Production of π-Mesons in Nucleon-Nucleon Collisions

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(Received October 16, 1950)

The production of π-mesons in nucleon-nucleon collisions as predicted by scalar meson theory, pseudoscalar meson theory with pseudoscalar and pseudovector coupling, and vector theory with vector coupling is compared with the experimental results of the workers at Berkeley. The calculations are made on the basis of third-order perturbation theory using the methods of Feynman and Dyson and also using a phenomenological treatment of the nucleon-nucleon interaction. Account is taken of the fact that the final nucleons are not in plane wave states. It is shown that pseudoscalar theory with pseudovector coupling gives qualitative agreement with experiment.

I. INTRODUCTION

A PROBLEM of considerable interest in meson theory is the production of mesons in nucleon-nucleon collisions. The intimate connection, predicted by meson theory, between this process and ordinary nucleon-nucleon scattering provides an excellent and basic opportunity for testing the fundamental assumptions of meson theory. The meson theory of nuclear forces assumes that π-mesons, found to interact strongly with nuclei, are responsible for the coupling between nucleons. Since this coupling via the meson field implies the existence of virtual mesons in the mutual field of two nucleons, it should be possible, if sufficient energy is available, for virtual mesons to be materialized as free and observable particles. A comparison of the predictions of the theory with the experimental measurements of nucleon-nucleon scattering and of meson production should indicate whether this basic assumption is quantitatively correct.

Experiments are now being carried out at Berkeley[1] which give information concerning the production of charged and neutral mesons in neutron-proton and proton-proton collisions. It is of interest to consider in a systematic manner the theory of the production in order to understand what can be learned from these experiments.

The production of mesons in nucleon–nucleon collisions has been studied in considerable detail by a number of theoretical workers.[2] The most thorough theoretical analyses which have been made can be divided into two types: (1) an application of the meson field theory of nuclear forces to the problem considered as a third-order process, and (2) an attempt to describe phenomenologically the scattering of the nucleons associated with the meson emission, and only the meson emission itself, by meson field theory. The first method suffers from the well-known failure of meson theory to describe nuclear forces in more than a qualitative way and from the more general failure of perturbation theory in the weak coupling approximation applied to problems in which the coupling is not weak. This method can, however, be applied rigorously in the lowest order and with this limitation gives a logically complete description of the process. Also, because of the close relation between high energy nucleon-nucleon scattering (virtual meson exchanges) and meson production (virtual meson exchanges together with a real meson emission), a theory which gives qualitatively correct results for the former process might be expected to be correct to a similar approximation for the latter.

The second method, based on a more phenomenological approach, describes the meson emission on the basis of field theory but separates the nucleon-nucleon scattering, which gives the momentum transfer necessary for over-all energy and momentum conservation, and attempts to describe this in terms of the experimentally measured potentials. This method is inadequate in as far as processes can occur in the meson production which cannot be described in terms of the scattering process preceded or followed by meson emission. Such processes are, for example, the interruption of a virtual meson exchange by a real meson emission. These processes are equivalent to scatterings which take place far off the energy shell, i.e., where energy and momentum are related very differently from the relationship for free particles. Only if such processes give unimportant contributions can the phenomenological approach be approximately correct. However, if this assumption is made, the calculation can be made in a manner much more independent of meson field theory than is the third-order perturbation calculation.

The rigorous treatment of meson production as a third-order process is complicated by mathematical difficulties in the integration of the cross sections over the momenta of the final nucleons to give a result which depends only on the meson momentum. This problem has been partially solved by Morette[2] for pseudoscalar mesons with pseudoscalar coupling; Morette, however, ignored the effects of the Pauli principle and so introduced a rather large error (about 50 percent) in the cross section near threshold. Her expressions for the cross sections also have been averaged over meson and

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[1] C. Richman and H. Wilcox, Phys. Rev. 78, 85(A) (1950). Bjorklund, Crandall, Moyer, and York, Phys. Rev. 77, 213 (1950). Cartwright, Richman, Whitehead, and Wilcox, Phys. Rev. 78, 823 (1950). V. Z. Peterson, Ph.D. thesis, U. of Calif., 1950. W. Cartwright and M. Whitehead, private communication.
[2] For references, see Cecile Morette, Phys. Rev. 76, 1432 (1949).

nucleon charges and, therefore, do not give separately the cross sections for charged and neutral mesons for neutron-proton and proton-proton collisions. Since these separate quantities are those which can be determined experimentally, it is of interest to calculate them. Therefore, we have in Sec. II considered the calculation of the transition matrix elements on the basis of rigorous third-order perturbation theory and obtained the expressions for scalar theory, pseudoscalar theory with pseudoscalar, and pseudovector theory with vector coupling. The calculations are made in the center-of-mass system for energies near threshold, where the velocities of the final particles are small. Corrections of the order of v^2/c^2 for the final nucleons and mesons are neglected, so the results are only applicable for incident nucleon energies of 350 to 400 Mev, corresponding to maximum meson energies in the center-of-mass system of 23 to 44 Mev.

The second method of calculation, treating the nucleon-nucleon interaction phenomenologically, has been carried out by Marshak and Foldy[3] for scalar mesons and for pseudoscalar mesons with pseudovector coupling. They found a zero cross section for scalar mesons and a very small cross section (about 10^{-31} cm^2 at 350 Mev) for pseudoscalar mesons. Both of these results disagree with the experimentally observed[1] large cross section which is of the order of 10^{-28} cm^2. However, their treatment is incorrect because they chose a charge independent nuclear potential which is not in agreement with later data on high energy scattering. Experimentally, high energy neutron-proton and proton-proton scattering are qualitatively different. Approximate agreement with the experimental results at high energy is given by the potentials[4]

$$-g_{NP}^2[(1+P_x)/2][\exp(-\mu r)/4\pi r] \quad \text{for } N\text{–}P \text{ scattering, } g_{NP}^2/4\pi=0.479, \quad (1)$$

and

$$+g_{PP}^2(\boldsymbol{\sigma}_1\cdot\nabla/\mu)(\boldsymbol{\sigma}_2\cdot\nabla/\mu)[\exp(-\mu r)/2\pi r] \quad \text{for } P\text{–}P \text{ scattering, } g_{PP}^2/4\pi=0.0209,$$

where P_x is the space exchange operator. The choice of the P–P potential is not unique; any potential which predicts a very singular and strong interaction in P states would give approximate agreement with the high energy scattering. This potential is chosen because it corresponds to pseudoscalar theory with pseudovector coupling. Using these potentials, it is of interest to carry out calculations similar to those done by Marshak and Foldy to see whether sufficiently large cross sections can be obtained to explain the experimental results.

