identifying magic numbers in the region $50<A<65$. The level scheme of Maria Mayer suggests that 28 and 32 are magic numbers in this region, whereas Haxel *et al.* suggest 28 and 34. Nordheim suggests 34, and Feenberg and Hammack hint that 32 and/or 34 may be magic. The minimum in the packing fraction curve in Fig. 1 is located at $_{28}Ni^{60}$.§ This coincides, according to the Mayer scheme, with a doubly-magic nuclide.

The heaviest stable isotope of silicon, $_{14}Si^{30}$, is of interest because of its exceptionally low packing fraction. Maria Mayer has shown that strong spin-orbit coupling can lead to a reversal of the $2s$, $3d_{5/2}$ level order with the result that 14 nucleons complete the $3d_{5/2}$ shell and constitute a particularly stable configuration. The 16-nucleon configuration, representing the completion of the $2s$ shell, should also be very stable. Thus, it may be that $_{14}Si^{30}$ owes its exceptional stability to a doubly-magic configuration. Its doubly-magic brother, $_{14}Si^{28}$, is somewhat less closely knit. The increase in stability resulting from an excess of neutrons over protons, which is so pronounced in the heavier nuclides, is presumably responsible for this difference.

It seems likely that additional mass measurements can be of considerable use in the identification of the ground states of nuclei, particularly in the case of even-even nuclei, where no information has so far been derived from spin measurements. In conclusion, one may venture to say that the mass evidence to date gives general support to the level scheme of Maria Mayer.

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§ *Note added in proof.*—Some recent experiments suggest that Ni^{62} may mark the minimum of the packing fraction curve. These experiments are being continued.

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Properties of Dirac Wave Functions in a Central Field*

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The boundary conditions for a Dirac particle in a central scalar field are discussed for both bound and continuum states. In this development, the methods employed are considerably different from those customarily used for the corresponding nonrelativistic case.

I. INTRODUCTION

WHILE the Dirac wave functions for a particle in a central field possess properties which are, in many instances, qualitatively similar to those exhibited by the corresponding solutions to the Schroedinger problem, there are several essential differences which have apparently not been considered very completely or stated explicitly in previous investigations. An important point of difference is concerned with the boundary conditions which, in the relativistic and nonrelativistic problems, must be discussed quite independently. Intimately connected with the formulation of boundary conditions is the question of which potential functions are admissible from a physical point of view. Here, the radically different answer provided by the relativistic problem is, in part, traceable to the energy doubling (existence of positive and negative energy states) and, in part, arises from the spin properties. A third point of interest is the study of the nodal properties (oscillation theorems, etc.) in the case of bound states. Despite the fact that the Dirac equations do not form a Sturm-Liouville system, several of the theorems concerning nodal properties are applicable.

The following is devoted primarily to a discussion of the three aforementioned problems: (1) boundary conditions, (2) admissible potentials, and (3) nodal properties. In connection with the study of nodal properties, we have found it necessary to develop methods somewhat different from those generally used in the treatment of Sturm-Liouville systems. Since these methods are also applicable to such systems, they may be of interest for classes of problems other than the one discussed here (Sec. VII).

The desirability of such a study was encountered in our program for computation of L-shell internal conversion coefficients, which required extensive numerical calculation of Dirac wave functions in a central noncoulomb field.[1] Some considerations which may be

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[1] Rose, Goertzel, Spinrad, Harr, and Strong, Phys. Rev. **76**, 1883 (1949).

useful for the numerical calculation of wave functions are given below.

The wave functions for a Dirac electron in a central field are of two types (doubling caused by spin-orbital coupling).[2] Denoting the two types by indices by a and b, the four components of the wave function, for a given total angular momentum quantum number j and given magnetic quantum number m, can be written in the form

$$\begin{aligned}
(\psi_{a1}, \psi_{b3}) &= \left(\frac{|\kappa|-m+\tfrac{1}{2}}{2|\kappa|+1}\right)^{\frac{1}{2}} Y_{|\kappa|}{}^{m-\frac{1}{2}}(if_\kappa, g_\kappa), \\
(\psi_{a2}, \psi_{b4}) &= \left(\frac{|\kappa|+m+\tfrac{1}{2}}{2|\kappa|+1}\right)^{\frac{1}{2}} Y_{|\kappa|}{}^{m+\frac{1}{2}}(if_\kappa, g_\kappa), \\
(\psi_{a3}, \psi_{b1}) &= \left(\frac{|\kappa|+m-\tfrac{1}{2}}{2|\kappa|-1}\right)^{\frac{1}{2}} Y_{|\kappa|-1}{}^{m-\frac{1}{2}}(g_\kappa, if_\kappa), \\
(\psi_{a4}, \psi_{b2}) &= -\left(\frac{|\kappa|-m-\tfrac{1}{2}}{2|\kappa|-1}\right)^{\frac{1}{2}} Y_{|\kappa|-1}{}^{m+\frac{1}{2}}(g_\kappa, if_\kappa).
\end{aligned} \tag{1}$$

Here, $Y_\lambda{}^\mu$ is a (normalized) surface harmonic of degree λ, order μ, and f_κ and g_κ satisfy the differential equations

$$\begin{aligned}
d(rf)/dr &= \kappa f-(W-1-V)rg, \\
d(rg)/dr &= (W+1-V)rf-\kappa g.
\end{aligned} \tag{2}$$

For type a, $\kappa=-(j+\frac{1}{2})$, so that κ is a negative integer. For type b, $\kappa=j+\frac{1}{2}$, so κ is a positive integer. W is the total energy (including rest energy), and V the potential energy in units of mc^2; r is measured in units of $\hbar/mc$. We shall rewrite Eqs. (2) in the form

$$F'=\alpha_{11}F+\alpha_{12}G, \quad G'=\alpha_{21}F+\alpha_{22}G, \tag{2a}$$

where $F=rf$, $G=rg$, $\alpha_{11}=-\alpha_{22}=\kappa/r$, $\alpha_{12}=-(W-1-V)$, $\alpha_{21}=W+1-V$; and throughout a prime denotes differentiation with respect to r.

