" experiments; at 5 Mev the experimental uncertainty of σ_{tot} and of E contribute about equally, while at the lower energies the error is mainly in σ_{tot}. The mean values of r_{0s} for the different potential shapes differ from each other only by amounts small compared with the probable error; this is mainly due to the fact that the more accurate measurements of Williams *et al.*[12] were done for neutron energies of the order of magnitude of 1 Mev for which the effect of potential shape on the value of r_{0s} is much less marked than for energies of about 3 Mev or more.

TABLE III. Values of r_{0s} obtained from measurements of the total n–p scattering cross section, by the Minnesota group, at various energies E.

E (in Mev)	σ_{tot}	r_{0s} (in 10^{-13} cm) for ERT	Yukawa	Exp.	Sq. well	Error (in 10^{-13} cm)
0.798	4.79	2.5_5	2.4_5	2.5_5	2.5_5	±1.0
1.078	4.09	2.4	2.3	2.3_5	2.4_5	±1.0
1.340	3.66	2.1_5	2.0	2.1	2.2	±1.0
1.578	3.32	2.5	2.3	2.4_5	2.5_5	±1.0
4.92	1.64	2.7	2.3	2.6	2.8	±1.5
4.97	1.56	4.3	4.3	4.3	4.2	±1.5
Weighted mean for r_{0s}		2.6	2.4_5	2.5_5	2.6_5	±0.5
b_s			2.2_5	2.4	2.5_5	±0.5

[12] Lampi, Freier, and Williams, Phys. Rev. **80**, 853 (1950).

In Table III we also give the values for the intrinsic singlet range b_s for the three potential shapes considered. For the Yukawa potential the value of the corresponding meson mass μ_s is

$$\mu_s=(365\pm 80)m_e. \quad (14)$$

For the exponential shape the intrinsic singlet and triplet ranges (and hence the radii of the singlet and triplet potentials) agree well. For the Yukawa and square well shapes, b_t and b_s differ (in opposite directions), but only by about the combined probable error of b_t and b_s.

III. PHOTOMAGNETIC CAPTURE

Let σ_H be the cross section for the photomagnetic capture of a slow neutron of velocity v by a stationary proton. We assume that a fraction ϵ_5 of the total cross section σ_H is due to the presence of a magnetic exchange moment (see below). The remainder of the cross section is then given by

$$\sigma_H v(1-\epsilon_5)=2\pi(e^2/Mc)(\hbar/Mc)(W_1/Mc^2)^{\frac{3}{2}} \times(\mu_P-\mu_N)^2[\gamma+\beta'-\gamma^2 D]^2/(1-\rho_t'\gamma)\beta'^2, \quad (15)$$

where $\rho_t'\equiv\rho_t(-\epsilon, -\epsilon)$ and the other symbols are as defined in **BL**. Equation (15) is the same as the approximate expression (**BL**, 47), except that ρ_t' has not been replaced by r_{0t} and D is the function defined in (**BL**, 25)

$$D=\int_0^\infty (\psi_g\psi_s-u_g u_s)dr. \quad (16)$$

For $\sigma_H v$ we take the value obtained from the experiments of Whitehouse and Graham,[13]

$$\sigma_H v=6.81\times 10^{-20}\ (\text{cm}^3/\text{sec})(1+\epsilon_4). \quad (17)$$

In **BL** the probable error, ϵ_4, in the measurement[13] of $\sigma_H v$ was taken to be ± 0.04 and the measurement agrees to within this accuracy with earlier results by Walker and Frisch.[14] On the other hand, Halban *et al.*[15] consider that the possibility of a somewhat larger error cannot be excluded. For γ, a_t, and a_s we use the values (7), (8), and (9), respectively, for ρ_t' the values given in Table I and for $(\mu_P-\mu_N)$ a value[16] of 4.706. The relation for D is then

$$D=\{[1.064+0.78\delta+6.8\epsilon_1+5.2\epsilon_2-0.92\epsilon_3] +[1-(1+\epsilon_4-\epsilon_5)^{\frac{1}{2}}][4.04-0.8\delta-10\epsilon_1-6\epsilon_2 -0.9\epsilon_3]\}, \quad (18)$$

where

$$\delta\equiv\rho_t'-\rho_t \quad (19)$$

and ϵ_1, ϵ_2, ϵ_3 are as defined in **B**.