This method is applied in Sec. III to the calculation of the transition matrix elements for scalar theory, vector theory with vector coupling, and pseudoscalar theory with pseudovector coupling. For pseudoscalar coupling, it is possible to generalize the P–P potential to its relativistic form, using the equivalence between pseudovector and pseudoscalar coupling pointed out by Nelson.[5] The equivalence theorem gives for the potential the result

$$g_{PP}^2(2M/\mu)^2(\gamma_5)_1(\gamma_5)_2\exp(-\kappa r)/4\pi r. \quad (2)$$

The calculation can then be carried out using this type of interaction. However, it will be shown that for pseudoscalar coupling processes occur in the production of mesons which cannot be described in terms of a potential interaction and which give the largest contribution to the matrix element. A treatment on the basis of a potential model, therefore, is not justified. For such a theory the methods of third-order perturbation theory should give more reliable results. These phenomenological calculations are made in the center-of-mass system and are restricted to energies near threshold, i.e., less than 400 Mev for the incident nucleon in the laboratory system.

An additional important effect, which has been ignored in these calculations, must be considered before comparison with experiment can be made. In the ordinary approach to the problem of meson production, the approximation is made of replacing the wave functions of the initial and final nucleons by plane waves. This approximation is fairly good for the initial nucleons, since it is equivalent to the use of the Born approximation in high energy scattering. However, because of the large amount of energy carried off by the meson in its rest mass, the final nucleons are moving slowly, particularly near thresholds. It was pointed out by Hart and Chew that as a consequence of this it is possible, if the final nucleons are a neutron and a proton, that a deuteron may be formed. In addition, even if the nucleons do not form a deuteron, use of the plane wave approximation for the final nucleons gives very inaccurate results. In the Appendix these effects are considered and are shown to be important.

II. THIRD-ORDER FIELD-THEORETIC CALCULATION OF TRANSITION MATRIX ELEMENTS

The third-order matrix element can be written down directly using the methods of Feynman and Dyson.[6] A Feynman-Dyson diagram for meson production is given in Fig. 1. Seven additional diagrams can be obtained for emission of the meson by the three other nucleons and corresponding diagrams for the cases in which the two initial or final nucleons are interchanged. The matrix element for the diagram of Fig. 1 is

$$f^\lambda(f^\rho)^2\bar{\psi}(3)\left[U_\mu{}^2\frac{\tau^\lambda 1\tau^\rho}{(p_1-q)_\mu\gamma_\mu-iM}U_\nu{}^1\right]\psi(1) \times\frac{\bar{\psi}(4)U_\nu{}^1\tau^\rho\psi(2)}{(p_4-p_2)^2+\mu^2}\phi_\mu{}^\lambda(q), \quad (3)$$

[3] R. Marshak and L. Foldy, Phys. Rev. **75**, 1493 (1949).
[4] R. Christian and E. Hart, Phys. Rev. **77**, 441 (1950); R. Christian and H. P. Noyes, Phys. Rev. **79**, 85 (1950).
[5] E. C. Nelson, Phys. Rev. **60**, 830 (1941).
[6] R. P. Feynman, Phys. Rev. **76**, 769 (1949). F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

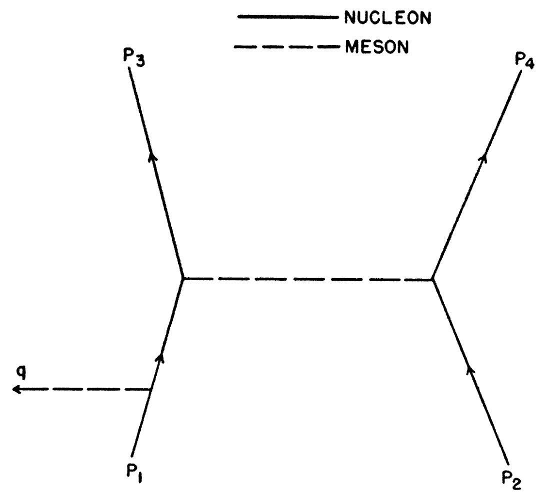

FIG. 1. Feynman-Dyson diagram for meson production by N–N collisions in lowest order. The solid lines represent the nucleons; the dashed lines represent mesons.

where $U_\mu^1=U_\mu^2=\delta_{\mu 0}$ for[7] S(III), and $=\gamma_5\delta_{\mu 0}$ for Ps.Ps(III); $U_\mu^1=(\delta_{\mu 0}\gamma_5/\mu)(P_4-P_2)_\nu\gamma_\nu$, and $U_\mu^2=(\delta_{\mu 0}\gamma_5/\mu)q_\nu\gamma_\nu$ for Ps.Pv(III); $U_\mu^1=U_\mu^2=\gamma_\mu$ for V.V(III); $\phi_\mu^\lambda=\phi^\lambda\delta_{\mu 0}$ for S(III), Ps.Ps(III), and Ps.Pv(III); and $\phi_\mu^\lambda=\phi_\mu^\lambda$ for V.V(III).

We use the symmetrical isotopic spin notation[8]

$$f^\lambda\tau^\lambda=f_1\tau_1+f_2\tau_2+f_3\tau_3+f_4\tau_4.$$

Charge symmetry requires that $f_1=f_2=f/\sqrt{2}$; the use of both f_3 and f_4 corresponds to two choices of sign for the coupling of the neutral mesons to the nucleons, since the expectation value of τ_3 is positive for a neutron and negative for a proton, while the expectation value of τ_4 is positive for both nucleon states.

The seven additional matrix elements can be obtained by various permutations of P_1, P_2, P_3, and P_4, changes of the sign of q_μ, and interchanges of U_μ^2 and U_ν^1, corresponding to an emission of the final meson after the exchange of the virtual meson. In the center-of-mass system these matrix elements are approximately

$$f^\lambda(f^\rho)^2[\psi^*(3)\tau^\lambda\tau^\rho U_1\psi(1)]\times[\psi^*(4)\tau^\rho U_2\psi(2)]/(2M^2\mu^5)^{\frac{1}{2}}, \quad (4)$$

where $U_1=U_2=1$, S(III); $U_1=1$, $U_2=(\boldsymbol{\sigma}\cdot\mathbf{p}_2)\mu/4M^2$, Ps.Ps(III); $U_1=(\boldsymbol{\sigma}\cdot\mathbf{q})(\boldsymbol{\sigma}\cdot\mathbf{p}_1)/\mu^2$, $U_2=(\boldsymbol{\sigma}\cdot\mathbf{p}_2)/\mu$, Ps.Pv(III); $U_1=|q|/\mu$, $U_2=1$, V.V(III).

