(B) Excitation Curves

The thick target x-ray excitation curve is obtained by plotting the normalized activity as a function of the electron beam energy. Since the activity has a short half-life, the varying target current was weighted exponentially from the end of the irradiation period. Thus, all activities were normalized to infinite irradiation with constant current. For each isotope, several irradiations were made at each voltage, and the activities were averaged to reduce the statistical error.

Figure 2 shows the lower energy portions of the excitation curves for Ag^{107} and Ag^{109}, respectively. The thresholds in both cases are below 800 kev, although the data do not permit accurate location. The breaks in the curves indicate an activation level in Ag^{107} at 1.285 ± 0.018 Mev and an activation level in Ag^{109} at 1.210 ± 0.018 Mev. Figure 3 shows both excitation curves over the entire energy range investigated and emphasizes the 75-kev difference between the corresponding levels in the two nuclei. In considering the close similarity between the two nuclei in other respects, it is to be expected that such a small difference between the energy levels exists. The approximate over-all cross section (i.e., per electron incident on a thick gold target) at 1.4 Mev is of the order of 10^{-34} cm^2 for both isotopes. In Figs. 2 and 3, to convert *activity* to *over-all cross section*, multiply the ordinates by 4×10^{-36} cm^2.

These excitation curves agree with the early work of Feldmeier[2] in this laboratory but are contrary to that of Wiedenbeck.[4] This experiment is further confirmation that, as has been previously discussed,[3] one cannot obtain more than a few energy levels by x-ray excitation.

ACKNOWLEDGMENTS

We wish to express our appreciation to Professor A. C. Helmholz of the University of California for supplying the separated silver isotopes. Assistance in taking data and operating the electrostatic generator was generously contributed by E. J. Schillinger and T. F. Ruane.

PHYSICAL REVIEW VOLUME 82, NUMBER 4 MAY 15, 1951

Effect of Finite Nuclear Size on the Elastic Scattering of Electrons*

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(Received August 15, 1950)

The effect of the finite size of the nucleus on the elastic scattering of electrons with energies in the neighborhood of 20 Mev has been calculated for two simple spherical nuclear models: a uniform charge distribution, and a uniform shell charge distribution, both of radius $R=1.45\times 10^{-13} A^{\frac{1}{3}}$ cm. An exact phase shift analysis has been made, the phase shifts differing appreciably from those for a point nucleus only in the $j=\frac{1}{2}$ state.

Phase shifts for elements of $Z=13$, 29, 50, and 79 have been plotted as a function of energy over the interval from 15 to 35 Mev, permitting the calculation of any desired cross section within this range. Representative curves of the ratio of the scattering cross section for the finite nucleus to that for a point nucleus have been plotted.

The difference in the scattering due to the two models is large enough so that accurate experiments might distinguish between them, the actual average nuclear charge distribution probably falling somewhere between these two cases. The ability to differentiate between the two models, however, depends on the accuracy with which nuclear radii are known.

I. INTRODUCTION

WHEN a beam of high energy electrons (~20 Mev) falls on an atom, a significant part of the scattering results from electrons which have actually penetrated the nucleus, so that the scattering cross section depends on the nature of the nuclear charge distribution. Previous calculations for lower energies have taken the nucleus to be simply a point charge. It is the purpose of this paper to calculate the effect of the finite nuclear size on the scattering of these high energy electrons. Two simple nuclear models are assumed: (1) a uniform spherical charge distribution, and (2) a shell charge distribution, with the charge distributed uniformly over the surface of the nucleus.

Rose[1] has already treated this problem by means of the Born approximation, which is valid for the light elements. Here exact results are obtained by means of a phase shift analysis.

For the energies of interest here, it is necessary to use the Dirac relativistic theory of the electron. Relativistic scattering theory, for spherically symmetric scattering centers, involves the solution of a pair of simultaneous first-order differential equations, the well-known Dirac

* This paper was supported in part by the joint program of the ONR and AEC.
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[1] M. E. Rose, Phys. Rev. **73**, 279 (1948).

radial equations.[2]

$$\begin{aligned}(1/\hbar)[(E-V/c)+mc]F_\kappa&\\ +(dG_\kappa/dr)-[(\kappa-1)/r]G_\kappa&=0,\\ -(1/\hbar)[(E-V/c)-mc]G_\kappa&\\ +(dF_\kappa/dr)+[(\kappa+1)/r]F_\kappa&=0,\end{aligned}\tag{1}$$

where $\kappa=\pm(j+\frac{1}{2})$, $j=l\pm\frac{1}{2}$.

From the asymptotic form of G_κ,

$$G_\kappa\sim[\sin(kr-\tfrac{1}{2}l\pi+\eta_\kappa)]/kr,\tag{2}$$

one can determine the phase shift η_κ, depending on the potential V of the scattering center, and the relativistic energy E of the incident electrons. The scattering cross section is given in terms of the phase shifts by

$$\sigma(\theta)=|f(\theta)|^2+|g(\theta)|^2,$$

$$\begin{aligned}f(\theta)&=\frac{1}{2ik}\sum_{\kappa=1}^{\infty}\kappa[(e^{2i\eta_\kappa}-1)P_{\kappa-1}(\cos\theta)\\&\qquad+(e^{2i\eta_{-\kappa}}-1)P_\kappa(\cos\theta)],\\ g(\theta)&=\frac{1}{2ik}\sum_{\kappa=1}^{\infty}[-e^{2i\eta_\kappa}P_{\kappa-1}^1(\cos\theta)+e^{2i\eta_{-\kappa}}P_\kappa^1(\cos\theta)],\end{aligned}\tag{3}$$

where P_κ^1 and P_κ are associated Legendre polynomials.