[13] W. J. Whitehouse and G. A. R. Graham, Can. J. Research **A25**, 261 (1947).
[14] R. L. Walker, MDDC—414, unpublished.
[15] G. R. Bishop *et al.*, Phys. Rev. **80**, 211 (1950); **81**, 219 (1951).
[16] J. E. Mack, Revs. Modern Phys. **22**, 64 (1950).

Some calculations have been carried out, and more are in progress by Austern and Sachs[17] and by Gray[18] to relate the contribution of magnetic exchange moments to n–p capture, ϵ_5, to the experimentally known value of the exchange moment in the ground state of H^3 and He^3 on a general phenomenological theory. After adjusting the parameters of such a theory to give the correct value for the H^3 exchange moment, the value obtained for ϵ_5 still depends to some extent on (a) the shape of the wave functions for the deuteron and for the triton and hence on the shape of the neutron-proton potential; (b) a distribution function $\Phi(\rho)$. This function $\Phi(\rho)$ enters in all calculations concerning exchange magnetic moments and is not known theoretically unless a specific meson theory is used. Austern and Sachs[17] have shown that, for specific deuteron and triton wave functions, the value obtained for ϵ_5 does not depend very much on the actual shape of $\Phi(\rho)$ as long as $\Phi(\rho)$ is appreciable only for $\rho<2\times10^{-13}$ cm, and ϵ_5 is then approximately $(+0.04)$. If $\Phi(\rho)$ has a "longer tail," larger values are obtained for ϵ_s, but it may be possible to get an upper limit for the size of the "tail" of $\Phi(\rho)$ for $\rho>2\times10^{-13}$ cm from other considerations (see Sec V). Separate values are being calculated by Gray[18] for ϵ_5 for various potential shapes, making some reasonable assumptions about $\Phi(\rho)$, and using consistent pairs of triton and deuteron wave functions corresponding to the same potential.

Because of the present uncertainties both in ϵ_4 and ϵ_5, we give values obtained for D for various values of $(\epsilon_4-\epsilon_5)$ in Table V, using (18) and the values obtained for δ for the three potential shapes from Table I.

To use the values obtained for D from Table V to calculate a value for r_{0s}, we still need an expression for D in terms of r_{0t} (or ρ_t) and of r_{0s}. In **BL** an approximation to such an expression is given (which we denote by ERT) which can be written in the following form

$$D=0.430+0.250r_{0s}+0.250(r_{0t}-1.72), \quad (20a)$$

where D, r_{0t}, and r_{0s} are all in units of 10^{-13} cm. The exact expression for D depends on r_{0t} and r_{0s} in a very complicated way, which is different for the different potential shapes, and moreover depends somewhat on the values of a_t and a_s. However, if a_t and a_s have values near those given in (8) and (9), if ρ_t lies between 1.5 and 2.0 and r_{0s} between 2 and 3 (ρ_t and r_{0s} again expressed in units of 10^{-13} cm), an approximate expression can be found for D, which is more accurate than (20a), in the form

