Electron-Neutrino Angular Correlation*

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Angular correlation functions are calculated for allowed and first forbidden beta-decay transitions including the effect of the nuclear coulomb field. The process of beta-decay is viewed as a kind of scattering in which an incident negative energy neutrino is transformed by the Fermi interaction in to scattered electron in a positive energy state. The outgoing electronic wave function is calculated, and the angular correlation functions are determined from the angular dependent electron probability current through sections of a sphere at large distances from the nucleus. For allowed transitions, the angular correlation is found to be independent of atomic number and identical with the results of previous calculations which assumed $Z=0$. The $Z=0$ approximation is shown to be unreliable for first forbidden transitions in which correlation terms proportional to the first and second powers of the potential energy of an electron at the surface of the nucleus appear. This effect is important even for moderately heavy nuclei.

I. INTRODUCTION

THE relation between the directions of emission of the electron and neutrino emitted by a β-active nucleus depends on the type of light particle-nucleon interaction assumed in the Fermi theory.[1] This was first pointed out by Bloch and Møller.[2] More recently, the electron-neutrino angular correlation was investigated theoretically for the five different types of β-decay coupling[3] by Hamilton.[4] His results, for allowed and first forbidden transitions, were based on the $Z=0$ approximate treatment of the Fermi theory. Rose[5] considered the effect of finite nuclear charge on the allowed transition β-neutrino angular correlation functions. We not only confirm his concludion that the effect is small, but also, as is shown in Sec. III (A), treat the problem more exactly and conclude that, aside from the appearance of the usual Z dependent Fermi function, *the allowed correlation functions are identical with those obtained by Hamilton.*

In all previous work on β-neutrino angular correlation that has come to our attention, the effect of the nuclear coulomb field on the outgoing electron has been either neglected or, as in reference 5, been treated approximately. The shapes of forbidden transition energy spectra may depend strongly on the nuclear charge. In Sec. III (B), the Z dependent angular correlation functions for first forbidden transitions are computed and are shown to differ from the $Z=0$ results of Hamilton in just those cases where the energy spectrum also exhibits a strong Z dependence.

II. ANGULAR DEPENDENT DECAY PROBABILITY

It is helpful to visualize the β-radioactive processes as a kind of scattering of a light Dirac particle by a heavy nucleon. The usual negatron decay process[6] can be described as the scattering of a light particle from its vacuum neutrino state of negative energy by a nucleon into its positive energy, negative charge, electron state. Charge is conserved in the process by the transformation of the target nucleon from its neutron into its proton state. The electron and neutrino are considered as two possible states of a light Dirac particle, differing as to charge and rest mass.

The interaction hamiltonian between nucleon and light particle is presumed to be some linear combination of the five Lorentz invarient operators designated as scalar (S), polar vector (V), axial vector (A), tensor (T), or pseudoscalar, (P).

$$H_X=G(\mathbf{O}_H\cdot\mathbf{O}_L)_X(\tau_H\tau_L+\tau_H^*\tau_L^*)\delta(\mathbf{x}_H-\mathbf{x}_L), \quad (1)$$

where

$$X=S,\ V,\ T,\ A,\ \text{or}\ P.$$

The strength of the interaction is determined by the Fermi constant G; H and L refer to nucleon and light particle operators, respectively; τ_H transforms a neutron into a proton; τ_L transforms a neutrino into an electron; τ_H^* and τ_L^* are the inverse operators making positron decay possible. The different coupling operators, $(\mathbf{O}_H\cdot\mathbf{O}_L)_X$, are constructed by the contraction of the usual contravariant light particle Dirac operators with their covariant heavy particle analogs.[7]

(A) Asymptotic Wave Function

We specify the initial system, β-radioactive nucleus plus incident vacuum neutrino of spin s, and momentum $-\mathbf{q}$, by the wave function,[8]

$$\psi_I=\psi_I(\mathbf{x}_H,\mathbf{x}_L)\exp(-iE_It),$$

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[1] E. Fermi, Z. Physik **88**, 161 (1934).

[2] F. Bloch and C. Møller, Nature **136**, 912 (1935).

[3] E. J. Konopinski, Revs. Modern Phys. **15**, 209 (1943).

[4] D. R. Hamilton, Phys. Rev. **71**, 456 (1947).

[5] M. E. Rose, Phys. Rev. **75**, 1444 (1949).

[6] To apply our results to positron decay, it is only necessary to change the sign of Z for the product nucleus where it appears in Eqs. (22) and (37) of Sec. III (B).

[7] The five operators, $(\mathbf{O}_H\cdot\mathbf{O}_L)_X$, as given explicitly in terms of β, $\boldsymbol{\sigma}$, $\boldsymbol{\alpha}$, and γ_5 by J. Tiomno and John A. Wheeler, Revs. Modern Phys. **21**, 144 (1949), were found most convenient. See also reference 3.