The terms in η_κ represent the contribution of the $j=l+\frac{1}{2}$ states to the scattering, while those in $\eta_{-\kappa}$ represent the $j=l-\frac{1}{2}$ states. (η_κ and $\eta_{-\kappa}$ correspond to η_l and η_{-l-2} in Mott's notation but refer to the same j value rather than the same l value.)

II. SOLUTION OF THE DIRAC RADIAL EQUATIONS

For energies of 20 Mev and above, the rest energy of the electron plays a small role, and the treatment of these equations is greatly simplified if it can be neglected completely. This is a better approximation, even, than it appears at first glance; for if kept, the rest mass appears in the results primarily in the form $E^2-m^2c^4$.

Making this simplification, the radial equations become,

$$\begin{aligned}(k-U)F_\kappa+G_\kappa'-[(\kappa-1)/r]G_\kappa&=0,\\ -(k-U)G_\kappa+F_\kappa'+[(\kappa+1)/r]F_\kappa&=0,\end{aligned}\tag{4}$$

where $k=E/\hbar c=p/\hbar$, $U=V/\hbar c$.

If one makes the substitution

$$F=ir^{-\frac{1}{2}}(A\psi-B\phi),\quad G=r^{-\frac{1}{2}}(A\psi+B\phi),\tag{5}$$

where A and B are constants, and then eliminates $B\phi$ from the pair of equations, one obtains the following second-order differential equation for ψ.

$$\psi''+\left[(k-U)^2+\frac{i(k-U-rU')}{r}-\frac{\kappa^2-\frac{1}{4}}{r^2}\right]\psi=0.\tag{6}$$

If ψ is eliminated rather than ϕ, the resulting equation is the complex conjugate of the above equation; thus, the problem is simplified by the fact that one can take $\phi=\psi^*$, and only one second-order equation need be solved.

The constants A and B enter the picture when the solution of the second-order equations is also required to satisfy the pair of first-order equations.

The equation for ψ is to be solved for a potential which, inside the nucleus, depends on the specific nuclear model chosen but which, outside the nucleus, equals the coulomb potential that would exist if all the charge were located at a point.

Separate solutions for these two regions are obtained with the requirement that their values and slopes coincide at the nuclear radius. This, together with the boundary condition that $\psi(0)=0$, (in order that G be finite at the origin) determines ψ uniquely, except for a constant multiplying factor which can be absorbed into A.

Thus, for $r>R$, region II, the potential is $U=-\alpha/r$, where $\alpha=Ze^2/\hbar c$,

$$\psi_{\mathrm{II}}''+[(k+\alpha/r)^2+ik/r-(\kappa^2-\tfrac{1}{4})/r^2]\psi_{\mathrm{II}}=0.\tag{7}$$

This equation reduces very simply to Whittaker's confluent hypergeometric equation, and the general solution[3] can be written

$$\psi_{\mathrm{II}}=A_1\psi_{c1}+A_2\psi_{c2},$$

$$\psi_{c1}=r^{\rho+\frac{1}{2}}e^{-ikr}\,{}_1F_1(\rho+i\alpha;\,2\rho+1;\,2ikr),$$

where $\rho=(\kappa^2-\alpha^2)^{\frac{1}{2}}$,

$$\psi_{c2}=r^{-\rho+\frac{1}{2}}e^{-ikr}\,{}_1F_1(-\rho+i\alpha;\,-2\rho+1;\,2ikr).\tag{8}$$

Matching the inside and outside wave functions and derivatives at the nuclear radius,

$$\psi_{\mathrm{I}}(R)=A_1\psi_{c1}(R)+A_2\psi_{c2}(R),$$
$$\psi_{\mathrm{I}}'(R)=A_1\psi_{c1}'(R)+A_2\psi_{c2}'(R).$$

Solving for the constants A_1 and A_2,

$$A_1=(\psi_{\mathrm{I}}\psi_{c2}'-\psi_{c2}\psi_{\mathrm{I}}'/\psi_{c1}\psi_{c2}'-\psi_{c2}\psi_{c1}'),$$

and

$$A_2=(\psi_{c1}\psi_{\mathrm{I}}'-\psi_{\mathrm{I}}\psi_{c1}'/\psi_{c1}\psi_{c2}'-\psi_{c2}\psi_{c1}'.\tag{9}$$

Since it is the asymptotic form of G which is of interest, it is the asymptotic form of ψ which will be considered here. Using the well-known asymptotic expressions for the confluent hypergeometric functions, one has

$$\begin{aligned}\psi_{c1}&\sim\frac{\Gamma(2\rho+1)}{\Gamma(\rho+1-i\alpha)}(-2ik)^{-\rho}r^{\frac{1}{2}}e^{-ikr-i\alpha\ln 2kr}e^{-\frac{1}{2}\pi\alpha},\\ \psi_{c2}&\sim\frac{\Gamma(-2\rho+1)}{\Gamma(-\rho+1-i\alpha)}(-2ik)^{\rho}r^{\frac{1}{2}}e^{-ikr-i\alpha\ln 2kr}e^{-\frac{1}{2}\pi\alpha}.\end{aligned}\tag{10}$$

[2] Mott and Massey, *Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), 2nd edition, p. 74.

[3] Whittaker and Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1927), p. 337.

