Heisenberg Operators in Quantum Electrodynamics. I

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A new analytic continuation principle is described, by means of which the calculation of matrix elements of Heisenberg operators in any quantized field theory is greatly simplified. By a "Heisenberg operator" is meant an average over a finite space-time region of a field operator in the Heisenberg representation of the theory. The analytic continuation is made by varying the characteristic masses of the fields through real values. In this way a Heisenberg operator with the physically occurring masses is derived from an operator calculated with very large fictitious masses. In the region of large masses, where real creation of particles is impossible, the operator is identical with the S-matrix for a suitably chosen scattering problem. The calculation reduces to the calculation of an S-matrix, to which the techniques of Feynman are directly applicable.

The S-matrix itself is a nonanalytic function of the masses in the region of values where thresholds for real processes occur. A special device is introduced in order to bypass the region of nonanalyticity. The Heisenberg operator is modified by the insertion of real exponential damping factors which have the effect of making all energy denominators in a perturbation expansion complex. The modified operator is an analytic function of the masses and of the damping coefficients, for all real positive values of the masses. The analytic continuation is made by varying the masses while the damping coefficients are non-zero, letting the damping coefficients tend to zero when the physical values of the masses have been reached.

I. INTRODUCTION

IT is generally believed that quantum electrodynamics, in spite of its inherent divergences, constitutes a consistent and meaningful theory. That is to say, given any directly measurable physical effect, the theory will predict for it a finite and unambiguous value, or statistical distribution of values. It has become possible to use the theory consistently, and to circumvent the divergence difficulties, by using the idea of renormalization of constants, an idea originally suggested by Kramers[1] and Bethe,[2] and worked out in detail by Schwinger,[3] Tomonaga,[4] Feynman,[5] and others. All these authors, however, have only applied the theory to particular problems. No general proof exists that the theory will give finite values for all measurable quantities in all situations.

A start in the construction of such a proof was made by the author,[6] who showed that all matrix elements of the S-matrix in quantum electrodynamics become finite after the renormalizations of mass and charge are carried out. This analysis of the S-matrix falls short of what is required in two important respects. (i) The S-matrix is written down as a formal series expansion in powers of the coupling constant e, assuming the series to be convergent after the renormalizations are carried out; the convergence of the series is never proved, and the series is in fact certainly not convergent in problems in which bound states are involved; thus, the analysis is applicable only to pure scattering processes between free particles. (ii) The S-matrix describes the results of measurements of the over-all behavior of a system integrated over space-time; it does not include a description of local measurements, for example, measurements of field-strengths and current-densities in finite space-time regions. The purpose of the present series of papers is to remove the defect (ii) of the previous work. It will be proved that quantum electrodynamics gives well-defined finite values for all locally measurable quantities, whenever the expansion in powers of e is convergent. Incidentally, it will appear that our methods provide a basis for removing defect (i) also from the analysis; this question will be discussed in later papers of the series.

By a locally measurable quantity in quantum electrodynamics[7] is meant an operator such as the field-average

$$\mathbf{F}_{\mu\nu}(R)=(1/V)\int_R \mathbf{F}_{\mu\nu}(x)d_4x, \tag{1}$$

where $\mathbf{F}_{\mu\nu}(x)$ is an electromagnetic field-strength operator in the Heisenberg representation, and R is a finite space-time region of 4-dimensional volume V. We use heavy type for Heisenberg representation operators, light type for interaction representation operators. Another example of a measurable quantity is

$$\mathbf{j}_\mu(S)=(1/V)\int \mathbf{j}_\mu(x)S(x)d_4x; \tag{2}$$

[1] H. A. Kramers, "Non-relativistic quantum-electrodynamics and correspondence principle," Solvay Conference Report, Brussels, 1948.
[2] H. A. Bethe, Phys. Rev. **72**, 339 (1947).
[3] J. Schwinger, Phys. Rev. **74**, 1439 (1948); **75**, 651 (1949); **76**, 790 (1949).
[4] T. Tati and S. Tomonaga, Prog. Theor. Phys. **3**, 391 (1948); Fukada, Miyamoto, and Tomonaga, Prog. Theor. Phys. **4**, 47 and 121 (1949). Also earlier papers in the same journal.
[5] R. P. Feynman, Phys. Rev. **76**, 749 and 769 (1949); **80**, 440 (1950).
[6] F. J. Dyson, Phys. Rev. **75**, 486 and 1736 (1949). These papers will be referred to hereafter as (A) and (B).
[7] The author is much indebted to Professor W. Pauli and Dr. Res Jost, who pointed out to him the necessity for working in terms of such field-averages. An unpublished calculation of Dr. Jost, communicated to the author by letter in March, 1950, included many of the ideas of the present series of papers.

here $\mathbf{j}_\mu(x)$ is a current-density operator in the Heisenberg representation, and $S(x)$ is a function which is zero outside R and unity inside R, except in a thin shell in the neighborhood of the boundary of R in which it changes smoothly from 0 to 1. As is well known,[8] the smoothness of $S(x)$ at the boundary of R is necessary in case (2) in order to obtain an operator whose mean-square fluctuation is finite. Operators such as (1) and (2) can be built up by linear superposition of the Fourier-transformed operators

$$\mathbf{F}_{\mu\nu}(p)=(1/2\pi)^4\int \mathbf{F}_{\mu\nu}(x)e^{-ip\cdot x}d_4x, \tag{3}$$

$$\mathbf{j}_\mu(p)=(1/2\pi)^4\int \mathbf{j}_\mu(x)e^{-ip\cdot x}d_4x, \tag{4}$$

where p is an arbitrary 4-vector in momentum-space. The program of this and the next paper in the series will be to prove in detail that the operator (4) is finite after all renormalizations have been carried out, including a renormalization of the unit of charge in terms of which (4) is measured. A simple argument will then show that (3) is finite after a corresponding renormalization of the unit of field-strength. It will be clear that the methods are of general validity and could be applied equally well to any locally measurable operator of the same form as (1) or (2).