The total transition matrix elements given by these expressions, squared and averaged over the spins of the nucleons, are given by the expressions

$$|H_{if}|^2=G_{\mathrm{III}}^6/2M^2\mu^5,\ \text{S(III), Ps.Ps(III)},$$
$$=(G_{\mathrm{III}}^6/2M^2\mu^5)\cdot(|q|^2/\mu^2),\ \text{Ps.Pv(III), V.V(III)}; \quad (5)$$

the values of G_{III}^6 for the various theories and processes are given in Table I. It is apparent that the results are sensitive to the relative choice of sign for the coupling of neutral mesons to neutrons and protons, since the use of f_3, with f_4 set equal to zero, corresponds to the opposite choice of sign for the couplings, while the use of f_4, with f_3 equal to zero, corresponds to the same choice of sign for the couplings. It is interesting to observe that the only theory which predicts the production of neutral mesons in P–P collisions with a cross section comparable with that for charged mesons is Ps.Ps(III). For the other theories, cancellations occur between matrix elements, corresponding to emission of the final meson by the initial or final nucleons, which make the cross section of the order of $(\mu/M)^2$ smaller than for charged meson production.

III. PHENOMENOLOGICAL CALCULATION OF PRODUCTION

In this type of calculation we assume that the process of production can be described by meson emission preceded or followed by the nucleon-nucleon scattering. This, however, is not quite true, since virtual processes occur which cannot be described in terms of such a separation. An analysis of the process, considered as taking place in third order, indicates three basic ways in which a typical process can take place. These are

$$\begin{array}{ll}\text{(a)} & \\ \text{(b)} & P_1+P_2\rightarrow \\ \text{(c)} & \end{array}\left\{\begin{array}{l}P_1+P'+\omega'\rightarrow P'+P_3\\ P_2+P'+\omega'\rightarrow P'+P_3\\ P_2+P'+\omega'\rightarrow P_2+P_4+\omega+\omega'\end{array}\right\}\rightarrow P_3+P_4+\omega. \quad (6)$$

It is clear that the virtual meson (ω') exchange in processes (a) and (b) is analogous to that which occurs in scattering; therefore, the exchange can be replaced by the effects of the potential which predicts the scattering. However, in process (c) the real meson (ω) is emitted between the emission and reabsorption of the virtual meson. Such a process cannot occur in the scattering of real nucleons; therefore, its effect cannot be given in terms of the potential model. However, it can be shown easily that such processes will give a rather small contribution, at least for theories in which negative energy states are not important for the virtual nucleons. The energy denominator for these matrix elements is given by

$$1/(E_0-E_1')(E_0-E_2'), \quad (7)$$

where E_0 is the total energy, and E_1' and E_2' are the energies of the two intermediate states. These denominators are (ignoring negative energy states), for processes (a) and (b),

$$1/(\tfrac{1}{2}E_0-E_4-\omega')(E_0-2E_4)=1/(\tfrac{1}{2}\omega-\omega')\omega \quad (8)$$

and, for process (c),

$$1/(\tfrac{1}{2}E_0-W_4-\omega')(\tfrac{1}{2}E_0-\omega-\omega'-E_3)=1(\tfrac{1}{2}\omega-\omega')(-\tfrac{1}{2}\omega-\omega'). \quad (9)$$

Now, since near threshold the energy ω' of the virtual

[7] We shall, in what follows, refer to the third-order computations for scalar, pseudoscalar with pseudoscalar or pseudovector coupling, and vector theory with vector coupling as S(III), Ps.Ps(III), Ps.Pv(III), and V.V(III), respectively.

[8] K. M. Case, Phys. Rev. 76, 1 (1949).

TABLE I. Values of $G_{\mathrm{III}}{}^6$ for production of charged mesons and neutral mesons of type 3 (coupled through τ_3) or of type 4 (coupled through τ_4).

		S(III), V.V.(III)	Ps.Ps(III)$\times\mu^3/16M^3$	Ps.Pv(III)$\times(M/\mu)^2$
N–P	π^+,π^-	$f^2(f^2-2f_3^2)^2$	$f^2(f^2+2f_4^2)^2$	$f^2((f^2-2f_4^2)^2\sin^2\theta+(f^2+2f_3^2)^2\cos^2\theta)$
	$\pi^\circ(3)$	0	$2f_3^2(f_3^2-f_4^2)^2$	$2f_3^2((f_3^2-f_4^2)^2\sin^2\theta+f^4\cos^2\theta)$
	$\pi^\circ(4)$	0	$2f_4^2((f_3^2-f_4^2)^2+f^4)$	$2f_4^2\sin^2\theta(f^2+f_3^2-f_4^2)^2$
P–P	π^+	$f^2(f^2-2f_3^2)^2$	$f^2(3f^4-4f^2f_4^2+12f_4^4)$	$f^2((f^2-2f_4^2)^2\sin^2\theta+(f^2+2f_3^2)^2\cos^2\theta)$
	$\pi^\circ(3)$	0	$64f_3^2(f_3^2+f_4^2)^2$	0
	$\pi^\circ(4)$	0	$64f_4^2(f_3^2+f_4^2)^2$	0

mesons is much larger than the energy ω of the real meson, these are approximately $-1/\omega\omega'$ for (a) and (b), and $1/\omega'^2$ for (c). Therefore, the contribution from process (c) is smaller than the contribution from processes (a) and (b) in the ratio $\omega/2\omega'$. Since the momentum of the virtual meson is equal to the difference of the momenta of the initial and final nucleons, which at threshold is about $(\mu M)^{\frac{1}{2}}$, the ratio $\omega/2\omega'$ is about $\frac{1}{2}(\mu/M)^{\frac{1}{2}}$ which is about 20 percent. This contribution is negligible only if the ratio of the masses of the meson and nucleon is small; actually, an error of the order of 20 percent in the matrix element can be expected if this term is ignored. A further error arises from the neglect of negative energy states for the virtual nucleons; however, because of the largeness of the energy denominators for such processes, the contribution is negligible for all theories except for pseudoscalar coupling. This case will be discussed in detail below.