The ratio of ψ_{c1} to ψ_{c2} is asymptotically a constant, which will be called κ_a.

$$\kappa_a=(-2ik)^{2\rho}\frac{\Gamma(-2\rho+1)}{\Gamma(2\rho+1)}\frac{\Gamma(\rho+1-i\alpha)}{\Gamma(-\rho+1-i\alpha)}. \tag{11}$$

Asymptotically, ψ can then be written,

$$\psi\sim A_1\psi_{c1}(1+\kappa_a A_2/A_1)\sim\psi_{c1}(1+\xi), \tag{12}$$

where the arbitrary constant in ψ_{I} has been chosen so that $A_1=1$, and

$$\xi=\kappa_a A_2/A_1$$
$$=-\kappa_a\frac{\psi_{c1}\,\psi_{\mathrm{I}}'/\psi_{\mathrm{I}}-\psi_{c1}'/\psi_{c1}}{\psi_{c2}\,\psi_{\mathrm{I}}'/\psi_{\mathrm{I}}-\psi_{c2}'/\psi_{c2}}\quad\text{evaluated at } r=R. \tag{13}$$

It is seen that only the logarithmic derivative of ψ_{I} appears.

The question now is how ξ is related to the phase shift η_κ, which is to be determined from the asymptotic form of G_κ. It becomes necessary to find the ratio A/B, which is obtained from one of the first-order equations.

Consider first the case of a point nucleus, the coulomb potential. The first radial equation gives

$$\frac{A_c}{B_c}=e^{2ikr}\frac{(-\rho+\kappa+i\alpha)F_\rho{}^*-rF_\rho{}^{*\prime}}{(\rho-\kappa+i\alpha)F_\rho+rF_\rho{}'}, \tag{14}$$

where $F_\rho={}_1F_1(\rho+i\alpha;\,2\rho+1;\,2ikr)$.

Since this must be a constant, it can be evaluated at the origin.

$$A_c/B_c=-(\rho-\kappa-i\alpha)/(\rho-\kappa+i\alpha). \tag{15}$$

The asymptotic form of $G_\kappa{}^{(c)}$ can then be written

$$G_\kappa{}^{(c)}\sim 1/kr\,\sin(kr+\alpha\ln 2kr-\tfrac{1}{2}l\pi+\chi_\kappa), \tag{16}$$

where

$$e^{2i\chi_\kappa}=\frac{\rho-\kappa+i\alpha}{\rho-\kappa-i\alpha}\frac{\Gamma(\rho+1-i\alpha)}{\Gamma(\rho+1+i\alpha)}e^{-\pi i(\rho-l)}. \tag{17}$$

The appearance of the log term in the asymptotic form arises from the fact that the potential falls off as $1/r$.

It remains to be seen how this expression is altered by the assumption of a finite nuclear size.

$$G_\kappa\sim r^{-1}(A\psi+B\psi^*)$$
$$\sim r^{-1}[A(1+\xi)\psi_{c1}+B(1+\xi^*)\psi_{c1}{}^*].$$

This can be written as

$$G_\kappa\sim 1/kr\,\sin(kr+\alpha\ln 2kr-\tfrac{1}{2}l\pi+\chi_\kappa+\delta_\kappa), \tag{18}$$

where

$$e^{2i\delta_\kappa}=\frac{A_c}{B_c}\frac{B}{A}\frac{1+\xi^*}{1+\xi}. \tag{19}$$

The phase shift for the case of a finite nuclear size is given by

$$\eta_\kappa=\chi_\kappa+\delta_\kappa. \tag{20}$$

The ratio A/B is still to be determined, and going back to the first radial equation, one obtains

$$A/B=-e^{2ikr}$$
$$\times\frac{[(rF_\rho{}'+A_\rho F_\rho)+(\xi/\kappa_a)r^{-2\rho}(rF_{-\rho}{}'+A_{-\rho}F_{-\rho})]^*}{(rF_\rho{}'+A_\rho F_\rho)+(\xi/\kappa_a)r^{-2\rho}(rF_{-\rho}{}'+A_{-\rho}F_{-\rho})}, \tag{21}$$

where $A_\rho=\rho-\kappa+i\alpha$, which can be written

$$\frac{A}{B}=\frac{A_c}{B_c}\cdot\frac{1+(A_2/A_1)^*r^{-2\rho}K^*}{1+(A_2/A_1)r^{-2\rho}K}, \tag{22}$$

where

$$K=\frac{rF_{-\rho}{}'+A_{-\rho}F_{-\rho}}{rF_\rho{}'+A_\rho F_\rho}.$$

Here it is most convenient to use the asymptotic expressions to evaluate A/B; however, one must take care to use the complete asymptotic form of the confluent hypergeometric functions when taking derivatives.

$${}_1F_1(a;c;z)\sim\frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}+\frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c}.$$

It can be shown that the expression for A/B reduces to

$$\frac{A}{B}=\frac{A_c}{B_c}\cdot\frac{1+\frac{1}{2}(\kappa_a+\kappa_b)^*(A_2/A_1)^*}{1+\frac{1}{2}(\kappa_a+\kappa_b)(A_2/A_1)}, \tag{23}$$

where

$$\kappa_b=(2ik)^{2\rho}\frac{\Gamma(1-2\rho)}{\Gamma(1+2\rho)}\frac{\Gamma(\rho+i\alpha)}{\Gamma(-\rho+i\alpha)}, \tag{24}$$

and the phase shift δ_κ is then given by

$$e^{2i\delta_\kappa}=\frac{1+\frac{1}{2}(\kappa_a+\kappa_b)(A_2/A_1)}{1+\frac{1}{2}(\kappa_a+\kappa_b)^*(A_2/A_1)^*}\cdot\frac{1+\kappa_a{}^*(A_2/A_1)^*}{1+\kappa_a(A_2/A_1)}. \tag{25}$$