The scope of the present paper is to formulate a general expansion theorem, expressing Heisenberg operators such as (3) and (4) in terms of interaction representation operators.[9] The second paper of the series will carry through the renormalization program for these operators, using the expansion theorem as a mathematical basis. The statement of the expansion theorem occupies Sec. VIII of the present paper. The theorem is surprisingly simple, being formally identical with the corresponding expansion theorem for the S-matrix which was the basis of the previous analysis. The calculation of Heisenberg operators is thereby made fully as easy as the calculation of the S-matrix. There is no longer any advantage, for practical calculations, in using the S-matrix method in preference to the original contact-transformation method of Schwinger, which works in terms of Heisenberg operators. In fact, the two methods of calculation differ only when in the S-matrix a higher order radiative effect is mixed with contributions from real effects occurring in lower order; and just in these circumstances the Heisenberg operators are simpler to use, since they isolate the true higher order effects from the others. It will be found that, in general, for radiation problems which are not direct calculations of scattering matrix elements, the Heisenberg operators are a more appropriate practical tool than is the S-matrix.

[8] W. Heisenberg, Leipzig. Ber. **86**, 317 (1934).

[9] Expressions for the Heisenberg operators in terms of interaction operators have been studied previously by G. Källén, Arkiv Fysik **2**, 187 (1950). D. Feldman and C. N. Yang, Phys. Rev. **79**, 972 (1950).

II. FORMAL MANIPULATIONS

The expression for $\mathbf{j}_\mu(x)$ in terms of interaction representation operators is

$$\mathbf{j}_\mu(x)=S^{-1}(t)j_\mu(x)S(t), \tag{5}$$

where t is the time at the space-time point x, and the transformation operator $S(t)$ is [10]

$$S(t)=\sum_{n=0}^{\infty}\left(-\frac{i}{\hbar c}\right)^n\int_{-\infty}^{t}dx_1\int_{-\infty}^{t_1}dx_2\cdots\int_{-\infty}^{t_{n-1}}dx_nH^I(x_1)H^I(x_2)\cdots H^I(x_n). \tag{6}$$

Here,

$$H^I(x)=H_1(x)+H_s(x), \tag{7}$$

$$H_1(x)=-(1/c)j_\mu(x)A_\mu(x), \tag{8}$$

$$H_s(x)=-\delta mc^2\bar{\psi}(x)\psi(x), \tag{9}$$

$$j_\mu(x)=iec\bar{\psi}(x)\gamma_\mu\psi(x), \tag{10}$$

all these being interaction representation operators. Expanding Eq. (5) gives an expression for $\mathbf{j}_\mu(x)$ as a series of multiple commutators

$$\mathbf{j}_\mu(x)=\sum_{n=0}^{\infty}\left(\frac{i}{\hbar c}\right)^n\int_{-\infty}^{t}dx_1\int_{-\infty}^{t_1}dx_2\cdots\int_{-\infty}^{t_{n-1}}dx_n[H^I(x_n),[\cdots,[H^I(x_2),[H^I(x_1),j_\mu(x)]]\cdots]]. \tag{11}$$

By a rearrangement of terms, this series may be written in the alternative form

$$\mathbf{j}_\mu(x)=\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\left(\frac{i}{\hbar c}\right)^q\left(-\frac{i}{\hbar c}\right)^p\frac{1}{q!p!}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_{q+p}\,\bar{P}(H^I(x_1),\cdots,H^I(x_q))P(H^I(x_{q+1}),\cdots,H^I(x_{q+p}),j_\mu(x)), \tag{12}$$

where the P-bracket denotes, as usual, a chronologically ordered product, and the $\bar{P}$-bracket is a product ordered antichronologically, reading the order of factors in both cases from right to left.

In an expression such as Eq. (11), involving products of many interaction representation operators, all kinds of higher order virtual processes are implicitly included. The aim of our analysis is to find an expansion of $j_\mu(x)$ in which virtual processes are represented explicitly, so that the divergences which arise from virtual processes can be recognized and eliminated. Stated precisely, our aim is to express $j_\mu(x)$ as a sum of "normal products" multiplied by c-number coefficients; a normal

[10] The notation is not exactly the same as in references (A) and (B).

product is defined to be a product of free particle emission and absorption operators in which all emission operators stand to the left of all absorption operators,[11] so that no virtual emission and reabsorption processes are implicitly represented in it. The free particle operators are defined to be the coefficients $A_\mu(k)$, $\psi_\alpha(k)$, $\bar{\psi}_\alpha(k)$ in the momentum representations

$$\left.\begin{aligned} A_\mu(x) &= \int dk A_\mu(k)\exp(ik\cdot x) \\ \psi_\alpha(x) &= \int dk \psi_\alpha(k)\exp(ik\cdot x) \\ \bar{\psi}_\alpha(x) &= \int dk \bar{\psi}_\alpha(k)\exp(ik\cdot x) \end{aligned}\right\} \tag{13}$$

of the interaction representation operators, the coefficients with positive values of the fourth component k_0 being by definition absorption operators, those with negative values of k_0 emission operators. The reasons for expressing $j_\mu(x)$ as a sum of normal products are the following. Given any initial and final states specified by definite numbers of free particles with definite polarizations and momenta, there exists precisely one normal product with a non-zero matrix element between these two states. Therefore, a decomposition of $j_\mu(x)$ into normal products is equivalent to a complete listing of the matrix elements of $j_\mu(x)$ in the representation in which free-particle occupation numbers are diagonal. The effects of virtual processes are represented explicitly in the numerical factors multiplying the various normal products.