$$D=a+br_{0s}+c(\rho_t-1.720)-d(\rho_t-1.720)r_{0s}. \quad (20)$$

The four coefficients a, b, c, and d were calculated, separately for each of the three potential shapes, by Mr. Newcomb by evaluacting D directly from the wave functions for two slightly different values of ρ_t and two different values of r_{0s} with a_t and a_s given by (8) and (9) in each case. For the square-well potential, D was calculated analytically, using exact wave functions, for values of ρ_t of 1.67 and 1.82 and for values of r_{0s} of 2.49 and 2.72. For the exponential potential, D was calculated by numerical integration, using exact wave functions, for values of ρ_t of 1.807 and 1.460 for which the bessel functions occuring in the wave functions are of order $\frac{1}{3}$ and $\frac{1}{4}$, respectively. The values used for r_{0s} were 2.0 and 2.7. For the Yukawa potential, the approximate wave functions (**BL**, 33c) and (**BL**, 34b), which are exact solutions for the Hulthén potential, were used, but for the constant P defined in (11), the value for the Yukawa potential itself was used. The two values of ρ_t and the two values of r_{0s} used for the Yukawa potential were approximately the same as those used for the exponential potential. The values of a, b, c, and d thus calculated are given in Table IV.

TABLE IV. Values of the coefficients in Eq. (20) for three potential shapes.

	a	b	c	d
Yukawa (Hulthén)	0.686	0.146	0.57	0.09
Exponential	0.649	0.154	0.53	0.06
Square well	0.579	0.176	0.55	0.06

In Table V we give the values obtained for r_{0s}, using Eqs. (18) and (20), for various values of $(\epsilon_4-\epsilon_5)$. Present indications are for a large value of r_{0s}, of the order of magnitude of 3×10^{-13} cm, but in view of the present uncertainties both in ϵ_4 and ϵ_5 no reliable conclusions can as yet be derived. The presence of the tensor force in the deuteron problem has the effect, as was shown in **BL**, of increasing the values obtained for r_{0s} in this section by about 0.1×10^{-13} cm. It should be emphasized that the calculations in this section take account of magnetic exchange effects but are based on the assumption that there are no other relativistic effects which alter the magnetic moments of the neutron and proton in the deuterium nucleus appreciably.

IV. PHOTODISINTEGRATION AT MODERATE ENERGIES

In **BL** a simple approximation is derived for σ_e, the photoelectric disintegration cross section of the deuteron, for γ-rays of a few Mev energy. We assume that the proton–neutron potential is zero for a P-state. We can

TABLE V. Values of r_{0s} and of D (both in 10^{-13} cm) for n–p capture cross section σ_H given by $\sigma_H v=6.81(1+\epsilon_4)$ cm³/sec and with a fraction ϵ_5 of this cross section due to magnetic exchange moments.

	D (in 10^{-13} cm) for				r_{0s} (in 10^{-13} cm) for			
$\epsilon_4-\epsilon_5$	ERT	Yuk.	Exp.	Sq. well	ERT	Yuk.	Exp.	Sq. well
−0.10	1.271	1.320	1.284	1.254	3.36	4.34	4.12	3.83
−0.05	1.166	1.217	1.179	1.148	2.95	3.64	3.45	3.22
0	1.064	1.116	1.077	1.046	2.54	2.94	2.79	2.66
+0.05	0.964	1.017	0.977	0.946	2.14	2.27	2.14	2.08
+0.10	0.867	0.922	0.881	0.848	1.75	1.62	1.51	1.53

[17] N. Austern and R. G. Sachs, Phys. Rev. 81, 710 (1951).
[18] E. P. Gray, private communication.

then write this approximation in the form[18a]

$$\sigma_e = \sigma_{e0}/(1-\rho_t'\gamma), \tag{21}$$

where σ_{e0} is the theoretical cross section for zero range, given in Eq. (**BL**, 7). In **BL** the further approximation was made of neglecting the differences between ρ_t, ρ_t' and r_{0t}. In Table VI we give values for σ_e for various γ-ray energies calculated from Eq. (21) but with the value for ρ_t' taken from Table I for each of the potential shapes, separately.

The only error in expression (21) is due to having replaced the correct wave functions in the matrix element (**BL**, 1) by their asymptotic expression. We have estimated this error once more by computing the matrix element (**BL**, 1) directly by using either exact wave functions (for the square well) or approximate wave functions remaining accurate at small distances (for exponential and Yukawa). The calculation showed that, if the potential in the P-state is really zero, and if the γ-ray energy is near threshold, the approximate formula (21) is correct to about 0.03 percent for the square well and to about 0.1 percent for the exponential and Yukawa potential; for γ-rays of 4 or 5 Mev the equivalent errors are about 0.1 percent and 0.4 percent, respectively, and increase approximately with the square of the γ-ray energy. The values for σ_e calculated from (21) are given, without any corrections, in Table VI, for a γ-ray energy of 6.14 Mev although the values may be wrong by more than 1 percent at such high energies.