Using the relation $\Gamma(1-z)\Gamma(z)=\pi/\sin\pi z$, κ_a and κ_b can be written,

$$\kappa_a=\frac{\sin\pi(\rho+i\alpha)}{\sin 2\pi\rho}\frac{\rho-i\alpha}{2\rho}\left|\frac{\Gamma(\rho+i\alpha)}{\Gamma(2\rho)}\right|^2 e^{-\pi i\rho}(2k)^{2\rho},$$
$$\kappa_b=-\frac{\sin\pi(\rho-i\alpha)}{\sin 2\pi\rho}\frac{\rho-i\alpha}{2\rho}\left|\frac{\Gamma(\rho+i\alpha)}{\Gamma(2\rho)}\right|^2 e^{\pi i\rho}(2k)^{2\rho}. \tag{26}$$

Thus, the phase shift δ_κ is expressed in terms of A_2/A_1, which in turn depends on the logarithmic derivative of the inside wave function evaluated at the nuclear radius.

The next step is to see how this phase shift enters into the expression for the scattering cross section.

III. SCATTERING CROSS SECTION

Letting $\eta_\kappa=\chi_\kappa+\delta_\kappa$, the expressions for f and g can be written as

$$f=f_c+\frac{1}{2ik}\sum_{\kappa=1}^{\infty}\kappa[e^{2i\chi_\kappa}(e^{2i\delta_\kappa}-1)P_{\kappa-1}+e^{2i\chi_{-\kappa}}(e^{2i\delta_{-\kappa}}-1)P_\kappa],$$

and

$$g=g_c+\frac{1}{2ik}\sum_{\kappa=1}^{\infty}[-e^{2i\chi_\kappa}(e^{2i\delta_\kappa}-1)P_{\kappa-1}{}^1+e^{2i\chi_{-\kappa}}(e^{2i\delta_{-\kappa}}-1)P_\kappa{}^1],\quad(27)$$

where f_c and g_c are the values of f and g which give the coulomb scattering.

It can be shown that $\chi_\kappa=\chi_{-\kappa}$ and $\delta_\kappa=\delta_{-\kappa}$ in the approximation of neglecting the rest mass; therefore, one has

$$f=f_c+\frac{1}{2ik}\sum_{\kappa=1}^{\infty}\kappa e^{2i\chi_\kappa}(e^{2i\delta_\kappa}-1)(P_\kappa+P_{\kappa-1}),$$

and

$$g=g_c+\frac{1}{2ik}\sum_{\kappa=1}^{\infty}e^{2i\chi_\kappa}(e^{2i\delta_\kappa}-1)(P_\kappa{}^1-P_{\kappa-1}{}^1).\quad(28)$$

Using recurrence relations between associated Legendre functions, one has

$$P_\kappa-P_{\kappa-1}=\kappa\tan\tfrac{1}{2}\theta\cdot(P_\kappa+P_{\kappa-1}).\quad(29)$$

Thus, g can be written in term of f

$$g-g_c=\tan\tfrac{1}{2}\theta\cdot(f-f_c).\quad(30)$$

The scattering cross section is then

$$\sigma=\sigma_c+2Re[(f_c+\tan\tfrac{1}{2}\theta\cdot g_c)(f-f_c)^*]+\sec^2\tfrac{1}{2}\theta\cdot|f-f_c|^2,\quad(31)$$

where $\sigma_c=|f_c|^2+|g_c|^2$.

McKinley and Feshbach[4] have given the coulomb scattering cross section in terms of two functions, F and G (not to be confused with the radial functions F_κ and G_κ).

$$\begin{aligned}kf_c&=G-iq'F,\\ kg_c&=\tan\tfrac{1}{2}\theta\cdot G+iq'\cot\tfrac{1}{2}\theta\cdot F,\\ k^2\sigma_c&=|G|^2\sec^2\tfrac{1}{2}\theta+q'^2|F|^2\csc^2\tfrac{1}{2}\theta,\end{aligned}\quad(32)$$

where $q'=(\alpha/\beta)(1-\beta^2)^{\frac{1}{2}}$, $\beta=v/c$; $q'\simeq\alpha mc^2/E$.

G and F are given by the following expressions (taking $\alpha/\beta\simeq\alpha$):

$$\begin{aligned}&G=G_0+G_1,\quad F=F_0+F_1,\\ &G_0=\tfrac{1}{2}\alpha\cot^2\tfrac{1}{2}\theta\exp(i\alpha\ln\sin^2\tfrac{1}{2}\theta)\cdot[\Gamma(1-i\alpha)/\Gamma(1+i\alpha)],\\ &F_0=(i/\alpha)\tan^2\tfrac{1}{2}\theta G_0,\\ &G_1=E(\theta)\alpha^2+H(\theta)\alpha^3+[I(\theta)+J(\theta)]\alpha^4,\\ &F_1=A(\theta)\alpha^2+B(\theta)\alpha^3+[C(\theta)+D(\theta)]\alpha^4,\end{aligned}\quad(33)$$

where the angular functions are tabulated in the paper cited.