There is a general theorem due to Wick[12] which states that every product Q of interaction representation operators $\bar{\psi}_\alpha(x)$, $\psi_\beta(y)$, $A_\mu(z)$ has a unique decomposition into a sum of normal products. The decomposition is a consequence only of the commutation relations between the factors of Q; it is an operator identity, independent of the particular states in which we may be interested. Just for this reason, the rules for decomposing Q into a sum of normal products are simpler than the rules given in the author's earlier papers[6] for writing down the matrix elements of Q between given states. The two sets of rules are equivalent, but there is a great advantage in working with the normal product decomposition because this method makes unnecessary any explicit reference to the vacuum state of the fields.

Wick has stated and proved his theorem only for a chronologically ordered product Q. The theorem and the method of proof are valid, however, for all products. We shall here state the theorem in its general form, referring to Wick's paper for the proof.

[11] The decomposition of an operator into normal products has been frequently used in the past, for example by E. G. C. Stueckelberg, Nature **153**, 143 (1944); A. Houriet and A. Kind, Helv. Phys. Acta **22**, 319 (1949).

[12] G. C. Wick, Phys. Rev. **80**, 268 (1950).

We define a "factor pairing" of Q in the following way. A certain number (in particular this number may be all, or none, or any intermediate even number) of the factors of Q are associated together into pairs, the members of each pair being either one $\bar{\psi}$ and one ψ, or two A_μ operators. The remaining factors are left unpaired. Two factor-pairings are regarded as distinct, even when the number of pairs and the nature of the factors composing them are the same in both, provided that the positions in Q of the paired factors are different in the two cases.

To each factor-pairing corresponds a "rearranged form" of Q as follows. First, one pair of factors P_1 is brought to the left side of Q, the order of factors in the pair being kept as it was in Q. Then another pair of factors P_2 is brought to the left and placed immediately to the right of the first pair, and so on. The order in which the pairs are chosen is not significant. When all the pairs have been picked out, the unpaired factors are left, forming a product U. The rearranged form $F=P_1P_2\cdots U$ is given the sign plus or minus according as the permutation of electron-positron factors in going from Q to F is even or odd.

Corresponding to any given product U we define the "normal product" $N(U)$. First, the factors of U are decomposed according to Eq. (13) into sums of particle emission and absorption operators; U is then a sum of products of these elementary operators. $N(U)$ is formed from U by rearranging independently each of the products of elementary operators so that absorption operators stand to the right of emission operators, giving to each rearranged product independently the sign plus or minus according as the permutation of electron-positron factors was even or odd.

Finally, a "normal constituent" of Q is obtained from each rearranged form F as follows. The unpaired factors U are replaced by $N(U)$. Each pair of factors $\psi_\alpha(x)\bar{\psi}_\beta(x')$ is replaced by the C-number

$$-iS_{\alpha\beta}^{+}(x-x') = [-i/(2\pi)^3]\int d_4k\theta(k)\delta(k^2+m^2) \times (k_\mu\gamma_\mu+im)_{\alpha\beta}\exp[ik\cdot(x-x')]. \tag{14}$$

Each pair of factors $\bar{\psi}_\beta(x')\psi_\alpha(x)$ is replaced by

$$-iS_{\alpha\beta}^{-}(x-x') = [i/(2\pi)^3]\int d_4k\theta(-k)\delta(k^2+m^2) \times (k_\mu\gamma_\mu+im)_{\alpha\beta}\exp[ik\cdot(x-x')]. \tag{15}$$

Each pair $A_\mu(x)A_\nu(x')$ is replaced by

$$i\hbar c\delta_{\mu\nu}D^{+}(x-x') = [\hbar c/(2\pi)^3]\delta_{\mu\nu}\int d_4k\theta(k) \times \delta(k^2+\lambda^2)\exp[ik\cdot(x-x')]. \tag{16}$$

Here, $\theta(k)$ is the step-function defined by

$$\theta(k)=1, \quad k_0>0, \qquad \theta(k)=0, \quad k_0<0, \tag{17}$$

where k_0 is the fourth component of the 4-vector k; m is the reciprocal of the electron Compton wavelength. λ is a real constant which has the value zero when A_μ is the electromagnetic field. It is essential to our method that we carry through the analysis also for non-zero values of λ; this means that we consider A_μ to be either the electromagnetic field or a neutral vector meson field with rest-mass $(\hbar\lambda/c)$.