We now consider processes leading from the initial state of two nucleons to the final state of two nucleons and a meson. This can take place in two ways, either through a scattering of the two initial nucleons followed by the meson emission, or with the order of these events reversed. We, therefore, have for the matrix element

$$\begin{aligned}&\int drdr'[\psi_F^*(r,r')H(r,r')\psi'(r,r')]\\&\quad\times\int\{drdr'[\psi'^*(r,r')S_{\mathrm{I}}(r,r')\psi_{\mathrm{I}}(r,r')]/(E_0-E')\}\\&\quad+\int drdr'[\psi_F^*(r,r')S_{\mathrm{II}}(r,r')\psi''(r,r')]\\&\quad\times\int drdr'[\psi^{*\prime\prime}(r,r')H(r,r')\psi_{\mathrm{I}}(r,r')]/E_0-E''.\end{aligned}\tag{10}$$

In this expression S_{I} and S_{II} are the potentials describing the interaction of the initial and final nucleons, H is the operator for meson emission, and E' and E'' are the energies of the two intermediate states. The energy denominators are

$$\begin{aligned}E_0-E'&=(P_1^2/2M)-(P_2^2/2M)-P_1'^2/2M\\&=(P_0^2-P'^2)/M,\\E_0-E''&=(P_0^2/M)-(P''^2/M)-\omega.\end{aligned}\tag{11}$$

(A) Scalar Meson Theory, Vector Meson Theory

We shall now consider the case of two initial protons P_1, P_2 going into a final neutron N_3, proton P_4, and positive meson q. For this case we have the scattering before meson emission in the P–P potential and the scattering after emission in the N–P potential. If we take scalar coupling for the meson, then

$$H=f^\lambda\tau^\lambda\exp(-i\mathbf{q}\cdot\mathbf{r})/(2\omega)^{\frac{1}{2}}.\tag{12}$$

The case of vector coupling can be considered simultaneously, since the coupling is

$$H=f^\lambda\tau^\lambda\gamma_\mu\phi_\mu(r),\tag{13}$$

where γ_μ is the 4-vector formed from the Dirac matrices. This coupling is approximately, for longitudinally polarized mesons,

$$f^\lambda\tau^\lambda|q|/\mu\exp(-i\mathbf{q}\cdot\mathbf{r})/(2\omega)^{\frac{1}{2}}\tag{14}$$

if corrections of the order of v/c for the nucleons are ignored. Therefore, we can obtain this result from that for scalar theory by multiplying the matrix element by $|q|/\mu$.

We find for the matrix element for S(phen)[9]

$$\begin{aligned}&-\sqrt{2}fg_{PP}{}^2M(1-P_{12})\\&\quad\times\frac{(\chi_3{}^*\boldsymbol{\sigma}\cdot(\mathbf{P}_1+\mathbf{P}_4)\chi_1)(\chi_4{}^*\boldsymbol{\sigma}\cdot(\mathbf{P}_4+\mathbf{P}_1)\chi_2)}{(2\omega)^{\frac{1}{2}}(P_1^2-P_4^2)[\mu^2+(P_1+P_4)^2]}\\&\quad+\sqrt{2}fg_{NP}{}^2(1-P_{12})\\&\quad\times\frac{(1+P_{34}{}^x)}{2}\frac{(\chi_3{}^*\chi_1)(\chi_4{}^*\chi_2)}{(2\omega)^{\frac{1}{2}}\omega[\mu^2+(P_1+P_4)^2]},\end{aligned}\tag{15}$$

where the operator P^x exchanges the momentum coordinate (corresponding to space exchanges in a coordinate representation), and P exchanges both the momentum and spin coordinates. Near threshold, $P_1\gg P_3$ or P_4. We can also disregard μ^2 relative to P_1^2. We then note that

$$\tfrac{1}{2}(1+P_{34}{}^x)\frac{(\chi_3{}^*\chi_2)(\chi_4{}^*\chi_1)}{\mu^2+(P_1+P_4)^2}\cong\frac{(\chi_3{}^*\chi_2)(\chi_4{}^*\chi_1)}{P_1^2}.$$

[9] We shall refer to the phenomenological treatment of scalar, vector, and pseudoscalar theories with pseudovector coupling as S(phen), V.V(phen), and Ps.Pv(phen), respectively.

TABLE II. Values of G_{phen}^6 for production of charged mesons and neutral mesons of type 3 and type 4.

Charged		S(phen), V.V(phen) $f^2[(M/\mu g_{PP}^2)^2+g_{NP}^4]$	Ps.Pv(phen) $f^2((M/\mu g_{PP}^2)+g_{NP}^2)^2$
$N-P$	$\pi^\circ(3)$	0	0
	$\pi^\circ(4)$	0	0
$P-P$	$\pi^\circ(3)$	0	0
	$\pi^\circ(4)$	0	0

The expression finally simplifies to

$$\frac{-(1-P_{12})f}{M\mu^2(\mu)^{\frac{1}{2}}}\left[g_{PP}^2\frac{(\chi_3{}^*\boldsymbol{\sigma}\cdot\mathbf{P}_1\chi_1)(\chi_4{}^*\boldsymbol{\sigma}\cdot\mathbf{P}_1\chi_2)}{\mu^2} - g_{NP}^2(\chi_3{}^*\chi_1)(\chi_4{}^*\chi_2)\right], \quad (16)$$

where we have set $(P_1{}^2/2M)-\mu/2$ and $\omega=\mu$, which are their values at threshold.

A similar expression is obtained for an initial neutron and proton going into two final protons and a meson. For processes involving the scattering of two neutrons, the result depends on the choice of the N–N interaction. However, in the absence of any direct information, we shall assume that this is the same as the P–P interaction. The results for these processes are then identical with those given earlier.

For the production of neutral mesons, the analysis is similar to that given here. In this case, however, the nucleon charge is unchanged by the meson emission, and the scattering takes place before and after emission in the same potential. We find that cancellation occurs between the terms representing scattering before or after emission, so that a zero cross section is predicted for neutral mesons in either N–P or P–P collisions.

(B) Pseudoscalar Meson with Pseudovector Coupling

The analysis for pseudovector coupling can be carried out in a similar way. Here we have

$$H=f^\lambda\tau^\lambda\boldsymbol{\sigma}\cdot\mathbf{q}/\mu\exp(-i\mathbf{q}\cdot\mathbf{r})/(2\omega)^{\frac{1}{2}}. \quad (17)$$

This gives, for the production of charged mesons,

$$-f(1-P_{12})\{g_{PP}^2(\chi_3{}^*\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{P}_1\chi_1)(\chi_4{}^*\boldsymbol{\sigma}\cdot\mathbf{P}_1\chi_2)/\mu^2 - g_{NP}^2(\chi_3{}^*\boldsymbol{\sigma}\cdot\mathbf{q}\chi_1)(\chi_4{}^*\chi_2)\}/M\mu^3(\mu)^{\frac{1}{2}}. \quad (18)$$

For neutral mesons produced in P–P and N–P collisions, we find that, as for scalar mesons, the cross sections are zero (to order $(\mu/M)^2$).

TABLE III. Ratio of third-order to phenomenological matrix elements averaged over angles for production of charged mesons.