II. BOUNDARY CONDITIONS

The functions F and G are to be subject to the conditions that they be real and single-valued, and, in addition, for bound states

$$0<\int_a^b (F^2+G^2)dr<\infty, \quad 0\leqq a<b\leqq\infty. \tag{3a}$$

In addition, for both bound and continuum states, we require that

$$\lim_{r\to 0}(F_iG_k-F_kG_i)=0, \tag{3b}$$

where i and k each represents a triad of quantum numbers (W, κ, and m), denoting constants of the motion defining a stationary state in a central field. The radial functions are independent of m but do depend on W and κ.

Continuity and finiteness of F and G at all ordinary points of (2a) are assured if the functions satisfy Eq. (2a).

Condition (3a) is required by the probability interpretation of the wave function modulus. Condition (3b) is a consequence of the requirement that a linear superposition Ψ of stationary states satisfy the continuity equation, as the three-dimensional time-dependent wave equation demands. The continuity equation is fulfilled for Ψ if it is fulfilled for the mixed current-charge densities; that is, if

$$\operatorname{div}(\Psi_i{}^*\boldsymbol{\alpha}\Psi_k)+\partial(\Psi_i{}^*\Psi_k)/\partial t=0, \tag{4a}$$

since for $i=k$, Eq. (4a) is automatically satisfied. The contribution to the particle flux per unit solid angle from the mixed states is

$$r^2J_r{}^{(ik)}=Q_{ik}(\vartheta, \varphi)(F_iG_k-F_kG_i), \tag{4b}$$

where $J_r{}^{(ik)}$ is the radial component of $\Psi_i{}^*\boldsymbol{\alpha}\Psi_k$, and the angular dependent coefficients Q_{ik} are sums of terms involving products of spherical harmonics of, in general, different degree and order and are dependent on κ, m. If condition (3b) is not fulfilled, it follows that $J_r{}^{(ik)}\sim 1/r^2$, since, as will be shown below, $\lim(F_iG_k-F_kG_i)$ is always finite at the origin. Then $\mathbf{J}^{ik}\sim\operatorname{grad}1/r$, and $\operatorname{div}\mathbf{J}^{(ik)}\sim\delta(\mathbf{r})$, in contradiction to (4a).

We shall have use, in what follows, for the concept of left-handed and right-handed solutions (L and R solutions). By a left-handed solution of (2a), we mean one satisfying the conditions:

$$0<\int_0^b (F_L{}^2+G_L{}^2)dr<\infty, \quad b>0 \tag{5a}$$

$$\lim_{r\to 0}(F_iG_k-F_kG_i)=0, \tag{5b}$$

which are equivalent to Eq. (3) with the point at infinity excluded. By a right-handed solution, we mean one which fulfills the boundary condition

$$0<\int_a^\infty (F_R{}^2+G_R{}^2)dr<\infty, \quad 0<a<\infty, \tag{6a}$$

which is equivalent to Eq. (3) with the origin excluded. A bound state wave function, that is, one satisfying Eqs. (3a) and (3b), is both a right-handed and a left-handed solution (LR solution).

For continuum states, we still require Eqs. (5a) and (5b), and no boundary condition at infinity is necessary except for normalization purposes.

These boundary conditions are the necessary and sufficient conditions (along with Eq. (2a)) for the complete determination of the wave functions in the central field Dirac problem. In addition, Eq. (3b) restricts the choice of potential function on the basis of

[2] M. E. Rose, Phys. Rev. 51, 484 (1937). In this reference the subscripts on the functions f and g have a different meaning from that used here.

the behavior of $V(r)$ near the origin. We shall designate potentials for which Eq. (3b) is fulfilled as admissible. For such potentials, it will be seen that discrete eigenvalues are always finite. For potentials which are admissible, condition (3a) alone determines the eigenfunctions.

III. ADMISSIBLE POTENTIALS AND BEHAVIOR NEAR SINGULAR POINTS

(A) Behavior Near the Origin

Here and in the following, we shall find it helpful to use the phase function $\varphi(r)$ and the amplitude function $A(r)$, defined by[3]

$$F=A\sin\varphi, \quad G=A\cos\varphi. \tag{7}$$

From Eq. (2a), it is evident that F and G cannot vanish simultaneously at an interior point of the interval $(0-\infty)$. It follows, then, that A cannot vanish except at the end points. A cannot have any singularities except at the origin or at a point where V has a singularity. Substituting Eq. (7) into Eq. (2a), we obtain

$$A'/A=-(\kappa/r)\cos2\varphi+\sin2\varphi, \tag{8}$$

$$\varphi'=(\kappa/r)\sin2\varphi-(W-V)+\cos2\varphi. \tag{9}$$

Putting Eqs. (8) and (9) into integral form,

$$\ln A_1-\ln A_0=-\kappa\int_{r_0}^{r_1}r^{-1}\cos2\varphi dr+\int_{r_0}^{r_1}\sin2\varphi dr, \tag{10a}$$

$$\varphi_1-\varphi_0=\kappa\int_{r_0}^{r_1}r^{-1}\sin2\varphi dr-W(r_1-r_0)+\int_{r_0}^{r_1}Vdr+\int_{r_0}^{r_1}\cos2\varphi dr, \tag{10b}$$

where r_1 is any interior point. In the following, we investigate the behavior of A_0 and φ_0 as their argument r_0 approaches zero.

We shall now consider two cases according to whether

$$\left|\lim_{r\to0}(rV)\right|>1 \quad \text{or} \quad \leqslant 1.$$

This includes fields more strongly divergent than the coulomb field ($|\lim(rV)|=\infty$) and fields which are convergent or less strongly divergent than the coulomb field ($|\lim(rV)|=0$).

(1). $|\lim(rV)|>1$. If we allow r_0 to approach the origin, then in Eq. (10b) the integral $\int_0 Vdr$ is a divergent term which cannot be cancelled, for *all* κ, by any other term on the right-hand side of Eq. (10b). Hence, φ becomes infinite at the origin, and F and G have an infinite number of nodes in any finite interval including the origin.