The correct expression for the *photomagnetic* cross section for disintegration of the deuteron, σ_m, by γ-rays of energy $(2.227+E)$ Mev involves the expression (16) for D where ψ_s and u_s now represent wave functions corresponding to an energy E. In general, therefore, the expression in terms of r_{0s} and ρ_t to be used for D will depend on E. Furthermore, the effect of the magnetic exchange moments on σ_m will, in general, vary with the energy E. However, it is probable that, for E small compared with the deuteron binding energy (which is the case for the Ga^{72}, ThC″, and Na^{24} γ-rays), the percentage of σ_m which is due to magnetic exchange moments is practically the same as that for zero energy ϵ_5. Similarly, Eq. (20), with the coefficients given in Table IV, should be a good approximation for D for such low values of E. We write for σ_m at an energy of E Mev (wave number k)

$$\sigma_m(E) = R(E)\times\sigma_{m0}(E), \tag{22}$$

where $\sigma_{m0}(E)$, the cross section for zero range, is given by the expression (**BL**, 56). Substituting the latest values for the constants in this expression we get, to an accuracy better than 0.2 percent,

$$\sigma_{m0}(E) = 6.94\times10^{-28}\frac{E^{\frac{1}{2}}}{(E+2.227)(E+0.0739)}\ \text{cm}^2. \tag{23}$$

Making the above approximations then,[18b]

$$R(E) = R(0)[1+k^2(\tfrac{1}{2}r_{0s}-D)/(\gamma+\beta'-\gamma^2D)] \times[1+(\tfrac{1}{4}r_{0s}^2k^4+\beta' r_{0s}k^2)/(k^2+\beta'^2)]^{-1}, \tag{24a}$$

$$R(0) = [1-\gamma^2D/(\gamma+\beta')]^2\times(1-\rho_t'\gamma)^{-1}\times(1-\epsilon_5)^{-1}. \tag{24b}$$

$R(0)$, the value of $R(E)$ for zero energy, is equal to the ratio of the actual total magnetic capture cross section at zero energy to the theoretical expression for zero range of forces. We therefore have as an alternative to (24b),

$$R(0) = 1.041(1+\epsilon_4)\{1-2.4\epsilon_1-0.8\epsilon_2-0.05\epsilon_3\}. \tag{25}$$

In Table VI we give values of σ_m, for the three potential shapes discussed, calculated with the help of Eqs. (22), (23), (24a), and (25) for some energies at which experimental results[15] are available. In (25) we have put ϵ_4 equal to zero and the values for D and r_{0s} used in (24a) are those obtained from Table V for $(\epsilon_4-\epsilon_5)$ equal to zero, since no very definite values are as yet available for ϵ_5 and since $(\epsilon_4-\epsilon_5)$ equal to zero gives values for r_{0s} in rough agreement with those obtained from n–p scattering. Because of the large uncertainty in ϵ_4 and ϵ_5 at the present time the values for σ_m in Table VI are not yet accurate enough for an accurate comparison with experimental results and are given mainly to show the dependence of the values on the potential shape. It is seen that the magnetic cross section is very nearly independent of the potential shape, mainly because it is tied to the capture cross section at low energies; the electric cross section varies by about 4 percent, this variation being due to the difference between ρ_t and ρ_t'. The uncertainty in the γ-ray energies contributes a probable error of about ±2 percent to σ_e, the uncertainty in ρ_t slightly less than ±2 percent. The probable error in σ_m is mainly due to the uncertainty in the capture cross section and is somewhat larger than ±5 percent. Values of σ_m are included for 6.14 Mev γ-rays, for the sake of comparison, also calculated using the above formulas, which are probably not good approximations at such high energies.