Scalar, vector	Pseudovector coupling
$\{(f^2-2f_3{}^2)/4\pi\}^2$	$\{\frac{1}{3}(f^2-2f_4{}^2)^2+\frac{2}{3}(f^2+2f_3{}^2)\}/(4\pi)^2$
$(0.499)^2$	$(0.093)^2$

(C) Pseudoscalar Meson with Pseudoscalar Coupling

For the case of pseudoscalar coupling, we have

$$H=f^\lambda\tau^\lambda\gamma_5\exp(-iq\cdot r)/(2\omega)^{\frac{1}{2}}. \quad (19)$$

We shall now use the relativistic generalization of the P–P potential, which is

$$g_{PP}^2(2M/\mu)^2(\gamma_5)_1(\gamma_5)_2\exp(-\mu r)/4\pi r. \quad (20)$$

Since negative energy states are important with this form of interaction, we must reconsider the energy denominators in their relativistic form. If the intermediate nucleon is in a negative energy state, its energy is approximately equal to the negative of its rest mass, and we have for the energy denominators for processes (a) and (b), as given in Eq. (6), approximately $1/2M^2$ and for process (c) $-1/2M\omega'$. We also must consider the behavior of the matrix elements describing the emission and reabsorption of the mesons. For transitions between positive energy states, $(\gamma_5)(\gamma_5)\sim(v/c)^2$, where v is the velocity of the nucleus. For transitions to and from negative energy states, $(\gamma_5)(\gamma_5)$ is about 1. Combining these results, we find, for the approximate magnitude of the matrix elements,

Process (a) and (b):

$$2(v/c)^2/\omega\omega'\sim 1/2M\omega'(\mu/M)^{\frac{1}{2}}\cong 0.40/2M\omega' \quad \text{(transitions to positive energy states);} \quad (21)$$

Process (c):

$1/2M\omega'$ (transition to negative energy state).

We see, therefore, that near threshold we cannot neglect the nonpotential-like terms corresponding to process (c) with the virtual nucleon undergoing transitions to negative energy states, since they give the largest contribution to the matrix element. We must refer to the third-order perturbation theory results for a more adequate description of the process of the anomalous case of pseudoscalar coupling.

(D) Summary of Results for Phenomenological Calculation

The squares of the magnitudes of the matrix elements, given by the application of the phenomenological method to the meson production problem, are given by

$$|H_{if}|^2=G_{\text{phen}}^6/2M^2\mu^5 \quad \text{S(phen)}$$
$$=(G_{\text{phen}}^6/2M^2\mu^5)q^2/\mu^2 \quad \text{Ps.Pv(phen), V.V.(phen).} \quad (22)$$

The values of G_{phen}^6 are listed in Table II. A comparison of these results with those obtained in Sec. II by the third-order calculation for the production of charged mesons is given in Table III. It is apparent that the results obtained by these two methods are approximately equal for not unreasonable choice of the coupling constants. The use of symmetric theory, in which $f=f_3$

TABLE IV. Values of square of magnitude of transition matrix elements, for P–P production of positive mesons, evaluated at zero relative momentum for the final nucleons, leading to singlet and triplet spin state. (All are to be multiplied by $\frac{1}{2}M^2\mu^5$.)

	S(III)	S(phen)	V.V(III)	V.V(phen)	Ps.Ps(III)	Ps.Pv(III)	Ps.Pv(phen)
$\lvert M_{if}(0)(\text{triplet})\rvert^2$	0	0	0	0	$\frac{2}{3}G_{\text{III}}^6$	$G_{\text{III}}^6 2T/\mu$	$G_{\text{phen}}^6 2T/\mu$
$\lvert M_{if}(0)(\text{singlet})\rvert^2$	G_{III}^6	G_{phen}^6	$G_{\text{III}}^6 2T/\mu$	$G_{\text{phen}}^6 2T/\mu$	$\frac{1}{3}G_{\text{III}}^6$	0	0

and f_4 is set equal to zero, however, would predict a zero cross section for[10] S(III) and V.V(III). It is certainly not necessary, however, that such a choice of the coupling constants be made.

IV. CALCULATION OF DIFFERENTIAL CROSS SECTION

We shall consider the specific case of production of positive mesons in P–P collisions. The generalization to other cases can easily be made. To the approximation used in these calculations, the differential cross section in the center-of-mass coordinate system for the production of two nucleons and a meson in a nucleon-nucleon collision is given by the expression

$$d\sigma/d\Omega dT = 2\sqrt{2}M^2\mu[T(T_m-T)]^{\frac{1}{2}}|H_{if}|^2/(4\pi)^3\pi, \quad (23)$$

where T is the meson kinetic energy, T_m is the maximum meson kinetic energy, and H_{if} is the transition matrix element including the effects, discussed in the Appendix, of the interaction of the final nucleons. Using the results of that section, we can write

$$|H_{if}|^2 = |\psi_t(r=0)M_{if}(0)(\text{triplet})|^2 + |\psi_s(r=0)M_{if}(0)(\text{singlet})|^2, \quad (24)$$

where $M_{if}(0)$(triplet) and $M_{if}(0)$ (singlet) are the matrix elements leading to final triplet and singlet spin states, and $\psi_t(0)$ and $\psi_s(0)$ are the wave functions for the final nucleons in triplet and singlet spin states, respectively, evaluated at $r=0$. Substituting the values for the wave functions, as given in the Appendix, this becomes (setting $E_f = T_m - T$)

$$d\sigma/d\Omega dT = 2(2)^{\frac{1}{2}}M^2\mu[T(T_m-T)]^{\frac{1}{2}} \times\{|M_{if}(0)(\text{triplet})|^2[(V_t+T_m-T)/(\epsilon_t+T_m-T)] + |M_{if}(0)(\text{singlet})|^2(V_s+T_m-T)/(\epsilon_s+T_m-T)\}. \quad (25)$$

The values of $M_{if}(0)$(triplet) and $M_{if}(0)$(singlet) are given in Table IV. The relation between the constants G_{III}^6 and G_{phen}^6 and the coupling constants f, f_3, and f_4 is given in Tables I and II.

The total cross section, then, is given approximately by the expressions

$$\frac{4\pi\sqrt{2}M^2\mu T_m^2}{(4\pi)^3}\{|M_{if}(0)(\text{triplet})|^2 \times[\tfrac{1}{4}+(V_t/T_m)(1-2(\epsilon_t/T_m)^{\frac{1}{2}})] + (\text{corresponding term for singlet})\}$$ for S. and Ps.Ps.

and

$$\frac{8\pi\sqrt{2}M^2\mu T_m^3}{(4\pi)^3}\left\{\left|\frac{M_{if}(0)(\text{triplet})}{T}\right|^2 \times[\tfrac{1}{16}+(V_t/T_m)(\tfrac{3}{8}+\tfrac{3}{2}(\epsilon_t/T_m)-(\epsilon_t/T_m)^{\frac{1}{2}})] + (\text{corresponding term for singlet})\right\}$$

for V.V and Ps.Pv.