Now consider Eq. (10a). The dominant term on the right is

$$-\kappa\int_{r_0}^{r_1}r^{-1}\cos2\varphi dr.$$

Without the $\cos2\varphi$ factor, this would diverge as $r_0\to0$, but the presence of the rapidly oscillating factor makes it converge.[4] Hence, $\ln A$ remains finite as $r\to0$, and A remains finite and greater than zero.

It is now seen that $A(0)\neq0$, and

$$\lim_{r\to0}(F_iG_k-F_kG_i)=\lim_{r\to0}A_iA_k\sin(\varphi_i-\varphi_k)$$

approaches a finite nonvanishing limit, since $\varphi_i-\varphi_k$ does approach a definite limit at the origin. Consequently, condition (3b) is violated. Therefore, potentials for which $|\lim(rV)|>1$ at the origin are inadmissible for all stationary as well as nonstationary states. This is in contrast to the statement made by Plesset[5] who, while noting the irregular singularity at the origin for $V(r)\sim r^{-\nu}$ (ν a positive integer>1), did not consider the flux boundary condition (3b) and, therefore, concluded that a stationary state continuum solution was admissible in this case.

(2). $|\lim(rV)|\leqslant1$. We prove by construction that in all such cases there exists a solution of Eq. (2a) which satisfies the boundary conditions at the origin. The first and third integrals on the right-hand side of Eq. (10b) may now diverge individually as $r_0\to0$. However, with $r_0=0$, it is now always possible to find a finite $\varphi_0=\varphi(0)$ such that the sum of these two terms and also, therefore, the right-hand side of (10b) converge and such that equation Eq. (10b) is satisfied. Thus, we set

$$\varphi=\varphi_0+\theta,$$

where θ is to vanish as r approaches zero. Then, in the neighborhood of the origin, with $\sin2\theta\sim2\theta$ and $\cos2\theta\sim1$, the required solution of Eq. (9) is

$$\theta=r^{-2\eta e^{-2r\sin2\varphi_0}}\int_0^r r^{2\eta e^{2r\sin2\varphi_0}}\times[V-W+\cos2\varphi_0+(\kappa/r)\sin2\varphi_0]dr. \tag{11}$$

In Eq. (11) $\eta=-\kappa\cos2\varphi_0$, and we fix φ_0 (to within an additive integral multiple of π which is always arbitrary

[3] H. Prüfer, Math. Annalen **95**, 499 (1926). For another application of the phase and amplitude functions in the Dirac equations, see G. Breit and G. E. Brown, Phys. Rev. **76**, 1307 (1949).

[4] The convergence of the integral does not follow simply from the rapid oscillation of $\cos2\varphi$, but its convergence, in this case, can be seen by changing the variable of integration,

$$\int_{r_0}^{r_1}r^{-1}\cos2\varphi dr=\int_{\varphi_0}^{\varphi_1}(rd\varphi/dr)^{-1}\cos2\varphi d\varphi,$$

where φ_0 is to approach infinity. Suppose $\lim rV$ is unbounded at the origin. Then, by Eq. (9) $rd\varphi/dr\sim rV$ and the integrand approaches zero so that the integral converges. The remaining possibility is $\infty>|\lim rV|=|\beta|>1$. In this case, the first term in Eq. (9) is of the same order as V. However, as is shown in paragraph (2) below, these terms can never cancel, for all κ, if $|\beta|>1$ and the integral is bounded.

[5] M. S. Plesset, Phys. Rev. **41**, 278 (1932).

by Eq. (2a)) by the conditions

$$\eta = -\kappa \cos 2\varphi_0 \gtrless 0, \tag{11a}$$

$$\kappa \sin 2\varphi_0 = -\lim_{r\to 0}(rV). \tag{11b}$$

n order that Eq. (11b) may be fulfilled for all κ, with real φ_0, the condition $|\lim(rV)|<1$ is necessary and sufficient. For definiteness, we take the principal value of φ_0 for L and LR solutions. Equation (11a) corresponds to the choice of the solution of Eq. (2a) which is regular at the origin, while (11b) is sufficient to make θ finite at $r=0$. The conditions (11a) and (11b) can always be fulfilled, for the present case, with real φ_0. For fields coulomb at $r=0$ (point charge and screened fields), $(V=-\beta/r)$, we obtain the well-known results

$$\sin 2\varphi_0 = \beta/\kappa, \quad \cos 2\varphi_0 = -\gamma/\kappa, \tag{11c}$$

with $\gamma=(\kappa^2-\beta^2)^{\frac{1}{2}}$ or $\tan\varphi_0=\beta/(\kappa-\gamma)$, while, for the case $|\lim(rV)|=0$, we find

$$\begin{aligned} &\varphi_0=\pi/2, && \kappa>0 \\ &\varphi_0=0, && \kappa<0. \end{aligned} \tag{11d}$$

It is now clear that θ approaches zero at the origin. If $\eta=0$ (which occurs only for $|\lim(rV)|=1$, $|\kappa|=1$), the result is obvious. For $\eta\neq 0$, θ is indeterminate at the origin, and by l'Hôspital's rule,

$$\theta \to r[V-W+\cos 2\varphi_0+(\kappa/r)\sin 2\varphi_0]/2\eta, \tag{11e}$$

which clearly goes to zero [for example, see Eq. (11b)].

In Eq. (10a) we can now observe that the dominant term for $r_0\to 0$ is the first one, and by Eq. (11a) the indicial behavior of the amplitude function is

$$A \sim r^{\eta}. \tag{12}$$

Hence, a set of functions F_L and G_L will always exist for which $(\eta\neq 0)$

$$F_L(0)=0, \quad G_L(0)=0. \tag{12a}$$

If $\eta=0$, F_L and G_L have finite nonzero limits, and $|F_L|=|G_L|$, since $\varphi_0=\pm\pi/4$. Our subsequent theorems can easily be demonstrated to hold in this limiting case but, for ease of discussion, we shall omit explicit consideration of the case $|\lim(rV)|=1$ in the following.