The theoretical formulas derived in this paper for the case of a Yukawa potential are of course not quite as accurate as the results of Hansson and Hulthén,[19] who perform more direct calculations using very good approximations for the neutron-proton wave functions. However, the differences between the numerical values for σ_m, etc., obtained in this paper and in that of

[18a] It has been brought to our attention that H. Hall [Revs. Modern Phys. 8, 358 (1936)] was the first to recognize the main feature of Eq. (21), namely, that the photoelectric cross section depends on the range of the forces only through the normalization of the ground state, and that this dependence is similar to the dependence of the elastic triplet cross section on range.

[18b] In the equivalent expression to (24a) in reference **BL**, (**BL**, 55), the last factor was inadvertently omitted.

[19] I. F. E. Hansson and L. Hulthén, Phys. Rev. 76, 1163 (1949); I. F. E. Hansson, Phys. Rev. 79, 909 (1950).

TABLE VI. Photoelectric and photomagnetic cross sections, σ_e and σ_m, in 10^{-28} cm^2 for various γ-ray energies and potential shapes. (ϵ is taken to be 2.227 Mev.)

γ-ray	Ga^{72}		ThC″		Na^{24}		F+H	
Energy (Mev)	2.507		2.615		2.757		6.14	
	σ_e	σ_m	σ_e	σ_m	σ_e	σ_m	σ_e	σ_m
ERT	5.91	3.99	8.51	3.42	11.39	2.91	21.0	0.55
Yukawa	6.07	3.97	8.74	3.40	11.70	2.90	21.6	0.59
Exponential	5.95	3.98	8.57	3.41	11.47	2.91	21.2	0.59
Square well	5.86	3.99	8.43	3.42	11.29	2.92	20.8	0.58
Experimental	7.2 ± 1	4.4 ± 1	10.1 ± 1.5	3.8 ± 1	12.5 ± 1	3.3 ± 0.5	20.9 ± 1.5	0.6 ± 1

Hansson and Hulthén,[19] are due mainly to the fact that Hansson and Hulthén assume the triplet and singlet intrinsic ranges to be the same (whereas in this paper the singlet and triplet ranges are determined separately from experiments) and to a much smaller extent (at least for low enough values of E) to the inaccuracy of the formulas of this paper.

In Table VII we give some of the experimental results for $(\sigma_m+\sigma_e)$ and for σ_m/σ_e available at present. Values for σ_e and σ_m are given in Table VI, using as experimental values for $(\sigma_m+\sigma_e)$ and σ_m/σ_e the weighted means of the appropriate values in Table VII. The four results quoted for σ_m/σ_e for the Na^{24} γ-ray do not all lie within each others stated experimental errors. Dr. W. M. Woodward advised us that an evaluation of the relative accuracies of these experimental results is rather difficult and hence their unweighted arithmetic mean was taken. The accuracies of these "experimental" values for σ_e and σ_m are now considerably less than of the equivalent "theoretical" values, but further experiments are in progress. For the three lower energies considered, all experimental cross sections appear to be higher than the theoretical ones, but the discrepancies are slightly less than the experimental errors at present.

V. DISCUSSION

We have seen in the foregoing sections that two independent methods are now available for the determination of the singlet effective range r_{0s} from experiment; the one method being based on precision measurements of the neutron–proton scattering cross sections for neutron energies up to 5 Mev, the other on precision measurements of the neutron–proton capture cross section at very low neutron velocities. We have also seen that the values obtained for r_{0s} from each of these two experiments depend somewhat on the shape of the neutron–proton potential. A given value for the scattering cross section for neutron energies of 3 to 5 Mev leads to a value for r_{0s} about 0.4×10^{-13} cm *larger* for a square-well potential than for a Yukawa potential, with an intermediate value for an exponential potential. A given value of the neutron–proton capture cross section, on the other hand, leads to a value for r_{0s} about 0.3×10^{-13} cm *smaller* for a square-well than for a Yukawa potential, the value for an exponential potential being again intermediate. The errors in r_{0s} due to to the experimental inaccuracies in ρ_t, a_t, and a_s at the present time are only about $(\pm0.2)\times10^{-13}$ cm for the determination from the scattering cross section and about $(\pm0.1)\times10^{-13}$ cm for that from the capture cross section. Furthermore, these errors in r_{0s} are in the same direction for the two determinations, and these errors will probably be reduced further later this year when the coherent scattering length will be remeasured.[9]