After these replacements, F becomes an operator E which is by definition the normal constituent of Q corresponding to the factor-pairing from which F was derived. Each E is thus a normal product multiplied by various coefficients (15), (16), (17), (14)

The theorem of Wick can now be stated concisely. *Every product Q is identically equal to the sum of its normal constituents.*

In order to decompose $j_\mu(x)$ into a sum of normal products, we choose any particular term in the expansion (11), for example,

$$I_n(x)=\int_{-\infty}^{t} dx_1 \int_{-\infty}^{t_1} dx_2 \cdots \int_{-\infty}^{t_{n-1}} dx_n [H_1(x_n), [\cdots, [H_1(x_1), j_\mu(x)]\cdots]]. \tag{18}$$

This term can be rewritten more symmetrically as

$$I_n(x)=(1/n!)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\cdots dx_n S, \tag{19}$$

where

$$S=\sum \theta(x-x_1)\theta(x_1-x_2)\cdots\theta(x_{n-1}-x_n)[H_1(x_n), [\cdots, [H_1(x_1), j_\mu(x)]\cdots]], \tag{20}$$

the summation being over the $(n!)$ permutations of the points $x_1, \cdots, x_n$. S is a sum of $(2^n n!)$ products, all of them made up of the same factors in various orders. By a factor-pairing of S we mean a simultaneous pairing of identical pairs of factors in each of the $(2^n n!)$ products. To each factor-pairing F of S there corresponds a normal constituent E which is the sum of $(2^n n!)$ normal constituents, one taken from each product. The enumeration of the factor-pairings of S is now identical with the enumeration of factor-pairings for the chronological product

$$P(H_1(x_n), \cdots, H_1(x_1), j_\mu(x)), \tag{21}$$

which was required for the calculation of the S-matrix.[6] Exactly as in the S-matrix analysis, each factor-pairing is represented by a Feynman graph G, with $(n+1)$ vertices $x, x_1, \cdots, x_n$ connected by lines indicating which operators are to be paired.

III. INTRODUCTION OF DOUBLED FEYNMAN GRAPHS

Since we are now dealing with multiple commutators rather than with simple products, the analysis of operators given by the Wick theorem and represented by ordinary Feynman graphs will no longer be sufficient. We shall develop a more refined analysis, which will lead to a graphical representation by means of a new kind of diagram called a "doubled Feynman graph."

For brevity, we use the word "exchange-term" to mean either a commutator of two A_μ operators or an anticommutator of a $\bar\psi$ and a ψ operator; we use "sum-term" to mean either an anticommutator of two A_μ or a commutator of a $\bar\psi$ and a ψ. We write also instead of Eq. (20),

$$S=\sum \Theta C, \tag{22}$$

C denoting any one of the multiple commutators in Eq. (20) and Θ the corresponding product of θ-functions which specifies a particular chronological order of the points $x_1, x_2, \cdots, x_n$. Let a factor-pairing F of S, with b factor-pairs, be given. Let the Feynman graph corresponding to F be G, and let S_G be the corresponding normal constituent of S. In constructing S_G, we express each of the functions (14), (15), (16) as half the sum or difference of an exchange term and a sum-term. Thus,

$$D^{\pm}(x)=\tfrac{1}{2}[D(x)\mp iD^1(x)], \qquad S^{\pm}(x)=\tfrac{1}{2}[S(x)\mp iS^1(x)], \tag{23}$$

where $D^1(x)$ is the even function

$$D^1(x)=(1/2\pi)^3\int d_4k\,\delta(k^2+\lambda^2)e^{ik\cdot x}, \tag{23.1}$$

and S^1 is similarly defined.[13] Then, S_G is of the form

$$S_G=N_G\sum_P PL_P. \tag{24}$$

Here, the summation is over the 2^b possible products P; each P is a product of b factors, one corresponding to each internal line of G; to an electron line of G corresponds either a factor S or a factor S^1 in P, and to a photon line corresponds either D or D^1. For each P, L_P is a certain linear combination of the products Θ appearing in Eq. (22). N_G is a normal product of the $(3n+2-2b)$ unpaired factors in F, and depends only on G and not on P. The representation of S_G in the form (24) is unique.

To obtain further information about the L_P, we use an alternative form of S from which Eq. (12) was derived, namely,

$$S=\sum_{q=0}^{n}(-1)^{n-q}\sum \bar P(H_1(x_1'), \cdots, H_1(x_q')) \times P(H_1(x'_{q+1}), \cdots, H_1(x_n'), j_\mu(x)). \tag{24.1}$$

The inner sum is here over the $[n!/q!(n-q)!]$ ways of

[13] J. Schwinger, Phys. Rev. **75**, 651 (1949), Appendix.

choosing q points $x_1', \cdots, x_q'$ out of the n points $x_1, \cdots, x_n$. It is important that in Eq. (24.1) the θ-functions no longer appear explicitly. When S_G is constructed from Eq. (24.1), each term of the inner sum gives rise to a product of functions S^+, S^-, D^+, S_F, $\bar{S}_F$, D_F, and $\bar{D}_F$. The function $D_F(x-x')$ arises from the replacement of $P(A_\mu(x), A_\nu(x'))$ according to Eq. (16), and is given by

$$D_F(x)=D^1(x)+i[2\theta(x)-1]D(x). \qquad (24.2)$$

This is the Feynman D-function, which was used extensively in (A) and (B). The function $\bar{D}_F(x-x')$ arises from the replacement of $\bar{P}(A_\mu(x), A_\nu(x'))$ and is given by

$$\bar{D}_F(x)=D^1(x)-i[2\theta(x)-1]D(x). \qquad (24.3)$$

S_F and $\bar{S}_F$ are similarly defined. Now using Eqs. (23), (24.2), and (24.3), we again obtain S_G in the form (24), but with θ-functions arising only from the factors (24.2), (24.3). Therefore, L_P is a linear combination with numerical coefficients of terms W, each W being a product of factors $\theta(x'-x'')$; we say that each term W is "correct," meaning by this that to each factor $\theta(x'-x'')$ in W there corresponds at least one exchange-term factor $S(x'-x'')$ or $S(x''-x')$ or $D(x'-x'')$ in P. Thus, S_G has been expressed in a form where every θ-function is accompanied by a corresponding exchange-term.