[8] All quantities are made dimensionless by using the usual relativistic electron units of energy (mc^2), time ($\hbar/mc^2$), and length ($\hbar/mc$).

where (2)

$$\psi_I = U(\mathbf{x}_H) B_s(\mathbf{q}) \exp(-i\mathbf{q}\cdot\mathbf{x}_L),$$

and $E_I = E_U - K$; E_U = energy of initial nucleus. The neutrino rest mass is assumed to be zero; thus, the anti-neutrino emitted in the β-decay has an energy $K = q \geqslant 0$. The spinor amplitude B_s is normalized to unit incident neutrino current density. We designate the initial and final nuclear isobars by U and V, respectively. The interaction (1) transforms one neutron of U into a proton of V. These nuclear states are orthogonal.

Possible final states of the system involve the continuum positive kinetic energy ($W \geqslant 1$) electron wave functions in the coulomb potential field of the product nucleus. These Dirac functions are conveniently tabulated by Rose.[9] We use his notation with one exception. The radial functions f and g are given a more appropriate subscript $\kappa = \pm(j+\frac{1}{2})$, where j is the total angular momentum. The angular dependent functions are spherical harmonics, their order depending only on the pair of quantum numbers, κ and $m = [-j, -(j-1), \cdots, +j]$.

For our purpose, it is convenient to factor the electron wave functions into radial and angular dependent parts.

$$\psi_f = R_\kappa(W, r) A_f(\theta, \varphi). \tag{3}$$

The subscript f stands for the discrete pair of eigenvalues κ and m. In our notation, R is the diagonal matrix:

$$R_\kappa = \begin{pmatrix} f_\kappa & 0 & 0 & 0 \\ 0 & f_\kappa & 0 & 0 \\ 0 & 0 & g_\kappa & 0 \\ 0 & 0 & 0 & g_\kappa \end{pmatrix}, \tag{4}$$

and the angular dependent spinors are:

$$A_f(\theta, \varphi) = \begin{matrix} i\left(\dfrac{\kappa+m-\frac{1}{2}}{2\kappa-1}\right)^{\frac{1}{2}} (Y_{-\kappa}{}^{m-\frac{1}{2}},\ Y_{\kappa-1}{}^{m-\frac{1}{2}}) \\ i\left(\dfrac{\kappa-m-\frac{1}{2}}{2\kappa-1}\right)^{\frac{1}{2}} (Y_{-\kappa}{}^{m+\frac{1}{2}},\ -Y_{\kappa-1}{}^{m+\frac{1}{2}}) \\ \left(\dfrac{\kappa-m+\frac{1}{2}}{2\kappa+1}\right)^{\frac{1}{2}} (Y_{-\kappa-1}{}^{m-\frac{1}{2}},\ Y_{\kappa}{}^{m-\frac{1}{2}}) \\ \left(\dfrac{\kappa+m+\frac{1}{2}}{2\kappa+1}\right)^{\frac{1}{2}} (-Y_{-\kappa-1}{}^{m+\frac{1}{2}},\ Y_{\kappa}{}^{m+\frac{1}{2}}) \end{matrix} \tag{5}$$

$$\kappa \leqslant -1, \quad \kappa \geqslant +1.$$

After a transition, the system may be in a superposition of the basic eigenstates F represented by the time dependent wave function,

$$\psi_F = \psi_F(\mathbf{x}_H, \mathbf{x}_L) \exp(-iE_F t), \tag{6}$$

where

$$\psi_F = V(\mathbf{x}_H)\psi_f(W, \mathbf{x}_L),$$

and

$$E_F = E_V + W; \quad E_V = \text{energy of final nucleus.}$$

The subscript F differs from f in that it includes the continuum eigenvalues W for the electron. The functions $\psi_f(W, \mathbf{x}_L)$ are normalized to one electron per unit energy range

$$\left(\text{i.e.} \int d\mathbf{x}_L \psi_f^*(W, \mathbf{x}_L)\psi_{f'}(W', \mathbf{x}_L) = \delta(W-W')\delta_{\kappa\kappa'}\delta_{mm'}\right).$$

If the system is initially in the state I, we obtain, upon applying the usual first-order time dependent perturbation theory, subsequent states of the system as a superposition of the wave functions (2) and (6).

$$\chi(\mathbf{x}_H, \mathbf{x}_L, t) = \left\{\psi_I + \sum_f \int dW (f|H|s) \times \left\{\frac{\exp[-i(W_0-W-K)t]-1}{W_0-W-K}\right\}\psi_F\right\} \exp(-iE_I t), \tag{7}$$

where the nuclear energy difference is $W_0 = E_U - E_V$, and the matrix element of the perturbation, Eq. (1), reduces to the time independent expression:

$$(f|H|s) = G\left[\int d\mathbf{x}_H (V^* \mathbf{O}_H \tau_H U)\psi_f^*(\mathbf{x}_H) \times \exp(-i\mathbf{q}\cdot\mathbf{x}_H)\right] \cdot \mathbf{O}_L B_s. \tag{8}$$

We obtain from Eqs. (7) and (3), since V and U are orthogonal, the time dependent superposition of electron states resulting from the "scattering" of a neutrino by a nucleon. We call this scattered wave function resulting in the product nucleus V,

$$\psi_{sc} \exp(-iE_I t) = \int d\mathbf{x}_H V^* \chi(\mathbf{x}_H, \mathbf{x}_L, t),$$

where $\psi_{sc} = \sum_f \int dW (f|H|s)$ (9)

$$\times \left\{\frac{\exp[-i(W_0-W-K)t]-1}{W_0-W-K}\right\} R_\kappa A_f.$$

These "scattered" electron waves contain both incoming and outgoing particles. However, we can show, by a relativistic generalization of the method developed by Bethe,[10,11] that only the outgoing part contributes to the transition probability. The radial functions (4) for large r can be expressed as:[9]

$$R_\kappa = (2r)^{-1}[N \exp(i[pr+\delta_\kappa]) + \text{c.c.}], \tag{10}$$

[9] M. E. Rose, Phys. Rev. **51**, 484 (1937).