We can now use this cross section, ignoring the possibility of deuteron formation, to estimate the values of the coupling constants. If we assume a cross section at 350 Mev of 2×10^{-28} cm^2 with the corresponding maximum value of the meson kinetic energy in the center-of-mass system of 24.8 Mev, the corresponding values of $G_{\text{III}}^6/(4\pi)^3$ and $G_{\text{phen}}^6/(4\pi)^3$ are given in Table V. The values of the coupling constants f, f_3, and f_4 calculated from these values of $G^6/(4\pi)^3$, by the relation given in Tables I and II, are given in Table VI. For comparison, the values of the coupling constants adjusted to agree with the magnitude of P–P scattering[4] at 350 Mev (about 25 millibarns) are also given. It is apparent that the coupling constants predicted by these two processes, meson production and high energy scattering, are within the same order of magnitude. Conversely, we can conclude that any of the third-order or phenomenological results can correctly predict the order of magnitude of the meson production cross section, for not unreasonable choices of the coupling constants.

Finally, we must consider the additional process which can contribute to meson production, the formation of a deuteron and a meson in a proton–proton collision. The cross section can be calculated easily; we have (see Appendix)

$$d\sigma/d\Omega = |M_{if}(0)(\text{triplet})\psi_D(r=0)|^2 Mq\mu/(8\pi^2P_0) = |M_{if}(0)(\text{triplet})|^2 4(2)^{\frac{1}{2}}(T_m\epsilon_t)^{\frac{1}{2}}V_tM^2\mu/(4\pi)^3. \quad (26)$$

Using the values of the matrix elements given in Table

TABLE V. Values of the constants $G_{\text{III}}^2/4\pi$ and $G_{\text{phen}}^2/4\pi$ giving a total cross section at 350 Mev at 2×10^{-28} cm^2 for production of a positive meson, neutron, and proton in a P–P collision.

	S(III), S(phen)	V.V(III), V.V(phen)	Ps.Ps(III)	Ps.Pv(III) * Ps.Pv(phen)
$G^2/4\pi$	0.268	0.423	0.298	0.513

* Averaged over angles.

[10] A similar result was obtained in reference 3.

TABLE VI. Values of the coupling constants f, f_3, and f_4 as given by Tables I, II, and V, and also as predicted by P–P scattering at 350 Mev.

	Meson production	P–P scattering
S(III)	$[f^2(f^2-2f_3^2)^2]^{\frac{1}{2}}/4\pi=0.268$	$(f_3^2+f_4^2)/4\pi=0.142$
S(phen)	$f^2/4\pi=0.0608$	
V.V(III)	$[f^2(f^2-2f_3^2)^2]^{\frac{1}{2}}/4\pi=0.423$	$(f_3^2+f_4^2)/4\pi=0.142$
V.V.(phen)	$f^2/4\pi=0.254$	
Ps.Ps(III)	$[f^2(3f^4-4f^2f_4^2+12f_4^4)^2]^{\frac{1}{2}}/4\pi=3.79$	$(f_3^2+f_4^2)/4\pi=3.73$
Ps.Pv(III)	$f^2[\frac{2}{3}(f^2-2f_4^2)^2+\frac{1}{3}(f^2+2f_3^2)^2]^{\frac{1}{2}}/4\pi=0.145$	$(f_3^2+f_4^2)/4\pi=0.0209$
Ps.Pv(phen)	$f^2/4\pi=0.354$	

IV, the cross sections are

$$d\sigma/d\Omega=(G^2_{\rm III,\,phen}/4\pi)^3(T_m/\mu)^{\frac{3}{2}}6.66\times10^{-26}\ {\rm cm}^2\quad \text{Ps.Pv(III), Ps.Pv(phen)}$$
$$=G_{\rm III}{}^2/4\pi)^3(T_m/\mu)^{\frac{1}{2}}2.22\times10^{-26}\ {\rm cm}^2\quad \text{Ps.Ps(III).}\qquad(27)$$

At 350 Mev, using the values of the constants $G^2/4\pi$ given in Table V, the total cross sections for formation of a deuteron and a meson are

$$\sigma=7.63\times10^{-28}\ {\rm cm}^2\ \text{Ps.Pv(III), Ps.Pv(phen),}$$
$$=2.57\times10^{-28}\ {\rm cm}^2\ \text{Ps.Ps(III).}\qquad(28)$$

These cross sections are larger than those in which the final nucleons are not bound (assumed to be 2×10^{-28} cm^2).

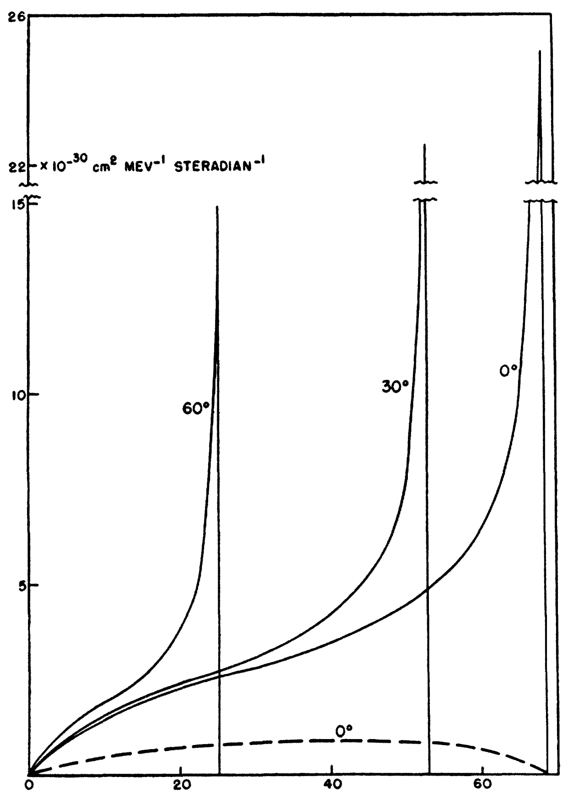

FIG. 2. Differential cross section at 350 Mev in the laboratory system for production of positive scalar mesons in P–P collisions. The solid curve includes the effects of the interactions of the final particles; the dashed curve is the result of the calculations using Born approximation throughout.

The differential cross sections at 350 Mev for proton-proton production of positive mesons are given in Figs. 2 and 3 for scalar theory and for the phenomenological treatment of pseudoscalar theory with pseudovector coupling. For comparison, the cross sections obtained when the Born approximation was made by treating the final nucleon wave functions as plane waves are also given (dashed curves). The very striking effects of the interactions of the final nucleons are obvious. The variation of the total cross sections with energy is given in Table VII, including the contribution to the cross section given when a deuteron is formed. The normalization is again to a total cross section of 2×10^{-28} cm^2 at 350 Mev for the production in which the two final nucleons are unbound.