The terms involving q' are negligible except for large angles, where F becomes much larger than G. Thus, for angles up to 150°, the coulomb cross section can be written

$$k^2\sigma_c=|G|^2\sec^2\tfrac{1}{2}\theta\quad(34)$$

with an error of less than one percent.

Noting that

$$f_c+g_c\tan\tfrac{1}{2}\theta=k^{-1}\sec^2\tfrac{1}{2}\theta\cdot G,$$

the ratio σ/σ_c becomes

$$\frac{\sigma}{\sigma_c}=1+\frac{2}{|G|^2}\sum_{\kappa=1}^{\infty}\kappa(P_\kappa+P_{\kappa-1})Re\left[Ge^{-2i\chi_\kappa}\left(\frac{e^{2i\delta_\kappa}-1}{2i}\right)^*\right]+\frac{1}{|G|^2}\left|\sum_{\kappa=1}^{\infty}\kappa(P_\kappa+P_{\kappa-1})e^{2i\chi_\kappa}\left(\frac{e^{2i\delta_\kappa}-1}{2i}\right)\right|^2.\quad(35)$$

The phase shifts, δ_κ, will decrease very rapidly with increasing κ since, classically speaking, those electrons whose angular momenta are such that they do not penetrate the nucleus itself cannot tell the difference between a finite nucleus and a point nucleus. For most of the energies considered here, in fact, only δ_1 will be important.

Now, $P_1+P_0=\cos\theta+1=2\cos^2\tfrac{1}{2}\theta$

$$\therefore\quad\frac{\sigma}{\sigma_c}=1+\frac{4\cos^2\tfrac{1}{2}\theta}{|G|^2}Re\left[Ge^{-2i\chi_1}\left(\frac{e^{2i\delta_1}-1}{2i}\right)^*\right]+\frac{4\cos^4\tfrac{1}{2}\theta}{|G|^2}\sin^2\delta_1.\quad(36)$$

This expression is valid for those energies for which δ_2 is negligible, for all Z, and for scattering angles up to 150°.

For low Z elements, such that α^2 is negligible, $G\simeq G_0$, and for energies such that δ_2 and $\delta_1{}^2$ are also negligible, the cross section becomes

$$\frac{\sigma}{\sigma_c}=1+\frac{8\delta_1}{\alpha}\sin^2\tfrac{1}{2}\theta\cos(\alpha\ln\sin^2\tfrac{1}{2}\theta+2\alpha).\quad(37)$$

Having seen how the scattering cross section depends on δ_κ, it is now necessary to find out just how large these phase shifts are for the different nuclear models.

IV. EVALUATION OF δ_κ FOR SPECIFIC NUCLEAR MODELS

To determine the phase shift, one must solve Eq. (6) for the region within the nucleus and evaluate $\psi_{\rm I}'/\psi_{\rm I}$ at the nuclear radius.

The easiest case to handle mathematically is the shell charge distribution, where the potential is simply

[4] W. A. McKinley, Jr. and H. Feshbach, Phys. Rev. 74, 1759 (1948).

TABLE I. $|G|$

Z	30°	60°	90°	120°	150°
13	0.467	0.0226	2.58×10^{-3}	2.91×10^{-4}	1.36×10^{-5}
29	2.54	0.1314	0.0156	1.80×10^{-3}	8.43×10^{-5}
50	8.28	0.491	0.0629	7.62×10^{-3}	3.55×10^{-4}
79	22.31	1.740	0.269	0.0370	1.788×10^{-4}

Table II. $Ge^{-2i\chi_1}$.

Z	30°	60°	90°	120°	150°
13	0.672	0.150	0.0508	0.01705	0.00368
	$-0.122i$	$-0.0096i$	$-0.00036i$	$0.00042i$	$0.00015i$
29	1.449	0.358	0.1248	0.0424	0.00915
	$-0.665i$	$-0.0588i$	$-0.0037i$	$0.0019i$	$0.00078i$
50	1.951	0.660	0.249	0.0873	0.0187
	$-2.114i$	$-0.237i$	$-0.026i$	$0.0029i$	$0.0021i$
79	0.0314	0.949	0.481	0.190	0.0423
	$-4.72i$	$-0.916i$	$-0.193i$	$-0.0314i$	$-0.0006i$

$U=-\alpha/R$. The wave equation then becomes

$$\psi_{\mathrm{I}}''+\left[\left(k+\frac{\alpha}{R}\right)^2+\frac{i(k+\alpha/R)}{r}-\frac{\kappa^2-\frac{1}{4}}{r^2}\right]\psi_{\mathrm{I}}=0. \quad (38)$$

This equation also reduces to Whittaker's confluent hypergeometric equation. The solution going to zero at the origin is

$$\psi_{\mathrm{I}}=e^{-i(k+\alpha/R)r}r^{|\kappa|+\frac{1}{2}} \times {}_1F_1(|\kappa|;2|\kappa|+1;2i(k+\alpha/R)r). \quad (39)$$

(The absolute signs about κ can be omitted since it is only positive κ that is now being considered.)