Since the difference in r_{0s} for "long tailed" and "short tailed" potentials are in opposite directions for the two different methods, we have, in principle, a way for estimating the shape of the neutron–proton potential. That is, we merely have to see for which of the three potential shapes discussed in this paper the two values for r_{0s}, obtained by the two methods, are in closest agreement. This, of course, would not determine the actual dependence of the neutron–proton potential on distance, but it would at least give a rough measure of the extent to which the potential is "long tailed" or "short tailed." This method gives a considerably more sensitive test for potential shape than an attempt to determine the variation of $\rho_t(0, E)$ with energy directly from neutron–proton scattering cross sections at different energies, since the corresponding variations of cross sections are still very small compared with the experimental errors. At present, the values of r_{0s} agree

TABLE VII. Experimental values for the total photodisintegration cross section σ_{tot} (in 10^{-28} cm^2) and for the ratio, σ_m/σ_e, of the magnetic to the electric cross section.

γ-ray	Ga^{72}	ThC″	Na^{24}	F+H
Energy (Mev)	2.507	2.615	2.757	6.14
$\sigma_{tot}\times(10^{28}\text{ cm}^{-2})$	(11.9 ± 0.8)[a] $(10.6\pm\ ?\ )$[b]	(13.9 ± 0.6)[a]	(15.9 ± 0.6)[a] (14.3 ± 1.5)[b] (17.2 ± 1.5)[d]	(21.5 ± 1.2)[e]
σ_m/σ_e	(0.61 ± 0.14)[c]	(0.37 ± 0.12)[h]	0.20[g], 0.27[f] 0.32[i], 0.26[j]	(0.03 ± 0.06)[k]

[a] See reference 15 (1950).
[b] Snell, Barker, and Sternberg, Phys. Rev. **80**, 637 (1950).
[c] See reference 15 (1951).
[d] Shinohara *et al.*, J. Phys. Soc. (Japan), **4**, 77 (1949).
[e] C. A. Barnes *et al.*, Nature **165**, 69 (1950).
[f] G. A. R. Graham and H. Halban, Revs. Modern Phys. **17**, 297 (1945).
[g] F. Genevese, Phys. Rev. **76**, 1288 (1949) (with corrections calculated by W. M. Woodward).
[h] Hammermesh and Wattenberg, Phys. Rev. **76**, 1408 (1949) (corrected).
[i] E. P. Meiners, Phys. Rev. **76**, 259 (1949).
[j] N. O. Lassen, Phys. Rev. **75**, 1099 (1949).
[k] P. V. C. Hough, thesis, Cornell 1950, unpublished.

best for a very short tailed potential (like the square well), but the present inaccuracies in both the scattering and capture cross sections are still somewhat too large to come to a definite conclusion about the potential shape. The following three improvements would suffice, at least for distinguishing between a very long tailed potential, like the Yukawa, and a very short tailed one, like the square well.

(1) Measurements of the neutron-proton scattering cross section would be needed at a few energies in the 3 to 5 Mev region. If each of these cross sections could be measured to an accuracy of about 1 percent and if the corresponding energies were also known to within 1 percent or better then r_{0s} would be known to within about $(\pm 0.3)\times 10^{-13}$ cm for each potential shape assumed. At these energies the value obtained for r_{0s} depends much more on potential shape than at lower energies. (2) The contribution to the neutron-proton capture cross section due to magnetic exchange moments would have to be known, theoretically, to within about 1 percent of the total cross section for each of the potential shapes considered. Calculations to this accuracy may be completed soon. (3) The ratio of the capture cross section of neutrons by hydrogen to that by boron would have to be remeasured, as well as the absolute value of the boron cross section. An accuracy of 1 or 2 percent in the neutron-proton capture cross section would give an error in r_{0s} of only about $(\pm 0.2)\times 10^{-13}$ cm.