V. CONCLUSIONS

The experimental results of the Berkeley workers[1] indicate that the cross section per nucleon for proton bombarding carbon is about 2×10^{-28} cm^2 for both charged and neutral mesons. For protons bombarding free protons, the cross section is about the same as for production of charged mesons but appears to be, perhaps, an order magnitude smaller for neutral meson production. It is apparent from the results of Secs. II, III, and IV that the relative size of these cross sections is predicted successfully only by the third-order result for pseudoscalar mesons with pseudovector coupling. The phenomenological result for scalar, vector, and pseudoscalar theory fails in that a zero cross section is predicted for production of neutral mesons in neutron–proton collisions. The third-order result for pseudoscalar theory with pseudoscalar coupling is of the right order of magnitude, except for neutral mesons produced in proton–proton collisions, where a cross section is predicted comparable with that for charged mesons in contradiction to the experimental result. The third-order result for scalar and vector theories is peculiar in that the cross section for neutral mesons vanishes for neutron–proton collisions and also vanishes for charged mesons if $f^2/2$ is taken equal to f_3^2 (Table I), corresponding to the use of symmetrical theory. It is interesting to observe that a small cross section is predicted for neutral meson production in proton-proton collisions by all of the theories except pseudoscalar theory with pseudoscalar coupling. It is somewhat questionable, however, that such cancellations as those which appear in these calculations are to be quantitatively believed, since it is possible that higher order virtual effects would remove the cancellation which appears in lowest order.[11]

The phenomenological calculations are successful in predicting a sufficiently large charged meson cross section for agreement with experiment, because an unsymmetrical choice of the neutron-proton and proton-proton potentials was made. The calculation made by Marshak and Foldy used a symmetrical interaction, and the cancellation which occurred reduced the cross section by 2 or 3 orders of magnitude. It is not necessarily true, however, that all symmetrical theories would give a similar result. A theory which while symmetrical could also predict the high energy nucleon-nucleon scattering would presumably give the same qualitative features as the potential models used in these phenomenological calculations. It is interesting to note that the use of symmetrical theory in the third-order calculation does predict correctly the general magnitudes of the cross sections for pseudovector coupling but gives zero cross sections for scalar and vector theory.

A more detailed prediction of these calculations which is of particular interest in the angular dependence of the cross sections. Only the third-order calculation for pseudoscalar theory with pseudovector coupling gives a non-isotropic angular distribution (in the center of mass system). This is due to the presence of a tensor force in the neutron-proton interaction which couples the orbital motion of the meson to that of the nucleons. This effect is absent in the phenomenological treatment because of the use of a central interaction to describe the neutron-proton interaction.

The experiments on meson production by protons bombarding free protons provide a good opportunity for verifying the detailed predictions of the differential energy spectra. The experimental results of Cartwright *et al.* for meson production in the beam direction are shown in Fig. 4 in comparison with the predictions of pseudoscalar theory with pseudovector coupling. It is apparent that agreement with experiment can be obtained only if the effects of the interactions of the final nucleons are taken into account. The predicted fine structure of the high energy peak resulting from the deuteron formation cannot be resolved with the present experimental data; presumably, an improvement of experimental techniques will make it possible to test this prediction of the theory. Additional data on the angular dependence of the cross sections will also provide a critical test of the theoretical predictions.

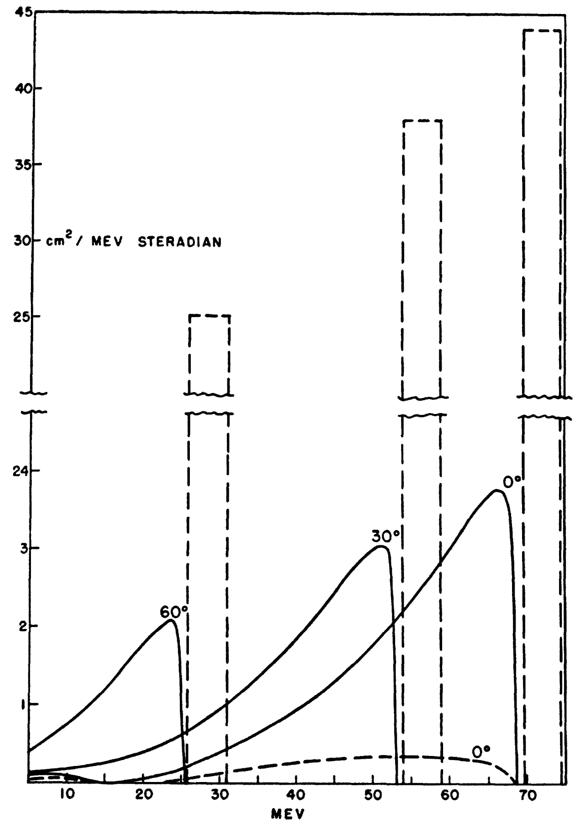

FIG. 3. Differential cross section at 350 Mev in the laboratory system for production of positive pseudoscalar mesons with pseudovector coupling in *P–P* collisions. The delta-function representing deuteron formation is averaged over a 5-Mev energy interval.

The author wishes to thank Professor Robert Serber and Dr. K. M. Watson for many interesting discussions of the theoretical results derived in this paper. He also wishes to thank the experimental workers at Berkeley, particularly Drs. Chaim Richman, Herb York, and Vince Peterson for continuous information about the preliminary results of their work and for their aid in interpreting the experiments. In particular, he wishes to thank William Cartwright and Marian Whitehead for permission to quote the results of their experiment in advance of publication.†

TABLE VII. Variation with energy of total cross section for positive meson production in *P–P* collisions, in units of 10^{-28} cm². The columns headed "Unbound" are for production leading to a neutron, proton, and meson; those headed "Deuteron" are for production leading to a deuteron and a meson.

	Scalar	Vector	Ps.Ps		Ps.Pv	
(Mev)	Unbound	Unbound	Unbound	Deuteron	Unbound	Deuteron
290	0	0	0	0	0	0
325	1.10	0.64	0.83	2.05	0.53	3.85
350	2.00	2.00	2.00	2.57	2.00	7.63
375	3.08	4.51	3.48	3.07	5.01	13.0

[11] K. A. Brueckner and K. Watson found for neutral meson production by photons that a similar cancellation in pseudoscalar theory was removed in higher orders (Phys. Rev. **79**, 187 (1950)).