[10] H. A. Bethe, Ann. d. Physik **4**, 443 (1930).

[11] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, New York, 1933), chapter XIV, section 3.

where

$$V=\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad a=i[(W-1)/\pi p]^{\frac{1}{2}}, \quad b=[(W+1)/\pi p]^{\frac{1}{2}},$$

and $p=(W^2-1)^{\frac{1}{2}}$ is the magnitude of the electron momentum far from the nucleus. Substituting expression (10) into (9), we evaluate the energy integral, by a method similar to Bethe's, thereby obtaining conservation of energy and the following asymptotic, time-dependent, scattered electron wave function:

$$\psi_{sc}\underset{W=W_0-K}{\overset{r\gg 1}{=}}\begin{cases}(\pi/r)\sum_f(f|H|s)NA_f\exp(i[pr+\delta_\kappa-\pi/2]), \\ \qquad\qquad r<(p/W)t \\ 0, \qquad\qquad r>(p/W)t=r_0.\end{cases} \quad (11)$$

The phase δ_κ is conveniently separated into two terms, one depending on κ, the other produced by the coulomb field and independent of κ. $\delta_\kappa=\epsilon_\kappa+\delta_c$,

$$\epsilon_\kappa=\eta_\kappa-(\pi/2)\gamma_\kappa-\arg\Gamma[\gamma_\kappa+i(\alpha ZW)/p], \quad (12)$$

where

$$\gamma_\kappa=[\kappa^2-(\alpha Z)^2]^{\frac{1}{2}},$$

and

$$\exp(2i\eta_\kappa)=-(\kappa-i\alpha Z/p)/(\gamma_\kappa+i\alpha ZW/p).$$

Here the asymptotic phase shift δ_c produced by the coulomb field depends on the atomic number Z and the fine structure constant α.

$$\delta_c=[\alpha ZW\ln(2pr)]/p. \quad (13)$$

In evaluating the integral (9), the coulomb factor $\exp(i\delta_c)$ need not be considered as a rapidly varying function of W for two reasons. First, the effect of orbital electron screening of the nuclear coulomb field would, if taken into account, cause the asymptotic value of δ_c to approach a constant. Secondly, even if the nucleus had been treated as stripped of orbital electrons and $\exp(i\delta_c)$, therefore, not removed from the integrand of (9), we would have obtained a different value for r_0 which would take into account the influence of the coulomb field on the electron velocity.[10]

(B) Electron Probability Current

To complete the proof that the wave function (11) contains only outgoing electrons and to obtain the final expression for the probability per unit time of observing an electron moving at an angle θ relative to the anti-neutrino momentum, it is convenient to make use of the Gordon decomposition[12] of the probability current density. At distances far from the nucleus, the electron current density per incident neutrino is:

$$\mathbf{J}=\tfrac{1}{2}[\psi_{sc}^*(i\nabla\beta\psi_{sc})-(i\beta\psi_{sc})(\nabla\psi_{sc}^*)] +\mathbf{curl}(\psi_{sc}^*\mathbf{M}\psi_{sc})+(\partial/\partial t)(\psi_{sc}^*\mathbf{P}\psi_{sc}). \quad (14)$$

The vector operators $\mathbf{M}$ and $\mathbf{P}$ are defined in reference 12. Clearly, the so-called "magnetic and electric polarization" current densities, involving $\mathbf{M}$ and $\mathbf{P}$, do not contribute to the radial probability current of electrons. By inserting (11) into (14) we can obtain, as follows, the probability per unit time for the emission of an electron outward from the nucleus along a radius vector $\mathbf{r}$ inclined at an angle θ to $\theta+d\theta$ relative to the anti-neutrino momentum direction which is selected as the polar axis x_3.

$$d\gamma=\int_0^{2\pi}d\varphi(\mathbf{J}\cdot\mathbf{r})r\sin\theta d\theta =\int_0^{2\pi}d\varphi\psi_{sc}^*(-p\beta)\psi_{sc}r^2\sin\theta d\theta. \quad (15)$$

An alternative expression for the radial electron current can be obtained by using the following relation,[13] valid for free electrons having energy $W\gtrsim 1$: $\psi^*(\beta W)\psi=-(\psi^*\psi)$. Thus, it follows that (15) has the alternative form:

$$d\gamma=(p/W)\int_0^{2\pi}d\varphi(\psi_{sc}^*\psi_{sc})r^2\sin\theta d\theta. \quad (16)$$

This latter expression clearly indicates that the radial current of electrons is positive definite.