The indicial behavior (12) may be particularized for the two subcases considered. For $\lim(rV)=-\beta$, we find from relationships (7), (11a), (11c), (11e), and (12)

$$F=O(r^{\gamma}), \quad G\sim O(r^{\gamma}), \tag{12b}$$

where the symbol O signifies "of order of," and the ratio F_L/G_L is given in Eq. (11c). For $\lim(rV)=0$, $\eta=|\kappa|$, we find[6]

$$\begin{aligned} &F_L=O(r^{\kappa}), && G_L=O(r^{\kappa+1}\alpha_{21}), && \kappa>0 \\ &F_L=O(r^{1-\kappa}\alpha_{12}), && G_L=O(r^{-\kappa}), && \kappa<0. \end{aligned} \tag{12c}$$

To summarize the results of this section, we conclude that if $|\lim(rV)|>1$ at the origin, there are no L solutions and, *a fortiori*, no eigensolutions in the discrete or continuous spectrum. If $|\lim(rV)|\leqslant 1$ at the origin, there is always an L solution, and for the inequality we have $A(0)=0$ [and, thus, $F(0)=G(0)=0$], and the same results must apply to eigensolutions. For an eigensolution, condition (3a) also requires that A, and therefore F and G, vanish at infinity. The explicit results concerning the behavior at the origin are given by expressions (11c), (11d), (12b), and (12c).

(B) Behavior at Infinity

In the following, we are concerned with bound state wave functions. We shall consider only potentials which are monotonic in the neighborhood of infinity. They may be bounded or unbounded. We do not consider periodic potentials, for example.

If $V(\infty)$ is infinite, we can neglect W and unity compared with V in the neighborhood of infinity, so that Eqs. (2a) become

$$F'=(\kappa/r)F+VG, \quad G'=-VF-(\kappa/r)G. \tag{13}$$

These have the asymptotic solutions

$$F=B(r)\exp(\pm i\int V dr), \quad G=\pm iF$$

(or the real parts thereof), with $|B'/B|\ll|V|$. By means of Eqs. (10a) and (10b), which are valid in all cases, it can be shown that $\lim(B(r))$ exists and does not vanish. Therefore, the solutions violate (6a). Hence, there are no bound states, but such solutions are admissible for continuum states. This differs from the result obtained with the Schroedinger equation and is a manifestation of the Klein paradox.

For bound states, then, $V(\infty)$ must be finite, and we can choose $V(\infty)=0$. Equations (2a) become, near infinity,

$$F'=(\kappa/r)F-(W-1)G, \quad G'=(W+1)F-(\kappa/r)G. \tag{14}$$

With $B(r)$ as above, these have the asymptotic solutions $F=B(r)\exp(\pm(1-W^2)^{\frac{1}{2}}r)$, $G=\pm(1+W)^{\frac{1}{2}}(1-W)^{-\frac{1}{2}}F$. If $|W|<1$, we can get a bound state by choosing the lower sign throughout; if $|W|\geqslant 1$, there are only continuum states. Hence, we have the well-known result that there is a continuous spectrum for $|W|\geqslant 1$; this differs from the Schroedinger case which, in the corresponding situation, has a continuous spectrum only for $W\geqslant 1$.

In summary, we get bound states if $V(\infty)=0$, $|W|<1$; a continuum if $V(\infty)=0$, $|W|\geqslant 1$; and only a continuum if $|V(\infty)|=\infty$, for any W, provided, of course, that the behavior, at points other than infinity, conforms to the boundary conditions imposed above.

From the results of discussion (A), we conclude that a potential $V(r)$ is admissible only if $|\lim(rV)|\leqslant 1$ at $r=0$, and (with the exception noted above) for all such admissible potentials, both F_L and G_L (and also the eigenfunctions F_{LR} and G_{LR}) vanish at $r=0$. We also note that for an admissible potential the energy is

[6] M. E. Rose, Phys. Rev. **82**, 389 (1951).

always finite:

$$W\int_0^\infty (F^2+G^2)dr=\int_0^\infty \{2\kappa FG/r+G^2-F^2 + V(F^2+G^2)+G'F-F'G\}dr, \quad (15)$$

where F and G are eigensolutions. By conditions (3a) and (3b), the integral on the left and the first four terms of the right-hand side are seen to converge. The last two terms of the integrand, involving the Wronskian of F and G, behave at the origin as follows:

$$FG'-GF'\sim r^{2\kappa}d(r\alpha_{21})/dr, \quad (\kappa>0),$$

and an integration by parts is seen to give a convergent integral at $r=0$. Convergence at ∞ follows from the asymptotic behavior already given. The case $\kappa<0$ gives the same conclusion. The convergence of the energy integrals is, of course, necessary but not sufficient, since $|W|<1$ is required. Actually, the fulfillment of Eqs. (2a) with $|W|<1$ and of Eqs. (3a) and (3b) is equivalent to this extended condition on the energy integrals.

IV. ORTHOGONALITY OF THE EIGENSOLUTIONS

The wave equations (2a) can be written in hamiltonian form by introducing the two component spinor

$$\Psi=\begin{bmatrix}F\\G\end{bmatrix}. \quad (16)$$

Then,

$$H\Psi=W\Psi, \quad (16a)$$

where

$$H=\sigma_x\frac{\kappa}{r}+i\sigma_y\frac{d}{dr}-\sigma_z+V=\begin{bmatrix}V-1 & \kappa/r+d/dr\\ \kappa/r-d/dr & V+1\end{bmatrix} \quad (16b)$$

is the self-adjoint hamiltonian operator, and the σ's are the usual Pauli matrices.