Let W_1 be any term in L_P which contains a set of factors forming a cycle, for example,

$$\theta(x_1-x_2)\theta(x_3-x_2)\theta(x_4-x_3)\theta(x_4-x_1). \qquad (24.4)$$

We may replace some of these factors by equivalent expressions according to the identity

$$\theta(x_i-x_j)=1-\theta(x_j-x_i). \qquad (24.5)$$

Then W_1 is split into a sum of terms W', one of which contains the factors

$$\theta(x_1-x_2)\theta(x_2-x_3)\theta(x_3-x_4)\theta(x_4-x_1) \qquad (24.6)$$

and is identically zero, while the others all contain fewer factors than does W_1. The terms W' obtained in this way are all correct. If some of them again contain cycles of factors, these W' can again be replaced by sums of terms with a smaller number of factors. In this way we shall finally arrive at an expression for L_P as a linear combination of products W, each W being correct and cycle-free.

Given any cycle-free W_2 which contains a chain of factors beginning with the point x, for example,

$$\theta(x_1-x)\theta(x_1-x_2)\theta(x_3-x_2)\theta(x_4-x_3),$$

we can replace certain factors as before by Eq. (24.5) in such a way that W_2 is replaced by a sum of correct cycle-free terms W'. One of the W' contains the chain of factors

$$\theta(x-x_1)\theta(x_1-x_2)\theta(x_2-x_3)\theta(x_3-x_4), \qquad (24.7)$$

while the others contain fewer factors than W_2. Proceeding in this way, we arrive ultimately at an expression for L_P as a linear combination of products W, each W being correct and cycle-free and also "ordered with respect to x." By this last phrase we mean that every chain of factors in W beginning with x is of the special form (24.7). A cycle-free W which is ordered with respect to x has the following property. The points $x_1, \cdots, x_n$ fall into two classes, C_1 and, C_2, one of which may be empty. There does not exist any factor $\theta(x'-x'')$ in W with x' and x'' in different classes. For every x_i in C_1, there exists one and only one chain of factors in W of the form

$$\theta(x-x')\theta(x'-x'')\cdots\theta(x'''-x_i), \qquad (24.8)$$

connecting the points x and x_i.

Each W may be expressed as a linear combination of products

$$Z_q=\theta(x-x_1')\theta(x_1'-x_2')\cdots\theta(x'_{q-1}-x_q')\theta(x'_{q+1}-x'_{q+2})\cdots\theta(x'_{n-1}-x_n'),$$

where q takes the values 0, 1, 2, $\cdots$, n, and $x_1', \cdots, x_n'$ are any permutation of the points $x_1, \cdots, x_n$. When W is cycle-free and ordered with respect to x, then an expansion of W into a sum of Z_q can be made, using only those Z_q for which the points $x_1', \cdots, x_q'$ make up the class C_1. It was proved earlier that L_P is a linear combination of the Θ occurring in Eq. (22), that is to say, a linear combination of Z_q with $q=n$. Also, L_P is a sum of W, each W being a linear combination of Z_q with q equal to the number of points in the class C_1 corresponding to W. But the representation of L_P as a linear combination of Z_q is unique; if a linear combination of the Z_q is zero, then every coefficient must be zero. Therefore, the sum of the terms W in L_P, for which the number of points in C_1 is not n, is identically zero. We need retain only the terms W for which the class C_2 is empty.

Summarizing the results of the analysis so far, we have expressed S_G in the form (24). L_P is a linear combination of products W, with numerical coefficients. Each W is a product of factors $\theta(x'-x'')$ which is correct, and cycle-free and ordered with respect to x, and connected. By connected we mean that a chain of factors (24.8) exists in W for every point x_i. This has the important consequence that *non-zero normal constituents S_G of S are obtained only from connected graphs G.* We shall suppose henceforward that G is connected, which implies $b \geqslant n$.

A "doubled Feynman graph," G_T, is defined as the graph G with a certain subset T of its internal lines drawn double. The figure formed by the doubled lines T must satisfy three conditions: (i) it is connected; (ii) it is simply connected, without closed cycles; (iii) at least one line of T is incident at every vertex of G. These conditions imply that there are just n lines in T.

To each product W we associate a doubled graph G_T in the following way. To each factor $\theta(x'-x'')$ in W there corresponds one or more exchange-term factors in P involving the two points x' and x''. If there is more than one such exchange-term, we choose one arbitrarily and ignore the others. To each factor of W corresponds then one factor in P, and hence one internal line of G. The set T of lines so obtained satisfies the three required conditions for a set of doubled lines. We call T the "order-type" of W, and G_T is the corresponding doubled graph.

The order-type T implies a partial ordering of the points $x, x_1, \cdots, x_n$ defined by the following two statements: (i) x is later than x_i for all i; (ii) x_i is later than x_j if x_i lies on a polygonal arc joining x_j to x in T. Thus, T **fixes** W uniquely. The partial ordering is a complete **ordering** only if T is a single continuous arc without **branches and** with one end point at x.