On insertion of (11) into (15) or (16), we obtain alternative explicit expressions for the angular dependent probability current of electrons.

$$d\gamma=2\pi^2\sum_{ff'}(f'|H|s)(s|H|f) \times\exp(i[\epsilon_{\kappa'}-\epsilon_\kappa])(f|f')\sin\theta d\theta. \quad (17)$$

Here the angular dependent factors are defined alternatively as

$$2\pi(f|f')=\int_0^{2\pi}d\varphi(A_f^*[1-\beta/W]A_{f'}) =\int_0^{2\pi}d\varphi(A_f^*[1-\beta W]A_{f'}). \quad (18)$$

Thus, it follows that the angular dependence of the decay probability arises from the spinor products $(A_f^*A_{f'})$ alone, because $(A_f^*[\beta/W]A_{f'})$ must vanish to satisfy the requirement that

$$(A_f^*[\beta/W]A_{f'})=(A_f^*[\beta W]A_{f'})$$

for $W\gtrsim 1$. The integral over the azimuthal angle of the electron, of course, vanishes except for those sets of quantum numbers f, f' in which $m=m'$. Therefore, (18) simply reduces to:

$$(f|f')\equiv(\kappa,m|\kappa',m)=(A_{\kappa m}^*A_{\kappa' m}), \quad (19)$$

and the summation in (17) is restricted to κ, κ', and m.

It must be kept in mind that $d\gamma$ gives the probability current for an electron produced when a neutrino of a

[12] E. L. Hill and R. Landshoff, Revs. Modern Phys. **10**, 116 (1938).

[13] H. A. Bethe, Z. Naturforsch. 3a, 470 (1948). This relation follows simply from the Dirac equation with $r\gg 1$. Our choice of sign for the operators α and β is opposite from Bethe's.

particular momentum and spin is absorbed. The matrix elements $(f|H|s)$, given by (8), depend explicitly on the neutrino momentum and spin. To find the angular and energy dependent rate of β-decay, we sum $d\gamma$, given by (17), over all possible states of the incident neutrino. This introduces the usual statistical factor $q^2dq/(2\pi)^3$ for a neutrino of a definite spin and momentum in the direction of the solid angle $d\Omega$. The summation over neutrino spin is independent of the angle between electron and neutrino. However, in integrating over neutrino directions $d\Omega$, we hold the electron-neutrino angle fixed and instead integrate over all directional orientations of the transforming nucleus. In this manner, we obtain the probability per unit time of observing an electron in the energy range W to $W+dW$, having an electron anti-neutrino angle between θ and $\theta+d\theta$.

$$P(\theta, W, W_0, Z)d\theta dW = q^2\sum_{ff'}[f|f'](f|f')\sin\theta d\theta dW, \quad (20)$$

where $q=W_0-W$. The bracketed factors are a simplified notation for the angular independent part of the β-spectrum.

$$[f|f'] \equiv [\kappa, m|\kappa', m] = \sum_s \langle (f'|H|s)(s|H|f)\rangle_{Av} \exp(i[\epsilon_{\kappa'}-\epsilon_\kappa]). \quad (21)$$

The brackets $\langle\ \rangle_{Av}$ indicate the nuclear orientation average, and s stands for the two spin values of the neutrino of momentum components $q_1=q_2=0$, $q_3=q$.

Terms in Eq. (20) which make important contributions to the angular dependent decay probability include the interference between outgoing electrons for which $\kappa\neq\kappa'$. If Eq. (20) is integrated over all angles of β-emission, only the $f=f'$ terms do not vanish, and one obtains the usual angular independent β-energy spectrum.

III. ANGULAR DEPENDENT CORRECTION FACTORS

In order to calculate the angular correlation functions for allowed and first forbidden transitions, we shall make use of the expression for the angular dependent decay probability in a slightly modified form. The expression (20) gives the angular correlation and energy spectrum as well. For allowed transitions, the energy spectrum is the same for all five types of coupling. To present the angular correlations in convenient form, we introduce the factors C_{nX} from which the factor giving rise to the allowed energy spectrum has been removed. The subscript n is zero for allowed transitions and one for first forbidden transitions, and the subscript X equals S, V, A, T, or P designating the interaction (1) considered. Rewriting (20) in the nth-forbidden approximation as

$$P_{nX}d\theta dW = C_{nX}(\tfrac{1}{2}\sin\theta d\theta)[(G^2/2\pi^3)F_0pWq^2dW], \quad (22)$$

we define the correction factors as

$$C_{nX} = [(G^2/4\pi^3)F_0pW]^{-1}\sum_{ff'}[f|f']_{nX}(f|f'), \quad (23)$$

where F_0 is the usual Fermi function.

Now to calculate the matrix element $(f|H|s)$ given by (8), we expand the electron wave function appearing therein[14] in the following manner:

$$\psi_f = \psi_{f0} + r^{-1}\sum_{i=1}^{3} x_i\chi_{fj} + \cdots. \quad (24)$$

Here the first-order spherical harmonics have been expressed in the cartesian coordinates x_1, x_2, x_3, and the terms x_j/r factored out. ψ_{f0} is that part of ψ_f containing only zero-order spherical harmonics. For values of $\kappa=\pm1$, the electron wave function contains only zero- and first-order spherical harmonics; for $\kappa=\pm2$, it contains first- and second-order spherical harmonics. However, second-order spherical harmonics contribute only to second and higher orders of forbidden transitions. The selection of the axis of quantization (x_3) to correspond to the momentum direction of the anti-neutrino enables us to expand the factor $\exp(-i\mathbf{q}\cdot\mathbf{x})$, also appearing in the matrix element (8), as $1-iqx_3+\cdots$.