If Ψ_1 and Ψ_2 correspond to $W=W_1$ and W_2, respectively, we find

$$\Psi_2H\Psi_1-\Psi_1H\Psi_2=(d/dr)(F_2G_1-F_1G_2)=(W_1-W_2)\Psi_1\Psi_2. \quad (17a)$$

The condition for orthogonality ($W_1\neq W_2$),

$$\int_0^\infty \Psi_1\Psi_2dr=0, \quad (17b)$$

is seen to be fulfilled if $[F_2G_1-F_1G_2]_0^\infty=0$. By the results of the previous section, the quasi-Wronskian[7] $F_2G_1-F_1G_2$ vanishes at the upper limit for all R solutions and at the lower limit for all L solutions. For eigensolutions, it vanishes at both limits and (17b) is fulfilled. If one of the Ψ functions corresponds to a continuum state and the other to the discrete spectrum, the same result applies. For both states in the continuum, Eq. (17b), of course, does not apply, and normalization in the scale of $f(W)$, where f is a function of W, is utilized in the usual manner.

V. UNIDIRECTIONAL ROTATION THEOREM

In this and the following two sections, we develop certain theorems describing the behavior of the nodes of left, right, and eigensolutions. Retaining the notation that left and right solutions are designated with the subscripts L and R, we write the wave functions without subscripts (a) whenever it is unnecessary to specify that the solution is L or R and (b) for eigensolutions. The context will distinguish between these two cases.

Consider a cartesian 2-space in which the coordinates are G and F, so that a vector from the origin to the point (G, F) makes an angle φ with the G axis. Then, the unidirectional rotation theorem states that as r changes, the aforementioned vector crosses the coordinate axis ($F=0$, $G=0$) with a one-signed sense of rotation. There is one exceptional case of minor interest, which is discussed later (footnote 8). Leaving this case aside at present and introducing the notation $S(x)$ = sign of x, the theorem states that

$$S(\partial\varphi/\partial r)_{G=0}=S(\partial\varphi/\partial r)_{F=0} \quad (18)$$

is always positive or always negative at every internal node of F and G. In the following, references to nodes will imply internal nodes, unless explicit statement to the contrary is made.

It is useful to introduce $\rho=F/G=\tan\varphi$. Then, ρ fulfills the Riccati equation

$$\rho'=2\kappa\rho/r+\alpha_{12}-\alpha_{21}\rho^2. \quad (19)$$

Since $\partial\varphi/\partial r=(1+\rho^2)^{-1}\partial\rho/\partial r$, we have

$$S(\partial\varphi/\partial r)=S(\partial\rho/\partial r),$$

so that the theorem is equivalent to the statement that $\partial\rho/\partial r$ is one-signed at the (simple) poles and nodes of ρ. From Eq. (19),

$$S(\partial\rho/\partial r)_{\rho=\infty}=-S(\alpha_{21}),$$

and

$$S(\partial\rho/\partial r)_{\rho=0}=S(\alpha_{12}). \quad (20)$$

We now introduce the transformation

$$\mathfrak{F}=r^{-\kappa}F, \quad \mathfrak{G}=r^{\kappa}G$$

which is to be applied only in the open interval $(0-\infty)$, so that $\mathfrak{F}$ and $\mathfrak{G}$ are finite. Moreover, the internal nodes of $\mathfrak{F}$, $\mathfrak{G}$ coincide with those of F, G, respectively. Then

$$\mathfrak{F}'=a(r)\mathfrak{G}, \quad \mathfrak{G}'=b(r)\mathfrak{F}, \quad (21)$$

where $a(r)=\alpha_{12}r^{-2\kappa}$, $b(r)=\alpha_{21}r^{2\kappa}$. It is clear that wherever $\mathfrak{F}$ or $\mathfrak{G}$ has a node[8]

$$S(\mathfrak{F}'/\mathfrak{F})=-S(\mathfrak{G}'/\mathfrak{G})$$

at points in the immediate vicinity of the node, as can be seen immediately by an elementary graphical repre-

[7] In the nonrelativistic limit, $\frac{1}{2}(F_2G_1-F_1G_2)$ reduces to the Wronskian $G_2'G_1-G_1'G_2$. For the case $|\lim(rV)|=1$, see the remarks following Eqs. (12a).

[8] It is obvious that the rotation theorem is valid for any node which occurs in the region in which $\alpha_{12}<0$ (since we take α_{21} to be positive definite) and that such a region must exist in order that bound states exist. It is evident also that inside the turning point ($\alpha_{12}<0$), the extrema of $\mathfrak{F}$ and $\mathfrak{G}$ are concave toward the r axis. However, outside the turning point ($\alpha_{12}>0$), it is possible that an extreme of $\mathfrak{G}$, with opposite curvature, and a node of $\mathfrak{F}$ may occur at one point only. At such a point, $S(\partial\varphi/\partial r)>0$. This exceptional case occurs only for L-solutions which are not at the same time eigensolutions and for only a restricted energy range between the eigenvalues. We shall not make explicit mention of this exceptional case in the following, since its consideration does not affect our main conclusions.

sentation of the functions. This implies that nodes can exist only where

$$\mathfrak{F}'\mathfrak{G}'/\mathfrak{F}\mathfrak{G}=ab<0,$$

or where

$$S(\alpha_{12})=-S(\alpha_{21}). \tag{22}$$

From the continuity of α_{12} and α_{21}, this applies at the position of the node as well as in the vicinity of the node. Therefore, from Eq. (20)

$$S(\partial\rho/\partial r)_{\rho=0}=S(\partial\rho/\partial r)_{\rho=\infty}$$

which is equivalent to the theorem as stated. For negative definite V at finite r, we have $\alpha_{21}>0$.[9] This implies a clockwise sense of rotation for φ in crossing the quadrant lines. For the sake of definiteness, we shall consider this case in all further discussion.[8]

An important consequence of the result just obtained is that we can establish the relation between the number of nodes of F and G for L, R, and LR solutions. For L solutions and for eigensolutions (LR) as well, the value of φ at the origin is given by Eqs (11c) and (11d) for admissible potentials. Thus, for potentials attractive at the origin, $\varphi_L(0)$ is in the first quadrant ($\pi/2\geqslant\varphi>0$) for $\kappa>0$, and in the fourth quadrant for $\kappa<0$, ($0\geqslant\varphi>-\pi/2$). In the following, the number of internal nodes of F and G is denoted by n_F and n_G, respectively.