The phase shift depends on A_2/A_1, which is given by Eq. (9). If one writes

$$\psi_{c1}=e^{-ikr}r^{\rho+\frac{1}{2}}F_\rho,$$
$$\psi_{c2}=e^{-ikr}r^{-\rho+\frac{1}{2}}F_{-\rho}, \quad (40)$$

and

$$\psi_{\mathrm{I}}=e^{-ik_\alpha r}r^{\kappa+\frac{1}{2}}F_\kappa,$$

where

$$F_\rho={}_1F_1(\rho+i\alpha;2\rho+1;2ikr),$$
$$F_\kappa={}_1F_1(\kappa;2\kappa+1;2ik_\alpha r),$$

and

$$k_\alpha=k+\alpha/R,$$

then

$$\frac{A_2}{A_1}=-R^{2\rho}\frac{RF_\rho'+(\rho-\kappa+i\alpha-RF_\kappa'/F_\kappa)F_\rho}{RF_{-\rho}'+(-\rho-\kappa+i\alpha-RF_\kappa'/F_\kappa)F_{-\rho}}, \quad (41)$$

and δ_κ is determined from Eq. (25).

For the uniform charge distribution, the potential within the nucleus is $U=-(\alpha/2R)(3-r^2/R^2)$, which leads to the equation

$$\frac{d^2\psi_{\mathrm{I}}}{dx^2}-\left[\frac{\kappa^2-\frac{1}{4}}{x^2}-i\frac{k_{\alpha'}R}{x}-(k_{\alpha'}R)^2+\tfrac{3}{2}i\alpha x+\alpha k_{\alpha'}Rx^2-\tfrac{1}{4}\alpha^2x^4\right]\psi_{\mathrm{I}}=0, \quad (42)$$

where $x=r/R$, $k_{\alpha'}=k+3\alpha/2R$.

This equation cannot be put into any of the standard forms, but noting that all that is required is $\psi_{\mathrm{I}}'(R)/\psi_{\mathrm{I}}(R)$, one can make the substitution

$$\Psi=(d\psi_{\mathrm{I}}/dx)/\psi_{\mathrm{I}} \quad (43)$$

which reduces the second-order equation to the Riccati first-order equation

$$d\Psi/dx+\Psi^2=J(x), \quad (44)$$

where $J(x)$ is the coefficient of ψ_{I} in Eq. (42).

This equation can be solved by the power series method, letting

$$\Psi=\frac{1}{\kappa}\sum_{n=0}^{\infty}a_nx^n \quad (45)$$

and evaluating the coefficients a_n by the usual procedure.

Evaluating Ψ at $r=R$; i.e., $x=1$,

$$\Psi_{r=R}=\sum_{n=0}^{\infty}a_n. \quad (46)$$

In terms of Ψ, A_2/A_1 is

$$\frac{A_2}{A_1}=-R^{2\rho}\frac{RF_\rho'+(\rho+\frac{1}{2}-ikR-\Psi)F_\rho}{RF_{-\rho}'+(-\rho+\frac{1}{2}-ikR-\Psi)F_{-\rho}}. \quad (47)$$

Calculations have been made for four elements, of atomic number 13, 29, 50, and 79, and for a number of energies between 15 and 35 Mev. The values of $|G|^2$ $Ge^{-2i\chi_1}$, for use in Eq. (36), are listed in Tables I and II.

Using the formula $R=1.45\times10^{-13}A^{\frac{1}{3}}$ cm for the nuclear radius, and with $k=E/\hbar c=0.507\times10^{11}\,E_{\mathrm{Mev}}$ cm^{-1}, kR is given by $kR=7.35\times10^{-3}\,E_{\mathrm{Mev}}A^{\frac{1}{3}}$.

The phase shifts, δ_1, are plotted against energy in Figs. 1 and 2. The scattering cross sections for the above Z at 20 and 30 Mev, and for energies of 15, 25, and 35 Mev at $Z=29$ for both the shell and uniform charge distributions, are plotted in Fig. 3 (a, b, and c).

V. DISCUSSION OF RESULTS

The general effect produced by the finite size of the nucleus is seen to be a reduction of the scattering cross section, predominating at large angles. This is in agreement with the wave picture of the scattering; for if the

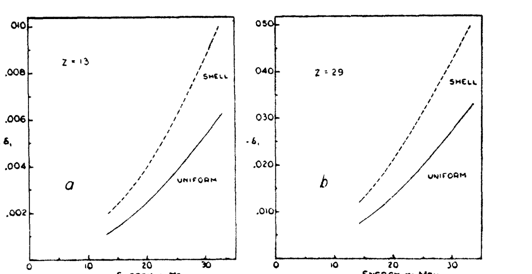

FIG. 1. Phase shift δ_1 as a function of energy for shell and uniform models for (a) $Z=13$, (b) $Z=29$.

nucleus is considered shrunk to a point, then the scattered waves from all elements of the nuclear charge will arrive at the observation point in phase, and, therefore, with maximum intensity. With a finite nucleus, the scattered waves from the different elements will not be in phase, interference will occur, and the intensity will usually be reduced. Furthermore, the forward scattering will suffer less reduction, since the path lengths of the different scattered waves will be more nearly the same and the waves more nearly in phase.

In these calculations, R has been taken as $1.45 \times 10^{-13} A^{\frac{1}{3}}$ cm, but varying the radius, with a fixed Z, produces exactly the same effect as varying the energy, because of the dependence of the phase shift on kR. Thus, the curves of phase shift as a function of energy give equally well their dependence on the radius.

The different charge distributions produce a considerable difference in the scattering, particularly for the higher energies and higher Z, thus raising the possibility that accurate scattering experiments will yield information regarding nuclear structure. However, this effect is greatest at large scattering angles, which makes it difficult to measure, since the coulomb cross section becomes very small at large angles.