† *Note added in proof:* Recent experimental results of Cartwright, Richman, and Whitehead show that the angular distribution of positive pi-mesons produced in *P–P* collisions at 343 Mev is almost entirely $\cos^2\theta$ in the center-of-mass system. This is

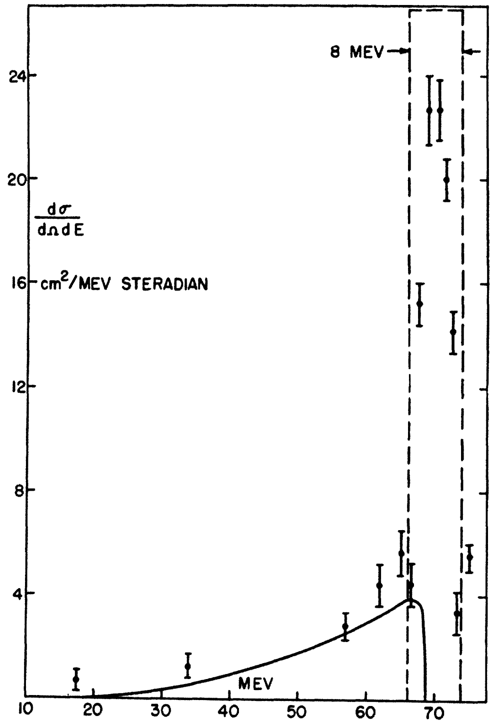

FIG. 4. Comparison of the experimental results of Cartwright *et al.* for production of positive mesons by 340-Mev protons bombarding free protons. The curve is for mesons produced in the direction of the proton beam in the laboratory system.

APPENDIX[12]

Effects of Interaction of Final Nucleons

For simplicity, we restrict ourselves to the case of two initial protons leading to a final neutron, proton, and positive meson. In the calculations described in Secs. II and III, we have made the approximation of representing the wave function of the final nucleons by plane waves. This is equivalent to using the Born approximation to describe the nucleon-nucleon scattering. However, near threshold, where the final nucleons have low energies, the scattering into these final states is poorly represented by the Born approximation applied to the potentials. The calculations can be done in a more satisfactory way if the actual wave function of the final nucleons is used. We can then resolve this into plane waves by the relation

$$\psi_F(r)=\int \alpha_k \exp(ik\cdot r)dk, \qquad (1)$$

where

$$\alpha_k=[1/(2\pi)^3]\int \psi_F(r)\exp(-ik\cdot r)dr. \qquad (2)$$

The calculations which we have made can then be considered to represent one of the fourier components of this momentum distribution. We can represent the transition matrix element which leads from the initial state to the final state, in which we have plane outgoing waves of relative momentum k, by $M_{if}(k)$. The transition matrix element to a state $\psi_F(r)$ then will be given by the expression

$$H_{if}=\int dk\alpha_k M_{if}(k). \qquad (3)$$

If we wish further to separate the final state into singlet and triplet spin states, we must consider separately the matrix elements of $M_{if}(k)$ leading to these spin states.

If we insert the definition of α_k, we have

$$H_{if}=[1/(2\pi)^3]\int\int drdk \exp(-\mathbf{i}\mathbf{k}\cdot\mathbf{r})\psi_F(r)M_{if}(k). \qquad (4)$$

Now if we define

$$[1/(2\pi)^3]\int dk \exp(-\mathbf{i}\mathbf{k}\cdot\mathbf{r})M_{if}(k)=M_{if}'(r), \qquad (5)$$

then we have for the matrix element

$$\begin{aligned} H_{if}&=\int \psi_F(r)M_{if}'(r)dr \\ &=\psi_F(r_{\mathrm{Av}})\int M_{if}'(r)dr \\ &=\psi_F(r_{\mathrm{Av}})M_{if}(0), \end{aligned} \qquad (6)$$

where $\psi_F(r_{\mathrm{Av}})$ is the value of the final wave function at an average value of r. This separation can be made more acceptable if we note that $M_{if}'(r)$ must be large only for r considerably less than the range of the forces, since the large momentum transfers necessary for meson production lead to a rather singular form for $M_{if}'(r)$. Then, since for low energy nucleons, $\psi_F(r)$ is slowly varying over a region of the size of the meson Compton wavelength, the results will not be sensitive to the particular value of r_{Av}, as long as r_{Av} is considerably less than $\hbar/\mu c$.

This simple result can be applied to the problem of interest. If we wish to calculate the probability that a deuteron is formed, we can take the expectation value of the transition matrix element $M_{if}(0)$ between the initial state of arbitrary spin and the final triplet state, and multiply this by the deuteron wave function evaluated at r_{Av}. Similarly, if we wish to include the effects of the interaction of the slowly moving final nucleons in a singlet or triplet state, we multiply the appropriate values of the matrix elements by the singlet or triplet wave functions evaluated at r_{Av}. We shall use the approximate wave functions for a square well[4] of range 1.53×10^{-13} cm, triplet depth V_t of 52.9 Mev, singlet depth V_s of 41.1 Mev. This gives simple analytic expressions for the wave functions of the deuteron and of the unbound system of neutron and proton of low relative momentum. The use of this nonsingular potential may underestimate the magnitude of the wave functions for small separations; however, the results will not be qualitatively incorrect. The approximate wave functions are

$$\begin{aligned} \psi_D(r)&=\sin[(MV_t)^{\frac{1}{2}}r]/r[(M\epsilon_t)^{\frac{1}{2}}/2\pi]^{\frac{1}{2}}, \\ \psi_{t,s}(r)&=\sin[(M(V_{t,s}-E_f))^{\frac{1}{2}}r]/r[M(\epsilon_{t,s}+E_f)]^{\frac{1}{2}}, \end{aligned} \qquad (7)$$

where ϵ_s, ϵ_t are the singlet or triplet binding energy (in magnitude), and E_f is the energy of the final nucleons. The magnitude of these wave functions is quite insensitive to the choice of r_{Av} for r_{Av} less than $\hbar/\mu c$; for simplicity, we shall evaluate them at r_{Av} equal to zero. We then have

$$\begin{aligned} \psi_D(0)&=(MV_t)^{\frac{1}{2}}[(M\epsilon_t)^{\frac{1}{2}}/2\pi]^{\frac{1}{2}}, \\ \psi_{t,s}(0)&=[(V_{t,s}+E_F)/(\epsilon_{t,s}+E_F)]^{\frac{1}{2}}. \end{aligned} \qquad (8)$$

It is apparent from these results that the value of the matrix element is considerably increased by the factor $\psi_F(0)$ near threshold, where the final nucleon energy E_f is considerably less than the well depth of about 50 Mev. The factor

$$(V_{s,t}E_f)/(\epsilon_{s,t}+E_f) \qquad (9)$$

approaches 1 only for $E_f\gg V$; this condition is not satisfied until the incident nucleon energies are of the order of a Bev. At 350 Mev, where the final nucleon energy varies from 0 to about 25 Mev, this factor varies from about 2 to 25 for the triplet state and from 2 to about 600 for the singlet state. This has the effect of raising the cross section by a factor of about 3 or 4. The effects of the interaction of the final nucleons, therefore, clearly are large and cannot be ignored.

given only by the third-order result for pseudoscalar theory with pseudovector coupling with $f^2=2f_4^2$ (see Table I), supporting the conclusions already reached in this paper.

[12] The material of this Appendix was developed in collaboration with Geoffrey Chew and Edward Hart, to whom the author wishes to express his appreciation.