Corollary 1. For L solutions: With $\kappa>0$, $n_F-n_G=0$ or 1 depending upon whether $\rho_L(\infty)$ is positive or negative, respectively. If $n_F=n_G$, the nodes of F and G can be set into one to one correspondence so that each F node precedes the corresponding node of G. If $n_F=n_G+1$, the same correspondence can be set up between the first n_F-1 nodes of F and n_G nodes of G, and again the nodes of F precede the nodes of G and the last G node (node of maximum r) is followed by the last F node. With $\kappa<0$, $n_G-n_F=0$ or 1, and the nodes of G precede the nodes of F in the sense described above. In both cases (κ greater than or less than zero), the nodes of F and G separate each other.

From the asymptotic behavior at ∞ (Sec. III(B)), we have

$$\rho_R(\infty)=-[(1-W)/(1+W)]^{\frac{1}{2}}$$

for R solutions (see paragraph following Eq. (14)). Therefore, $\varphi_R(\infty)$ is in either the second or fourth quadrant. The clockwise rotation of φ, with increasing r at the nodes, then immediately leads to the following result.

Corollary 2. For R solutions, $n_F-n_G=0$ or 1, and the nodes of F precede the nodes of G as r decreases from infinity. Also, the F and G nodes separate each other. These results apply for both signs of κ.

Combining the results of Corollaries 1 and 2, we obtain the following.

Corollary 3. For an eigensolution,

$$\begin{aligned} &n_F=n_G+1, && \kappa>0\\ &n_F=n_G, && \kappa<0. \end{aligned} \tag{23}$$

For $\kappa>0$, the first and last nodes (considering F and G together) are F nodes. For $\kappa<0$, the nodes of G precede those of F as r increases. The nodes of F and G separate each other in both cases (κ greater than or less than zero).

VI. OSCILLATION THEOREMS

We consider a field such that at least two eigensolutions exist. The eigenvalues may be arranged in order of increasing numerical value W_0, W_1, ..., and the corresponding eigenfunctions are Ψ_0, Ψ_1, ..., where

$$\Psi_m=\begin{bmatrix}F_m\\ G_m\end{bmatrix}.$$

[9] This is, of course, a sufficient condition but not a necessary one.

We proceed to establish the following oscillation theorem. If the number of nodes of the eigenfunctions F_m and G_m are denoted by $n_F(m)$ and $n_G(m)$, then

$$n_G(m+1)=n_G(m)+1, \quad n_F(m+1)=n_F(m)+1. \tag{24}$$

From Corollary 3 of the preceding section, only one of the results, Eqs. (24), need be established; the other then follows immediately from Eqs. (23).[10]

In order to establish the oscillation theorem (24), we make use of the following lemma. For L solutions, φ is a monotonic decreasing function of W, while for R solution, φ is a monotonic increasing function of W. We apply the adjoint theorem (17a) to two functions Ψ_1 and Ψ_2 corresponding to energies W and $W+dW$, respectively; then, dropping the subscript 1 on the wave functions,

$$\left[F\frac{\partial G}{\partial W}-G\frac{\partial F}{\partial W}\right]_{r_1}^{r_2}=\int_{r_1}^{r_2}\Psi^2dr. \tag{25}$$

For L solutions, we apply Eq. (25) in the interval 0 to r and note that (see Sec. III(A))

$$\lim_{r\to 0}(G_L^2\partial\rho_L/\partial W)=0$$

for all admissible potentials. Then

$$G_L^2\partial\rho_L/\partial W=-\int_0^r\Psi^2dr. \tag{26a}$$

For R solutions, we apply Eq. (25) in the interval r to ∞ and again note that (see Sec. III(B))

$$\lim_{r\to\infty}(G_R^2\partial\rho_R/\partial W)=0$$

in all cases. Then

$$G_R^2\partial\rho_R/\partial W=\int_r^\infty\Psi^2dr. \tag{26b}$$

In terms of φ, Eqs. (26a) and (26b) become

$$\partial\varphi_L/\partial W=-\Psi_L^{-2}\int_0^r\Psi_L^2dr \tag{27a}$$

and

$$\partial\varphi_R/\partial W=\Psi_R^{-2}\int_r^\infty\Psi_R^2dr \tag{27b}$$

which establishes the lemma.

It follows that $\Delta\varphi(r)=\varphi_R(r)-\varphi_L(r)$ is a bounded differentiable and monotonic function of W. If we select a particular internal point r_0, the existence of an eigensolution, which requires that $\Psi_L(r_0)=\Psi_R(r_0)$, implies that

$$\Delta\varphi(r_0)=\nu\pi, \tag{28}$$

where ν is an integer. We designate by W_ν the value of W for which Eq. (28) is fulfilled. Then, as W increases, the next eigenfunction will be realized when $W=W_{\nu+1}$, at which point

$$\Delta\varphi(r_0)=(\nu+1)\pi. \tag{28a}$$

The existence of $W_{\nu+1}$ is assured by the premise that at least two eigenfunctions and the corresponding eigenvalues exist.

Taking for $\varphi_L(0)$ its principal value, it is clear that G_L will have n nodes if $(n-\frac{1}{2})\pi\leqslant-\varphi_L(r_0)<(n+\frac{1}{2})\pi$, where the equality holds if r_0 is a node of G_L. Consequently, the number of G_L nodes in the interval $0-r_0$ is[11]

$$n_G(0-r_0)=\langle-\pi^{-1}\varphi_L(r_0)-\tfrac{1}{2}\rangle, \tag{29}$$

where the symbol $\langle x\rangle$ is the smallest integer *greater* than x. Similarly, taking $\varphi_R(\infty)$ to have its principal value, the number of nodes of G_R in the interval r_0 to ∞ is

$$n_G(r_0-\infty)=\langle\pi^{-1}\varphi_R(r_0)-\tfrac{1}{2}\rangle.$$

[10] The subscript m on W and Ψ is not necessarily equal to the number of nodes of F or G. See text following Eq. (32).