It is also true that one cannot separate the two effects of type of charge distribution and size of nucleus. This comes from the fact that only one phase shift, δ_1, is necessary to describe the scattering. In general, the phase shift obtained from the uniform model at a given energy and Z is also given by the shell model using a radius about three-fourths that of the uniform model. Furthermore, the variation of the phase shift with energy for the two models does not differ enough to distinguish between them. However, radii are known

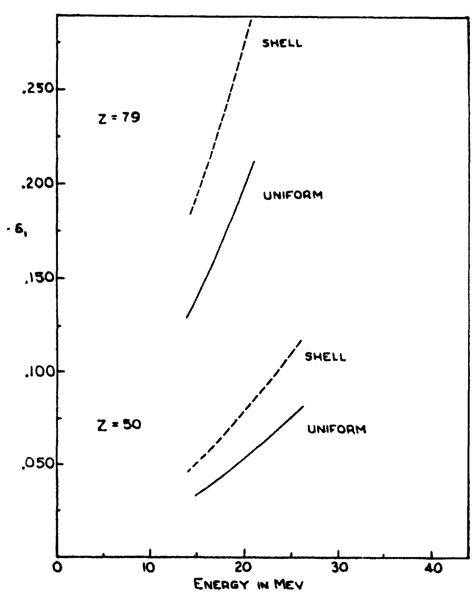

FIG. 2. Phase shift δ_1 as a function of energy for shell and uniform models for $Z=50$ and 79.

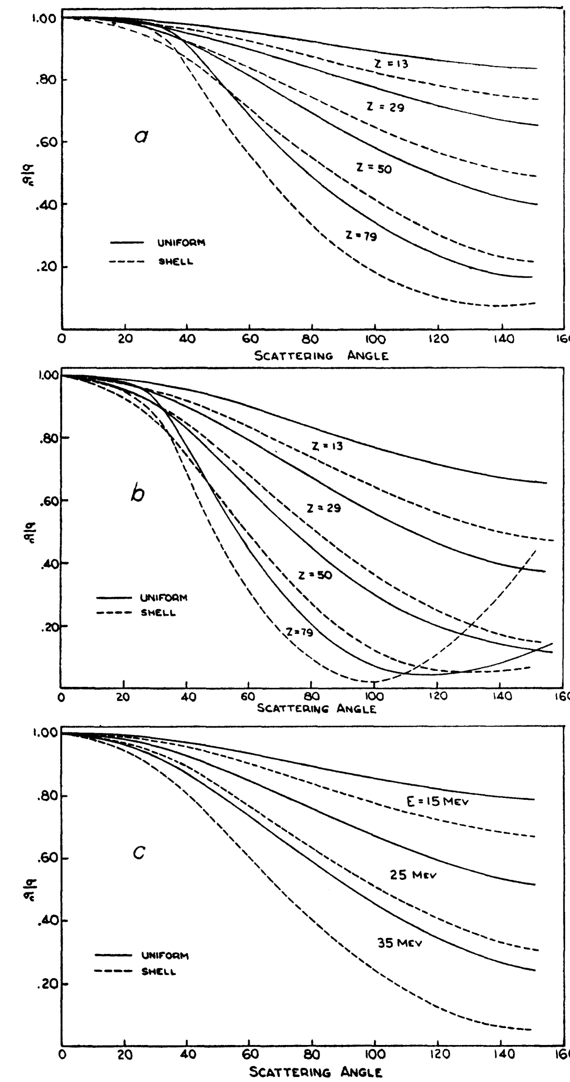

FIG. 3. (a) Scattering of 20-Mev electrons by shell and uniform nuclear models for various Z. (b) Scattering of 30-Mev electrons by shell and uniform nuclear models for various Z. (c) Scattering of electrons of 15, 25, and 35 Mev by shell and uniform nuclear models for $Z=29$.

accurately enough so that some indication of the nature of the nuclear charge distribution should be obtainable.

The second phase shift, δ_2, varies from 0.4 percent of δ_1 for $Z=13$ and $E=15$ Mev, to 3 percent for $Z=79$ and $E=35$ Mev and thus is negligible, except perhaps for very high Z and high energy.

With the phase shifts plotted against E, one can find the scattering for any energy in the neighborhood of 20 Mev. Owing to computational difficulties, the present method is not practicable for calculating phase shifts for kR much greater than unity. Moreover, at very high energies the present treatment is inadequate in principle, for it does not suffice to consider the interaction of the electrons with the nucleus as a whole;

one must consider interactions with the individual nucleons.

The present treatment is also limited by the fact that only spherically symmetric charge distributions can be considered, and thus the effect of nuclear quadrupole moment cannot be calculated. Magnetic forces have not been considered, although they may not be completely negligible at these energies. Furthermore, it is only the cross section for elastic scattering that has been investigated here. While the total cross section for nuclear excitation is small,[5] 10^{-4} barn for 16-Mev electrons (the coulomb differential cross section varying from a few barn at 30° to 10^{-3} barn at 150°), it is necessary to say a word about the effect of radiation.

Schwinger[6] has calculated the radiative correction to the coulomb cross section for essentially elastic scattering of electrons. One can write $\sigma/\sigma_c=1-\delta(\theta, \Delta E)$, where ΔE is the energy radiated. It is only for small ΔE that δ is important, and the scattering differs appreciably from the coulomb scattering. Representative values of δ are, for $\theta=\frac{1}{2}\pi$ and $E=20$ Mev,

$$\delta=0.071 \text{ if } \Delta E=0.022,$$

and

$$\delta=0.018 \text{ if } \Delta E/E=0.20.$$

Thus, the effect of radiation can be minimized in experimental work by the use of a wide range detector.