The arguments in this section are, of course, based on the assumption that there are no very strong relativistic effects which alter the magnetic properties of bound neutrons and protons. Even if such effects are present there is at least some hope that their neglect will be compensated, to a large extent, by the semi-empirical treatment of "exchange moments" discussed above. It has been shown[20] that, at low enough energies, the effects of tensor forces are similar to those of an "equivalent central potential." No detailed calculations have as yet been done, but it seems likely that the presence of tensor forces does not appreciably alter the results obtained in this paper, except that the value obtained for r_{0s} from the n-p capture cross section is increased by about 0.1×10^{-13} cm (and that any conclusions about potential shape now refer to the total equivalent central potential).

Another way of relating the shape of the neutron-proton potential to a property of nuclear magnetic moments is offered by a study of the hyperfine structure of hydrogen and deuterium. The ratio of the hyperfine structure separation has recently been measured very accurately by Prodell and Kusch.[21] These measurements, together with the latest value of the ratio of the magnetic moment of the deuteron to that of the proton,[22] gives the very accurate value of

$$\Delta \equiv 1-\left[\frac{(\nu_{\mathrm{H}}/\nu_{\mathrm{D}})_{\mathrm{exp}}}{(\nu_{\mathrm{H}}/\nu_{\mathrm{D}})_{\mathrm{theor}}}\right]=(1.703\pm 0.007)\times 10^{-4}, \quad (26)$$

where $(\nu_{\mathrm{H}}/\nu_{\mathrm{D}})_{\mathrm{theor}}$ is the theoretical result obtained on the assumption that the nucleus of deuterium, as well as of hydrogen, is a point particle without any structure. Low[23] has made a detailed study of the value which can be derived for Δ from our present knowledge of nuclear forces. Using the more accurate values quoted in this paper for r_{0t}, Low's result can be written in the form

$$\Delta=(1.88+\epsilon_6+\epsilon_7+\epsilon_8)\times 10^{-4}. \quad (27)$$

Here ϵ_6 is a term which depends on the shape of the neutron-proton potential and can be calculated fairly accurately for each of the potential shapes commonly discussed. For a short tailed potential ϵ_6 is of the order of magnitude of $(+0.15)$, for a long tailed potential (-0.15). Any finite spread of the magnetic moment of a bare neutron or a bare proton (or at least of the "anomalous" part of the proton moment) gives a negative value for ϵ_7. This value[24] is roughly proportional to the radius up to which the proton moment is spread out, and for a radius of 1×10^{-13} cm, ϵ_7 is of the order of magnitude of (-0.4). The remaining inaccuracies in the calculation and uncertainties in the theory contribute the term ϵ_8, which is somewhat smaller than (± 0.1). Using (26) and (27) we have

$$\epsilon_6+\epsilon_7=-0.18\pm 0.08. \quad (28)$$

We therefore have a means for finding an estimate of the spread of the magnetic moment of a bare proton corresponding to each shape we might consider for the neutron-proton potential. It might in turn be possible to find a relation between this spread of the moment of a bare nucleon with the distribution function[17] $\Phi(\rho)$ for the exchange moment between two nucleons; such a relation might be less sensitive to the exact nature of the theory used than the results obtained for either quantity.

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[20] R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950); H. A. Bethe (unpublished work).

[21] A. G. Prodell and P. Kusch, Phys. Rev. **79**, 1009 (1950).

[22] Smaller, Yasaitis, and Anderson, Phys. Rev. **80**, 137 (1950).

[23] F. Low, Phys. Rev. **77**, 361 (1950).

[24] A. Bohr, Phys. Rev. **73**, 1109 (1948).