Regrouping the terms of the sum (24), we have

$$S_G=N_G\sum_T S_1(T)S_2(T), \tag{25}$$

where the summation is over the order-types T compatible with G. Each $S_1(T)$ is a product of n factors of the form

$$\theta(x'-x'')S(x'-x''), \quad \theta(x'-x'')S(x''-x') \text{ or } \theta(x'-x'')D(x'-x''), \tag{26}$$

one corresponding to each doubled line of G_T. Each $S_2(T)$ is a linear combination of products of $(b-n)$ factors of the form

$$S(x'-x''), \quad D(x'-x''), \quad S^1(x'-x''), \text{ or } D^1(x'-x''), \tag{27}$$

one factor corresponding to each undoubled internal line of G_T.

Equation (25) is the conclusion of this lengthy analysis of S_G by means of doubled graphs. The only purpose of the analysis is to show that the θ-functions in Eq. (20) can be so rearranged that they occur always in conjunction with exchange-terms as in (26). Fortunately, the subsequent arguments will be of such a kind that we shall never need to carry out the rearrangement in practice. The final results will be very simple and will not involve θ-functions explicitly; but it seems to be essential, in order to justify the simple rules of calculation which we shall later formulate, to prove that the rearrangement of θ-functions is possible in principle.

For completeness, we must here observe that Eq. (18) is not the most general kind of term in the expansion of Eq. (11). In addition to Eq. (18), we shall need to consider terms in which a multiple commutator C_r appears involving, say, r operators H_1 and $(n-r)$ operators H_s. When Eq. (7) is substituted into Eq. (11), such commutators always appear in symmetrical combinations; thus,

$$\sum C_r=\sum[H_{1,s}(x_n), [\cdots, [H_{1,s}(x_1), j_\mu(x)]\cdots]], \tag{28}$$

where the summation is over the $\binom{n}{r}$ ways of choosing r suffixes 1 and $(n-r)$ suffixes S. The points $x_1, \cdots, x_n$ are not permuted in the different terms of Eq. (28). When we pass to the symmetrical form analogous to Eq. (19), we must now write

$$I_n(x)=(1/n!)\sum\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1\cdots dx_nS_r, \tag{29}$$

$$S_r=\sum\theta(x-x_1)\cdots\theta(x_{n-1}-x_n)[H_{1,s}(x_n), [\cdots, [H_{1,s}(x_1), j_\mu(x)]\cdots]]. \tag{30}$$

Here, the summation in Eq. (29) is over the $\binom{n}{r}$ ways in which we can associate with each of the points $x_1, \cdots, x_n$ either H_1 or H_s. The summation in Eq. (30) is over the $(n!)$ permutations of $x_1, \cdots, x_n$, the operators H_1 and H_s being permuted simultaneously so that each x_i carries its own operator around with it. By splitting up the double summation in this special way, we have arranged that S_r is a sum of products of the same factors in various orders. The different terms in Eq. (29), on the other hand, involve products of different sets of factors and are represented by different sets of graphs G. Therefore, we make the analysis into normal constituents for each S_r separately. In all other respects, the terms involving S_r can be treated in precisely the same way as the terms in S. To avoid needless complications, we dismiss the S_r with these brief remarks and carry through the analysis only for S.

IV. MOMENTUM SPACE REPRESENTATIONS

We wish to calculate the coefficient multiplying a particular normal product such as

$$N(\bar\psi_\alpha(p_1)\bar\psi_\beta(p_2)\cdots\psi_\delta(p_i)\cdots A_\lambda(p_j)\cdots A_\nu(p_l)) \tag{31}$$

in the expansion of $\mathbf{j}_\mu(p)$. Here $p, p_1, p_2, \cdots, p_l$ are given 4-vectors and $\alpha, \beta, \cdots, \lambda, \cdots, \nu$ are given spinor and vector suffixes. Let this coefficient be M. Then M is a sum of contributions $M(S)$ from different integrals of the general form

$$I_n(p)=(1/n!)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1\cdots dx_n\int_{-\infty}^{\infty}dxe^{-ip\cdot x}S, \tag{32}$$

with S given by Eq. (20).

We now take the decisive step which will have the effect of simplifying the calculation of $M(S)$. We consider instead of $M(S)$ the corresponding coefficient $M_\Gamma(S)$ in the expansion of the generalized operator

$$I_n(\Gamma, p)=(1/n!)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1\cdots dx_n\int_{-\infty}^{\infty}dxS \times\exp[-ip\cdot x+\Gamma_1\cdot(x-x_1)+\Gamma_2\cdot(x-x_2) +\cdots+\Gamma_n\cdot(x-x_n)]. \tag{33}$$

Here, the Γ_i are n 4-vectors with components

$$\Gamma_i=(0, 0, 0, \Gamma_{i0}) \tag{34}$$

and the Γ_{i0} are any n complex numbers with positive real parts. In Eq. (33) the integrals over $x_1, \cdots, x_n$ are exponentially convergent. The integrals in Eq. (32) are not convergent but either oscillate or diverge at infinity. We shall see later[14] that there are good physical reasons for *defining* the value of Eq. (32) to be the limit of Eq. (33) as all the Γ_{i0} tend to zero in a certain way. That is to say, we use the Abel summation method to give a meaning to Eq. (32). We shall discuss only very incompletely the physical meaning of Eq. (33) for finite Γ_{i0}, which will form the subject of a separate paper. The $I_n(\Gamma, p)$ are here used simply as mathematical auxiliaries in the evaluation of $I_n(p)$.