The author is greatly indebted to Dr. H. Feshbach for suggesting this problem and for continued help and encouragement. Thanks are due to Dr. M. E. Rose for several helpful discussions, and, for computational aid, the assistance of Ann Moldauer is gratefully acknowledged.

APPENDIX

Numerical Calculations

To evaluate the confluent hypergeometric functions, it is necessary to use their series representations.

$$ {}_1F_1(a;b;z)=1+\frac{a}{b}z+\frac{a(a+1)}{b(b+1)}\frac{z^2}{2!}+\cdots = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(b+n)}\frac{\Gamma(b)}{\Gamma(a)}\frac{z^n}{n!}.$$

For the coulomb solutions, one has

$$F_\rho=\sum_{n=0}^{\infty}\alpha_\rho^{(n)}z^n,\quad RF_\rho'=\sum_{n=0}^{\infty}n\alpha_\rho^{(n)}z^n,$$

where

$$\alpha_\rho^{(n)}=\frac{\Gamma(\rho+n+i\alpha)}{\Gamma(2\rho+1+n)}\frac{\Gamma(2\rho+1)}{\Gamma(\rho+i\alpha)}\frac{1}{n!},$$

$z=2ikR$, $\alpha=Z/137$, $\rho=(\kappa^2-\alpha^2)^{\frac{1}{2}}$.

Then A_2/A_1 for the shell model can be written

$$\frac{A_2}{A_1}=-R^{2\rho}\frac{\sum_{n=0}^{\infty}(n+\rho-\kappa+i\alpha-RF_\kappa'/F_\kappa)\alpha_\rho^{(n)}z^n}{\sum_{n=0}^{\infty}(n-\rho-\kappa+i\alpha-RF_\kappa'/F_\kappa)\alpha_{-\rho}^{(n)}z^n},$$

while, for the uniform model,

$$\frac{A_2}{A_1}=-R^{2\rho}\frac{\sum_{n=0}^{\infty}(n+\rho+\frac{1}{2}-ikR-\Psi)\alpha_\rho^{(n)}z^n}{\sum_{n=0}^{\infty}(n-\rho+\frac{1}{2}-ikR-\Psi)\alpha_{-\rho}^{(n)}z^n}.$$

This method is only practical for those values of kR for which the above series converge within a reasonable number of terms, which essentially limits one to $kR<1$.

To evaluate RF_κ'/F_κ, one can make use of Kummer's second transformation, which yields

$$R\frac{F_\kappa'}{F_\kappa}=\frac{\kappa}{2\kappa+1}(2ik_\alpha R)\frac{e^{ik_\alpha R}\,{}_0F_1(\kappa+\frac{3}{2};-\frac{1}{4}(k_\alpha R)^2)}{{}_1F_1(\kappa;2\kappa+1;2ik_\alpha R)}.$$

To evaluate Ψ, one has

$$\Psi=\sum_{n=0}^{\infty}a_n,$$

$$a_0=\tfrac{1}{2}[1+(1+4c_0)^{\frac{1}{2}}],$$
$$a_1=c_1/2a_0,$$
$$\vdots$$
$$a_n=\left[c_n-\sum_{l=1}^{n-1}a_la_{n-l}\right]\Big/(2a_0+n-1),\quad n=1,2,\cdots,$$

where c_n is given by

$$J(x)=\frac{1}{x^2}\sum_{n=0}^{6}c_nx^n\quad[\text{see Eq. (44)}],$$

$$c_n=0\qquad \text{For } n>6.$$

The calculations have shown that $Re(A_2/A_1)=-\alpha/\rho\, Im(A_2/A_1)$ for both models.

Utilizing this result, the phase shift δ_κ can be written as

$$\delta_\kappa=-\arctan\frac{e^{\pi\alpha}\frac{\kappa^2}{2\rho^2}\left|\frac{\Gamma(\rho+i\alpha)}{\Gamma(2\rho)}\right|^2(2k)^{2\rho}Im\frac{A_2}{A_1}}{2-(e^{\pi\alpha}\cot 2\pi\rho-e^{-\pi\alpha}\csc 2\pi\rho)\times\frac{\kappa^2}{2\rho^2}\left|\frac{\Gamma(\rho+i\alpha)}{\Gamma(2\rho)}\right|^2(2k)^{2\rho}Im\frac{A_2}{A_1}}.$$

The complex gamma-function is not tabulated, and the best way to evaluate it is to use the asymptotic formula in conjunction with repeated use of the recurrence relation $\Gamma(z+1)=z\Gamma(z)$.

The resulting expressions for magnitude and phase are

$$|\Gamma(\rho+i\alpha)|^2\simeq\frac{2\pi e^{-2x-2\alpha\phi}r^{2(x-\frac{1}{2})}(1+x/6r^2+\cdots)}{\kappa^2(\kappa^2+2\rho+1)(\kappa^2+4\rho+4)(\kappa^2+6\rho+9)},$$

$$\arg\Gamma(\rho+1+i\alpha)=\Phi-\sum_{n=1}^{3}\arctan(\alpha/\rho+n),$$

$$\Phi=(x-\tfrac{1}{2})\arctan(\alpha/x)+\alpha(\ln r-1)-\alpha/12r^2,$$

where $x=\rho+4$, $r^2=x^2+\alpha^2$, $\phi=\arctan(\alpha/x)$.

[5] Skaggs, Laughlin, Hanson, and Orlin, Phys. Rev. 73, 420 (1948).

[6] J. Schwinger, Phys. Rev. 75, 898(L) (1949).