(A) Allowed Transitions

The matrix element for allowed transitions is, using the conventional approximation,

$$(f|H|s)_{0A} = G\left(\int \mathbf{O}_H\right)\cdot(\psi_{f0}{}^*\mathbf{O}_L B_s)_\rho, \quad (25)$$

in which the light particle wave functions are removed from the integral and evaluated at the nuclear radius ρ. For simplicity, the integral over nucleon coordinates is abbreviated $\int\mathbf{O}_H$ as is usual.

To indicate the method employed in determining the allowed correlation factors C_{0X}, we shall consider a representative example, axial vector coupling. In this case $\mathbf{O}_H\cdot\mathbf{O}_L$ may be set equal to $\boldsymbol{\sigma}_H\cdot\boldsymbol{\sigma}_L$, since $\int\gamma_5=0$ for allowed transitions with this coupling. The factors $[f|f']_{0A}$ may then be calculated as follows: Make use of the Casimir trick for evaluating the neutrino spin sum and obtain

$$\sum_s(f'|H|s)(s|H|f) = G^2\sum_{j,k}\left(\int\sigma_j\right)^*\left(\int\sigma_k\right)[\tfrac{1}{2}\psi_{f0}{}^*\sigma_j(1-\alpha_3)\sigma_k\psi_{f'0}]. \quad (26)$$

Average the product of nuclear matrix elements over all possible orientations of the nucleus,[15] multiply by the phase factor, and obtain

$$[f|f'] = \tfrac{1}{3}G^2\left|\int\boldsymbol{\sigma}\right|^2 \times[\tfrac{1}{2}\psi_{f0}{}^*(3-\alpha_3)\psi_{f'0}]\exp(i[\epsilon_{f'}-\epsilon_f]). \quad (27)$$

The quantum numbers designated by f and f' can have the values $\kappa=\pm1$ and $m=\pm\frac{1}{2}$ for allowed transi-

[14] Clearly, the subscript H can now be dropped, nucleon coordinates being understood in Eq. (8).

[15] The products of nuclear matrix elements are averaged over nuclear orientations by the method discussed on p. 337 of Max Born, *Optik* (Verlag. Julius Springer, Berlin, 1933).

tions. Thus, since $m=m'$, the sum over f and f' involves eight terms. In four of these terms, $\kappa=\kappa'$; hence, $f=f'$. The factors $(f|f')$ in these four cases are

$$(f|f)=(2\pi)^{-1} \text{ for } (f|=(\pm 1, \pm\tfrac{1}{2}|. \tag{28}$$

In the four remaining terms, $\kappa=-\kappa'$, and the factors $(f|f')$ are

$$\begin{aligned}(\pm 1, \tfrac{1}{2}|\mp 1, \tfrac{1}{2})&=(2\pi)^{-1}\cos\theta,\\ (\pm 1, -\tfrac{1}{2}|\mp 1, -\tfrac{1}{2})&=-(2\pi)^{-1}\cos\theta.\end{aligned} \tag{29}$$

The exponential factors in Eq. (27) reduce to $\exp(i[\eta_{\kappa'}-\eta_\kappa])$, since for $|\kappa|=|\kappa'|$, $\gamma_\kappa=\gamma_{\kappa'}$. This can be seen by reference to Eq. (12).

Now summing over f and f', we obtain, on inserting (27), (28), and (29) into Eq. (23),

$$C_{0A}=\tfrac{1}{3}(F_0pW)^{-1}\left|\int\boldsymbol{\sigma}\right|^2\left[\frac{3\pi}{2}(f_1^2+g_{-1}^2)\right.$$
$$\left.-\frac{\pi}{2}(if_1g_{-1}\exp(i[\eta_1-\eta_{-1}])+\text{c.c.})\cos\theta\right]. \tag{30}$$

The combinations of radial functions and phase factors, enclosed in parentheses in this expression, are common to all five types of coupling. The choice of a particular type of coupling affects only the sign of the term containing $\cos\theta$ and a numerical factor.

The correlation functions C_{0X} are given below for the five types of coupling.

$$\left.\begin{aligned}C_{0S}&=\left|\int\beta\right|^2\tfrac{1}{2}(1+\gamma_1)[1-(p/W)\cos\theta]\\ C_{0V}&=\left|\int 1\right|^2\tfrac{1}{2}(1+\gamma_1)[1+(p/W)\cos\theta]\\ C_{0A}&=\left|\int\boldsymbol{\sigma}\right|^2\tfrac{1}{2}(1+\gamma_1)[1-\tfrac{1}{3}(p/W)\cos\theta]\\ C_{0T}&=\left|\int\beta\boldsymbol{\sigma}\right|^2\tfrac{1}{2}(1+\gamma_1)[1+\tfrac{1}{3}(p/W)\cos\theta]\\ C_{0P}&=\left|\int\beta\gamma_5\right|^2\tfrac{1}{2}(1+\gamma_1)[1-(p/W)\cos\theta].\end{aligned}\right\} \tag{31}$$

The common factor $\frac{1}{2}(1+\gamma_1)$ is independent of W and for the light elements is nearly unity. The angular correlations are clearly different in the first four coupling cases. The scalar and pseudoscalar couplings give the same angular correlation, but, on the other hand, their parity selection rules are different and their nuclear matrix elements are of different orders of magnitude. Hence, it is possible to distinguish between the five types of coupling. In all cases shown above, the angular dependent term is proportional to $p/W=v/c$ which indicates that the angular correlation is essentially a relativistic effect.