Let G be any Feynman graph with the correct external lines to give rise to normal constituents with the operator factor (31). Then, $M_\Gamma(S)$ is a sum of contributions $M_\Gamma'(S_G)$ from all such G. Each $M_\Gamma'(S_G)$ may be further subdivided into contributions $M_\Gamma(S_G)$ arising from a graph G with particular association of each factor in Eq. (31) to one external line of G, so that each external line is restricted to represent a particle carrying a certain definite momentum p_r. Finally, each $M_\Gamma(S_G)$ is a sum of contributions $M_\Gamma(S, G_T)$, one from each doubled graph G_T derived from G.

Let t_i be the algebraic sum of the momenta carried by the external lines of G at the vertex x_i. Then $M_\Gamma(S, G_T)$ is apart from numerical factors an integral

$$M_\Gamma(S, G_T)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\cdots dx dx_1\cdots dx_n S_1 S_2 \times\exp[-ip\cdot x-it_1\cdot x_1\cdots-it_n\cdot x_n+\Gamma_1\cdot(x-x_1) +\cdots+\Gamma_n\cdot(x-x_n)], \tag{35}$$

where S_1 is a product of n factors (26), and S_2 is a sum of products of $(b-n)$ factors (27). The "numerical factors" which are omitted from Eq. (35) depend on G but are independent of T. They are unimportant because we shall finally arrive at a much simpler formula for the precise evaluation of $M_\Gamma(S_G)$ including numerical constants; we shall use Eq. (35) only in order to determine the analytic behavior of $M_\Gamma(S, G_T)$ as a function of the Γ_{i0}. Now, write f for $(b-n)$, and let λ_a, $a=n+1, \cdots, b$, be the undoubled internal lines of G_T. The electron lines of G have arrows giving them a definite sense; it is convenient also to insert arbitrarily an arrow in each photon line, so that every line has a sense. Let the end points of λ_a be $x_a'x_a''$, with the arrow pointing from x_a'' towards x_a'. Suppose that $\lambda_{n+1}, \cdots, \lambda_{n+g}$ are electron lines and $\lambda_{n+g+1}, \cdots, \lambda_b$ are photon lines. Then S_2 can be written as an f-fold integral in momentum-space

$$S_2=\int_{\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dk_{n+1}\cdots dk_b C(k_{n+1}, \cdots, k_b) \times\prod_{a=n+1}^{n+g}(k_{a\mu}\gamma_\mu+im)\delta(k_a^2+m^2)\prod_{a=n+g+1}^{b}\delta(k_a^2+\lambda^2) \times\prod_{a=n+1}^{b}\exp[ik_a\cdot(x_a'-x_a'')]. \tag{36}$$

Here $C(k_{n+1}\cdots, k_b)$ is some numerical function which takes only 2^f distinct values, since it depends only on the signs of the fourth components of $k_{n+1}, \cdots, k_b$. The values of $C(k_{n+1}, \cdots, k_b)$ will depend on the precise combinations of D^1, D, S^1, and S functions occurring in S_2. They are pure numbers depending only on the shape of G_T and independent of all variables such as p_i and Γ_i. The product

$$\prod(k_{a\mu}\gamma_\mu+im)$$

is a symbolic notation for a product of dirac matrices which, multiplied with a similar product arising from S_1, makes up a tensor with the correct vector and spinor suffixes to serve as a coefficient for Eq. (31) in $\mathbf{j}_\mu(p)$.

Next we introduce momentum representations for the factors of S_1. Let λ_i be the doubled line of G_T joining x_i to a point x_j or x which is later than x_i in T. Let y_i be the point x_j or x in question, and let

$$z_i=y_i-x_i. \tag{37}$$

If x' and x'' are any two vertices of G, let $\theta_T(x', x'')$ denote 1 if either $x'=x''$ or x' stands later than x'' in T, and zero otherwise. Then, by Eq. (37),

$$x-x_i=\textstyle\sum_j \theta_T(x_j, x_i)z_j. \tag{38}$$

This implies the identities

$$\left.\begin{aligned}\textstyle\sum_i \Gamma_i\cdot(x-x_i)&=\textstyle\sum_j \Delta_j\cdot z_j,\\ \textstyle\sum_i t_i\cdot(x-x_i)&=\textstyle\sum_j u_j\cdot z_j,\\ \textstyle\sum_a k_a\cdot(x_a'-x_a'')&=\textstyle\sum_j q_j\cdot z_j,\end{aligned}\right\} \tag{39}$$

where

$$\left.\begin{aligned}\Delta_j&=\textstyle\sum\theta_T(x_j, x_i)\Gamma_i, \quad u_j=\textstyle\sum\theta_T(x_j, x_i)t_i,\\ q_j&=\textstyle\sum_a[\theta_T(x_j, x_a'')-\theta_T(x_j, x_a')]k_a=\sum_a c_{ja}k_a.\end{aligned}\right\} \tag{40}$$

Always, the suffixes i and j run from 1 to n, and a runs from $n+1$ to b.

Now let

$$S_1'=\exp[\Gamma_1\cdot(x-x_1)+\cdots+\Gamma_n\cdot(x-x_n)]S_1. \tag{41}$$

By Eqs. (26) and (39), S_1' is a product of n factors, one corresponding to each doubled line λ_i. When λ_i is an electron line with the arrow in the sense x_iy_i, the factor is

$$S_{\Delta i}(z_i)=\exp(\Delta_i\cdot z_i)\theta(z_i)S(z_i). \tag{42}$$

When λ_i is an electron line with arrow in the sense y_ix_i, the factor is

$$S_{\Delta i}^*(z_i)=\exp(\Delta_i\cdot z_i)\theta(z_i)S(-z_i). \tag{43}$$

[14] In the next paper of this series. The idea of using a limiting process of this kind is due to B. Ferretti, Nuovo Cimento 7, 79 and 375 (1950).