[11] Obviously, we can always choose r_0 so that $G(r_0)\neq 0$. However, with the meaning assigned to $\langle x\rangle$, it is evident that Eq. (29) is valid in any case.

It follows that the total number of nodes of G is

$$n_G=\langle-\pi^{-1}\varphi_L(r_0)-\tfrac{1}{2}\rangle+\langle\pi^{-1}\varphi_R(r_0)-\tfrac{1}{2}\rangle+\epsilon, \tag{30}$$

where $\epsilon=0$, if $G(r_0)\neq 0$, and $\epsilon=-1$, if $G(r_0)=0$, so that the node at r_0 is counted only once. In the former case, we write

$$\begin{aligned}-\pi^{-1}\varphi_L(r_0)-\tfrac{1}{2}&=\nu_1+\delta,\\ \pi^{-1}\varphi_R(r_0)-\tfrac{1}{2}&=\nu-\nu_1-2+1-\delta\end{aligned} \tag{31}$$

which agrees with Eq. (28). In Eqs. (31), ν_1 is an integer ($\geqslant -1$, see Sec. V), and $0<\delta<1$. It follows then that

$$n_G=\nu_1+1+\nu-\nu_1-1=\nu.$$

If r_0 is a node, $\delta=0$ and $n_G=\langle\nu_1\rangle+\langle\nu-\nu_1-1\rangle-1=\nu$. Thus, in any case, the number of nodes of G is given by

$$n_G=\Delta\varphi(r_0)/\pi. \tag{32}$$

Consequently (see Eq. (28a)), the number of nodes of G and, by Corollary 3, of F each increases by unity as W increases from one eigenvalue to the next.

The eigenfunction G, corresponding to the eigenvalue W_ν, is constructed in the following way. For $r\leqslant r_0$, $G=G_L$; for $r>r_0$, $G=G_R$ if ν is even, and $G=-G_R$ if ν is odd.

It must not be concluded that for the nth eigenfunction G will have $n-1$ nodes. This is actually the case for Schroedinger problem, where the lowest eigenfunction is always nodeless. However, in the relativistic case, there need not be a nodeless eigenfunction for a particular hamiltonian. For example, for $\kappa=1$, an attractive square well of unit radius and depth $V_0=-10$ has a lowest eigenfunction with two internal nodes. In this case, the well has been made so deep and broad that the first two values of W, for which the wave functions F and G are continuous at the radius of the well, have been pushed below $W=-1$. The essential difference between the relativistic and nonrelativistic cases is the existence of a lower bound for the discrete energy spectrum which must be recognized in the case of the former (see also Sec. VII).

A second oscillation theorem is concerned with the direction in which the nodes of F and G move as W increases. Let r_n be a node of F. Then in the vicinity of r_n

$$\rho=(r-r_n)\alpha_{12}(r_n)$$

and

$$(\partial\rho/\partial W)_{r_n}=(\partial\varphi/\partial W)_{r_n}=-\alpha_{12}(r_n)\partial r_n/\partial W.$$

From the results of Sec. V, we conclude that

$$S(\partial\varphi(r_n)/\partial W)=S(\partial r_n/\partial W). \tag{33}$$

Similarly, let r_m be a node of G. Then in the vicinity of r_m

$$1/\rho=(r-r_m)\alpha_{21}(r_m)$$

and

$$[(\partial/\partial W)/(1/\rho)]_{r_m}=-\alpha_{21}(r_m)(\partial r_m/\partial W)$$

or

$$\partial\varphi(r_m)/\partial W=\alpha_{21}(r_m)(\partial r_m/\partial W).$$

Thus,

$$S(\partial\varphi(r_m)/\partial W)=S(\partial r_m/\partial W). \tag{33a}$$

It follows from Eqs. (26a) and (26b) that as W increases the nodes of both F and G move to the left for L solutions and to the right for R solutions.

VII. NONRELATIVISTIC LIMIT

Certain of the properties of the relativistic wave functions, as discussed in the foregoing, are known to apply to the Schroedinger wave functions. There are certain questions connected with the Schroedinger problem, in particular boundary conditions and admissible potentials, which cannot be discussed as direct reductions of the preceding analysis to the nonrelativistic limit.

In the limit $|W-1|\ll 1$, $|V|\ll 1$, but $W-1=E\sim V$; the first-order Eqs. (2a) go over to the second-order nonrelativistic wave equation with G equal to r times the radial Schroedinger function. Here, one makes the identification $\kappa(\kappa+1)=l(l+1)$; that is, $\kappa=l$ or $\kappa=-l-1$. To effect a close parallelism with the preceding discussion, we can take $\kappa=-l-1$, so that $\kappa\neq 0$. However, it is sometimes convenient to take $\kappa=0$ (see Eq. (35) and the subsequent discussion). Since $\alpha_{21}\to 2$, and is, therefore, positive definite in the closed interval $(0,\infty)$, the zero of α_{21} which occurs in the relativistic case at $W=-1$, $r=\infty$ has disappeared. For this reason, the discrete eigenvalue spectrum in the nonrelativistic case has only an upper bound ($E=0$), and, by the usual argument, there is always a nodeless eigenfunction.

Although the Schroedinger equation is a limiting form of the Dirac equation, the corresponding statement for the boundary conditions does not seem to be true. Instead, the boundary conditions for the nonrelativistic problem are:

(a) Square integrability:

$$0<\int_a^b G^2dr<\infty,\quad 0\leqslant a<b\leqslant\infty. \tag{34a}$$

This is the limiting form of condition (3a), since F, as defined above, is the so-called "small component" which vanishes in the nonrelativistic limit.

(b) Lower bound to the continuous spectrum: Without such a condition, the only stable state would have energy $E=-\infty$.