A comparison of the angular correlation factors (enclosed in brackets) with those calculated by Hamilton[4] for $Z=0$ shows that the nuclear coulomb field produces no effect. The correlation factors given above are the same as Hamilton's.

The allowed correlation functions C_{0X} do not agree exactly with those calculated by Rose.[5] This is because in his calculation the phase factor, $\exp(i[\eta_{\kappa'}-\eta_\kappa])$, was treated approximately.

(B) First Forbidden Transitions

The method of computing the factors C_{1X} for first forbidden transitions parallels the calculation of C_{0X}, although it is much more tedious because the matrix elements $(f|H|s)$ contain many more terms. Consider the matrix element for first forbidden transitions, using axial vector coupling again as an example,

$$(f|H|s)_{1A}=G\left[\sum_{jk}\left(\int\sigma_jx_k\right)P_k^*\sigma_jB_s\right.$$
$$\left.-\left(\int\gamma_5\right)\psi_{f0}\gamma_5B_s\right], \tag{32}$$

where

$$P_k=\rho^{-1}\chi_{fk}+iq_k\psi_{f0}.$$

Notice here that the anti-neutrino moves along the x_3 axis; thus, $q_k=q\delta_{3k}$.

We express the nuclear matrix elements in Eq. (32) in terms of irreducible tensor components[16] as follows:

$$\int\sigma_jx_k=\tfrac{1}{2}Q_{jk}+\tfrac{1}{2}A_{jk}+\tfrac{1}{3}S_{jk}. \tag{33}$$

Under space rotations S, A, and Q transform, respectively, like the spherical harmonics of order 0, 1, and 2, and, consequently, give rise to different selection rules. Explicitly, the tensor components are[17]

$$\begin{aligned}Q_{jk}&=\int[\sigma_jx_k+\sigma_kx_j-\tfrac{2}{3}(\boldsymbol{\sigma}\cdot\mathbf{r})\delta_{jk}],\\ A_{jk}&=\int[\sigma_jx_k-\sigma_kx_j],\\ S_{jk}&=\int(\boldsymbol{\sigma}\cdot\mathbf{r})\delta_{jk}.\end{aligned} \tag{34}$$

Similar tensors, $\hat{A}$, etc., can be formed from the combinations $P_k^*\sigma_j$. We express the first term in (32) as a contraction of the tensors A, Q, and S, formed from the nuclear matrix elements, with the tensors $\hat{A}$, etc.,

[16] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).

[17] E. Greuling, Phys. Rev. **61**, 568 (1942).

formed from the electron-neutrino factors. These tensors cannot mix upon contraction, because Q is symmetric with trace zero, A is antisymmetric, and S is a scalar times a unit tensor. The second term in (32) does not contain the coordinates x_n in the nuclear matrix element and is, thus, no more complicated than the allowed matrix elements.

When the product $(f'|H|s)(s|H|f)$ is formed, cross products of the type occurring between the first and second terms of (32) will not be considered. In the case of axial vector coupling, they would have the form $(\int \boldsymbol{\sigma}\cdot\mathbf{r})^*(\int \gamma_5)f(W,\theta)$+complex conjugate. This particular cross term may be expected to vanish, because γ_5 and $\boldsymbol{\sigma}$ do not transform in the same way under a time reversal.[18]

The procedure used to obtain the factors $[f|f']$ is involved because of the large number of terms, but the steps are straightforward. As in allowed transitions, we average over all orientations of the nucleus[19] and make use of the Casimir trick in performing the neutrino spin sum.

In order to show the types of terms which appear, we shall sketch the calculation of one term in $[f|f']$. Choosing as a sample term of $(f'|H|s)(s|H|f)$, the product $A\,\hat{A}$, the nuclear orientation average yields

$$\sum_{jk}\sum_{j'k'}\langle A_{jk}^* A_{j'k'}\rangle_{\mathrm{Av}}\hat{A}_{jk}^* B_s B_s^* \hat{A}_{j'k'} = \tfrac{2}{3}\left|\int \boldsymbol{\sigma}\times\mathbf{r}\right|^2 \sum_{jk}\hat{A}_{jk}^* B_s B_s^* \hat{A}_{jk}. \quad (35)$$

Here we have written the equivalent form $2|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ for $\sum|A_{jk}|^2$. Performing the spin sum, we obtain for this term of $[f|f']$, neglecting a numerical coefficient and the phase factor,

$$\left|\int\boldsymbol{\sigma}\times\mathbf{r}\right|^2\Big[2\mathbf{P}^{*\prime}\,\mathbf{P}+\sum_{i\neq j}P_i^{*\prime}\sigma_i\sigma_j P_j+(\mathbf{P}^{*\prime}\cdot\boldsymbol{\alpha})P_3 + P_3^{*\prime}(\boldsymbol{\alpha}\cdot\mathbf{P})-P_1^{*\prime}\sigma_1\alpha_3\sigma_2 P_2-P_2^{*\prime}\sigma_2\alpha_3\sigma_1 P_1\Big], \quad (36)$$

where the prime indicates that the electron wave function has the quantum numbers f', otherwise f.

For first forbidden transitions, f (and f') can have the following values: $\kappa=\pm2$, $m=\pm\frac{1}{2}$, $\pm\frac{3}{2}$ (for $j=\frac{3}{2}$), and $\kappa=\pm1$, $m=\pm\frac{1}{2}$ (for $j=\frac{1}{2}$). There are 40 possibilities for the combination f, f' with $m=m'$. It is convenient to divide these possibilities into eight groups, making use of the fact that the angular dependent factors $(f|f')$ consist of eight different types. The groups are chosen so that all the members of a particular group have the same angular dependence. In three of the groups $f=f'$; thus, the exponential phase factors of $[f|f']$ are unity. In the remaining groups, three have $\kappa=-\kappa'$ so that the exponential has the form $\exp(i[\eta_{\kappa'}-\eta_\kappa])$; and two groups have $|\kappa|\neq|\kappa'|$ so that the complete factor $\exp(i[\epsilon_{\kappa'}-\epsilon_\kappa])$ is required.

To evaluate C_{1X} from Eq. (23), the summation over f and f' is done by summing first over the terms in each group and then summing over the eight groups themselves. The results of this involved process are as follows:

$$C_{1S}=\tfrac{1}{3}\left|\int\mathbf{r}\right|^2 B_S{}^{(1)},$$

$$C_{1V}=\tfrac{1}{3}\left|\int\mathbf{r}\right|^2 A_V{}^{(1)}+\left|\int\boldsymbol{\alpha}\right|^2\tfrac{1}{2}(1+\gamma_1)\times[1-\tfrac{1}{3}(p/W)\cos\theta],$$

$$C_{1A}=(1/60)\sum_{i,j}|Q_{ij}|^2 A_A{}^{(2)}+\tfrac{1}{6}\left|\int\boldsymbol{\sigma}\times\mathbf{r}\right|^2 A_A{}^{(1)}+\tfrac{1}{9}\left|\int\boldsymbol{\sigma}\cdot\mathbf{r}\right|^2 A_A{}^{(0)}+\left|\int\gamma_5\right|^2\tfrac{1}{2}(1+\gamma_1)[1+(p/W)\cos\theta], \quad (37)$$

$$C_{1T}=(1/60)\sum_{i,j}|Q_{ij}|^2 B_T{}^{(2)}+\tfrac{1}{6}\left|\int\boldsymbol{\sigma}\times\mathbf{r}\right|^2 B_T{}^{(1)}+\tfrac{1}{9}\left|\int\boldsymbol{\sigma}\cdot\mathbf{r}\right|^2 B_T{}^{(0)}+\left|\int\boldsymbol{\alpha}\right|^2\tfrac{1}{2}(1+\gamma_1)[1+\tfrac{1}{3}(p/W)\cos\theta].$$

Pseudoscalar coupling gives the same angular correlation as scalar coupling. The coefficient C_{1P} can be obtained by replacing $|\int\mathbf{r}|^2$ in C_{1S} by $|\int\gamma_5\mathbf{r}|^2$. The correlation factors[20] A_X and B_X, which are given in the approximation $\alpha Z\ll1$, are adequate for elements as heavy as RaE, for example, for which the error intro-

[18] C. L. Critchfield, Phys. Rev. **63**, 423 (1943), has pointed out that no such interference terms should appear because of the time reversal parity property of nuclear states. Matrix elements of γ_5 and $(\boldsymbol{\sigma}\cdot\mathbf{r})$ are nonvanishing if the isobaric states have opposite space parity and have, respectively, the same and opposite "time parity."

[19] The nuclear orientation average for the products of irreducible tensors can be obtained from Born's *Optik* (reference 15, p. 348). Using Born's method, we obtain

$$\langle A_{ij}^* A_{i'j'}\rangle_{\mathrm{Av}}=\tfrac{1}{3}\delta_{ii'}\delta_{jj'}\,\Sigma_{m,n}|A_{mn}|^2,$$
$$\langle Q_{ij}^* Q_{i'j'}\rangle_{\mathrm{Av}}=\tfrac{1}{5}\delta_{ii'}\delta_{jj'}\,\Sigma_{m,n}|Q_{mn}|^2.$$

The fractional factor in these equations is just the reciprocal of the number of independent components in the tensor.