(c) Invariance under extended point transformations: At singular points of such a transformation, a function which satisfies the wave equation in one coordinate system may not satisfy it in others. An example of a singular point is the origin in the transformation from Cartesian to polar coordinates. This condition excludes[12] a singularity as strong as r^{-1} for the Schroedinger ψ, and hence,

$$\lim_{r\to 0}G=0. \tag{34b}$$

This condition also applies to the Dirac equations, but, as has been demonstrated in Sec. III, Eq. (34b) is automatically fulfilled when conditions (3a) and (3b) are met, and Eq. (34b) was unnecessary. It will be noted that while finiteness of $\psi=G/r$ is not required *a priori*, the indicial behavior of G (see below) is such that Eq. (34b) is equivalent to the statement that ψ is finite at $r=0$.

Studying the Schroedinger equation by our methods requires changing from a second-order equation to a pair of first-order equations. This process is not unique. For studying nodal properties, as in Secs. V and VI, the most convenient choice is:

$$F'=-\chi G,\quad G'=F, \tag{35}$$

[12] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, New York, 1947), third edition, p. 155.

where $\chi=2(E-V)-l(l+1)/r^2$. A left-handed solution is one for which $G(0)=0$, so that square integrability is also assured; a right-handed solution is one which is square integrable near infinity, that is, in condition (34a), $a>0$, $b=\infty$. Corollaries 1 and 2, and the first of Eqs. (23) under Corollary 3, which is valid for $\kappa=0$, are immediate consequences of the continuity of G and G' and of $G_L(0)=G_R(\infty)=0$. The results of Sec. VI also apply, if $\kappa=0$, and are immediately applicable.

For studying admissible potentials a more convenient set of first-order equations is

$$F'=-(\chi'/2\chi)F-\chi^{\frac{1}{2}}G, \quad G'=\chi^{\frac{1}{2}}F. \tag{36}$$

Introducing A and φ as in Eqs. (7), we get

$$\begin{aligned}\ln A_1-\ln A_0&=-\int_{r_0}^{r_1}(\chi'/2\chi)\sin^2\varphi dr,\\ \varphi_1-\varphi_0&=-\int_{r_0}^{r_1}(\chi'/2\chi)\sin\varphi\cos\varphi dr-\int_{r_0}^{r_1}\chi^{\frac{1}{2}}dr.\end{aligned} \tag{37}$$

We shall consider only attractive potentials; one readily sees that there is always one and only one left-handed solution for repulsive potentials. We consider three cases based upon the behavior of $\lim(r^2V)$ as $r\to 0$.

(1) $\lim(r^2V)=-\infty$. Then $\lim(r^2\chi)=\infty$, for any l, and for sufficiently small r, $\chi>0$, $\chi'<0$. Then $\int_{r_0}^{r_1}\chi^{\frac{1}{2}}dr$ in Eqs. (37) diverges as $r_0\to 0$, and φ is infinite at the origin. Further, from the signs of χ' and χ, we see that $\ln A\to-\infty$, and A approaches zero at the origin. Hence, all solutions of (36) are L solutions. We can always find at least one R solution for the Schroedinger equation. Hence, for all E, the wave equation and boundary conditions (a) and (c) given above can be satisfied. Therefore, (b) is violated, and attractive potentials more singular than r^{-2} are inadmissible.

(2) $\lim(r^2V)=0$. Here χ has a different behavior for $l=0$ and $l>0$.

(2a) $l=0$. Then $\lim(r^2\chi)=0$. Proceeding as with the Dirac equations in Sec. III(A), we can find a finite φ_0 at the origin, with $\sin 2\varphi_0=0$. Setting $\varphi=\varphi_0+\theta$, and making the choice $\cos 2\varphi_0=-1$, we find, near the origin $\theta=-r\chi^{\frac{1}{2}}$, which approaches zero in agreement with assumption. This choice further makes A_0 approach zero and gives an L solution. The other possible choice ($\sin 2\varphi\to 0$, $\cos 2\varphi\to 1$ corresponding to the second linearly independent solution) makes A remain greater than zero at the origin, and Eq. (34b) is violated. Hence, there is only one L solution. For this solution, $A\sim\chi^{-\frac{1}{4}}$, $\cos\varphi\sim r\chi^{\frac{1}{2}}$, and $G\sim r$.

(2b) $l>0$. Then $\lim(r^2\chi)=-l(l+1)$, and φ takes on complex values sufficiently near the origin. The phase-amplitude method is not convenient for this case. However, the behavior of G near the origin now depends only upon l and not upon V. The well-known indicial behavior of G is r^ν, where

$$\nu=-l,\ l+1.$$

Only one solution, with $\nu=l+1$, satisfies the boundary conditions.

Hence, if $\lim(r^2V)=0$, there is one and only one L solution for all l, and such potentials are admissible. G behaves like r^{l+1} near the origin.

(3) $\lim(r^2V)=-\beta\neq 0$. The indicial behavior of G is r^ν, where

$$\nu=\tfrac{1}{2}+[\tfrac{1}{4}-2\beta+l(l+1)]^{\frac{1}{2}}.$$

With $l=0$, the real part of ν is positive for all β, and all solutions are left-handed. Therefore, boundary condition (b) is violated, and all such potentials are inadmissible.

Thus, on the basis of a consideration of bound states, the only admissible potentials which are attractive at the origin are those for which $\lim(r^2V)=0$ as $r\to 0$, whereas no restrictions need be placed on repulsive potentials. This is in contrast to the Dirac result (Sec. III), where the potentials could be classified as admissible or otherwise, without regard to the sign of V. This symmetry between attractive and repulsive potentials in the relativistic case is, of course, understandable in terms of energy doubling (quadratic energy-momentum relationship).[13]

At first glance one might conjecture that, at least in the Schroedinger case, the physical basis of admissibility of a potential is the requirement that the attraction shall not predominate over the centrifugal repulsion. The foregoing shows that this interpretation is not correct, and, in the Dirac case, such an interpretation would be definitely ambiguous.

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[13] The second-order equations for the relativistic F and G are independent of the sign of V, in the region where V is large, in all terms including those arising from the effect of spin.