[20] The operator β does not appreciably affect the magnitude of the nuclear matrix elements and has been omitted from them; however, this operator is important in the electron-neutrino portion of $(f|H|s)$ because of its commutation properties. The notation $A_X(n)$ and $B_X(n)$ is chosen so that B indicates that the coupling contains β and A indicates that it does not; $(n)=(0)$, (1), (2) indicates that the corresponding quantity multiplies the square of a 0th, 1st, or 2nd rank tensor nuclear matrix element, respectively.

duced by this approximation amounts to 15 percent or so.

$$\left.\begin{matrix}A_V^{(1)}\\ B_S^{(1)}\end{matrix}\right\}=\tfrac{3}{4}V^2[1\mp\tfrac{1}{3}(p/W)\cos\theta]$$
$$+V[(p/W)(p+q\cos\theta)\pm(q+p\cos\theta)]$$
$$+[1\pm(p/W)\cos\theta]D,$$
$$\left.\begin{matrix}A_A^{(2)}\\ B_T^{(2)}\end{matrix}\right\}=[5\mp(p/W)\cos\theta]D\pm pq(p/W)\sin^2\theta,$$
$$\left.\begin{matrix}A_A^{(1)}\\ B_T^{(1)}\end{matrix}\right\}=\tfrac{3}{2}V^2[1\mp\tfrac{1}{3}(p/W)\cos\theta] \qquad (38)$$
$$+2V[(p/W)(p+q\cos\theta)\mp(q+p\cos\theta)]$$
$$+[1\mp(p/W)\cos\theta]D\mp pq(p/W)\sin^2\theta,$$
$$\left.\begin{matrix}A_A^{(0)}\\ B_T^{(0)}\end{matrix}\right\}=(9/4)V^2[1\pm(p/W)\cos\theta]$$
$$+3V[(p/W)(p+q\cos\theta)\pm(q+p\cos\theta)]$$
$$+[1\pm(p/W)\cos\theta]D\pm 2pq(p/W)\sin^2\theta.$$

Here $D=p^2+q^2+2pq\cos\theta$, and the factor $V=\alpha Z/\rho$ is the magnitude of the potential energy of an electron at the surface of the nucleus.

For elements at light as Be^8, $V=1.8$ Mev, and for A^{36}, $V=10$ Mev. Thus, for all but the lightest elements the angular correlation term proportional to V^2 will be the dominant one in the correlation factors A_X and B_X. The terms proportional to V^2 resemble the allowed correlation functions. The terms proportional to V are of two types, depending on the sign in front of $(q+p\cos\theta)$. For the plus sign, emission of the anti-neutrino and the electron in the same hemisphere is favored; this is generally referred to as a forward correlation. For the minus sign, a comparatively much weaker correlation results which favors emission of the anti-neutrino and electron in opposite hemispheres; a so-called backward correlation. The terms independent of V represent the $Z=0$ limit. They are identical with the first forbidden correlation functions calculated by Hamilton.[4]

Of the correlation functions shown in Eqs. (38), only the second rank tensor factors $A^{(2)}$ and $B^{(2)}$ show no coulomb effect. Both $A^{(2)}$ and $B^{(2)}$ give a forward correlation. The factor $B^{(2)}$ favors a forward anti-neutrino somewhat more strongly than $A^{(2)}$, but it would be difficult to distinguish between them experimentally.

In Eqs. (38) the correlation factors $A^{(1)}$, $B^{(1)}$, $A^{(0)}$, and $B^{(0)}$ show a strong Z effect. The V^2 term is larger than the other terms in these correlation factors for elements with atomic number $Z>Z_c\sim 1.6W_0^{\frac{3}{2}}$, where W_0 is the maximum electron energy.[3] If $Z\gg Z_c$, then only the V^2 term is of consequence.

IV. CONCLUSIONS

Some general conclusions can be drawn concerning the angular correlation functions, if we consider together the allowed transitions and the V^2 terms in the first forbidden transitions. First, scalar nuclear matrix elements are multiplied by correlation factors proportional to $1\pm(p/W)\cos\theta$. Second, vector nuclear matrix elements multiply factors of the form $1\pm\frac{1}{3}(p/W)\cos\theta$. Furthermore, for $Z\gg Z_c$ the electron energy independent V^2 terms are dominant, and, therefore, some first forbidden β-emitters may have allowed-looking energy spectra. If $Z\gg Z_c$, then, clearly, from C_{0X} and C_{1X} it is possible for any one of the five types of coupling to give allowed-looking spectra and either a forward or a backward angular correlation. To interpret angular correlation data in this case ($Z\gg Z_c$), either the spin change must be known or the half-life must be so short that the transition can unambiguously be classified as allowed, as, for example, is the case with He^6.

Experimental evidence (i.e., unique first forbidden spectra and allowed transitions with spin change one) favors tensor or axial vector coupling. These two couplings give the Gamow-Teller selection rules. To distinguish between them, angular correlation data for a spin change ($0\leftrightarrow 1$) or ($0\leftrightarrow 0$) would be particularly valuable. These transitions should be either allowed or first forbidden, depending on the parity change. In the case of ($0\leftrightarrow 1$) transitions, axial vector coupling gives a backward angular correlation in both C_{0A} and C_{1A}, and tensor coupling gives a forward correlation in C_{0T} and C_{1T}. In the case of ($0\leftrightarrow 0$) transitions, axial vector coupling gives a forward correlation in C_{1A}, and tensor coupling gives a backward correlation in C_{1T}. Thus, it should be possible to distinguish between tensor and axial vector coupling by the general forwardness or backwardness of the correlation, provided the spin change is definitely ($0\leftrightarrow 1$) or definitely ($0\leftrightarrow 0$).

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