When λ_i is a photon line, the factor is

$$D_{\Delta i}(z_i)=\exp(\Delta_i\cdot z_i)\theta(z_i)D(z_i). \tag{44}$$

We denote the products of these three sets of factors in S_1' by

$$\prod_e,\ \prod_e',\ \text{and}\ \prod_p, \tag{45}$$

respectively.

Now it happens that the special functions $S_\Delta(z)$, $S_\Delta{}^*(z)$, $D_\Delta(z)$ have particularly simple representations in momentum-space, when Δ is a vector of the form

$$\Delta_i=(0,\,0,\,0,\,\Delta_{i0}) \tag{46}$$

according to Eq. (40), and Δ_{i0} has a positive real part. For example, we shall prove that

$$D_\Delta(z)=-(1/2\pi)^4\int_{-\infty}^{\infty}dk e^{ik\cdot z}/[(k+i\Delta)^2+\lambda^2]. \tag{47}$$

Let I be the right-hand side of Eq. (47). The integrand is nonsingular for real k, and so there is no ambiguity in defining the integral to be along the real axis for all four components of k. We may evaluate I by using Cauchy's theorem in the complex k_0 plane, where k_0 is the fourth component of k. The poles are then at the two points

$$k_0=-i\Delta_0\pm(k_1{}^2+k_2{}^2+k_3{}^2+\lambda^2)^{\frac{1}{2}}. \tag{48}$$

When z_0 is negative, the integrand vanishes for large k_0 in the upper half-plane. Since the poles (48) lie below the real axis, the value of I is then zero. When z_0 is positive, I is given by the sum of the residues at the two poles, namely,

$$I=\exp(-\Delta_0 z_0)D(z).$$

This completes the proof of Eq. (47). In exactly the same way we find

$$S_\Delta(z)=[-i/(2\pi)^4]\int_{-\infty}^{\infty}dk e^{ik\cdot z}/[(k+i\Delta)_\mu\gamma_\mu-im], \tag{49}$$

$$S_\Delta{}^*(z)=[-i/(2\pi)^4]\int_{-\infty}^{\infty}dk e^{+ik\cdot z}/[(k+i\Delta)_\mu\gamma_\mu+im]. \tag{50}$$

Using now Eqs. (39), (41), (45), (47), (49), and (50) and omitting a numerical factor gives the momentum representation of S_1' as

$$S_1'=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dk_1\cdots dk_n\prod_e\left[\frac{1}{(k_i+i\Delta_i)_\mu\gamma_\mu-im}\right]\times\prod_e'\left[\frac{1}{(k_i+i\Delta_i)_\mu\gamma_\mu+im}\right]\prod_p\left[\frac{1}{(k_i+i\Delta_i)^2+\lambda^2}\right]\prod_{i=1}^{n}\times\exp(ik_i\cdot z_i). \tag{51}$$

It remains only to substitute Eqs. (36) and (51) into Eq. (35). Using Eqs. (38) and (39), we may express $M_\Gamma(S, G_T)$ as an integral over the 4-vectors

$$x,\ z_1,\ \cdots,\ z_n,\ k_1,\ \cdots,\ k_b.$$

The integrations over x and the z_i are trivial, yielding only a product of $(n+1)$ δ-functions,

$$\delta(p+\textstyle\sum t_i)\delta(k_1+q_1+u_1)\cdots\delta(k_n+q_n+u_n). \tag{52}$$

The first factor in Eq. (52) is a constant multiplying factor expressing over-all conservation of energy and momentum. This factor will be replaced by a nonsingular function $f(\sum t_i)$ when the various Fourier components $\mathbf{j}_\mu(p)$ are superposed to form the physically measurable quantity (2). We omit this factor together with the other numerical factors in the discussion of Eq. (35). The remaining factors in Eq. (52) are just sufficient to express the momenta $k_1,\ \cdots,\ k_n$ as linear combinations of the $k_{n+1},\ \cdots,\ k_b$ according to Eq. (40). The resulting formula for $M_\Gamma(S, G_T)$ is

$$M_\Gamma(S, G_T)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dk_{n+1}\cdots dk_b C(k_{n+1},\ \cdots,\ k_b)\times\prod_e\left[\frac{1}{(-q_i-u_i+i\Delta_i)_\mu\gamma_\mu-im}\right]\times\prod_e'\left[\frac{1}{(q_i+u_i-i\Delta_i)_\mu\gamma_\mu-im}\right]\times\prod_p\left[\frac{1}{(-q_i-u_i+i\Delta_i)^2+\lambda^2}\right]\prod_{a=n+1}^{n+g}[(k_{a\mu}\gamma_\mu+im)\delta(k_a{}^2+m^2)]\prod_{a=n+g+1}^{b}\delta(k_a{}^2+\lambda^2). \tag{53}$$

V. COMPLEX INTEGRAL REPRESENTATIONS

Consider the rational function

$$\Phi\equiv\Phi(k_{n+1},\ \cdots,\ k_b)=\prod_e\left[\frac{(-q_i-u_i+i\Delta_i)_\mu\gamma_\mu+im}{(-q_i-u_i+i\Delta_i)^2+m^2}\right]\times\prod_e'\left[\frac{(q_i+u_i-i\Delta_i)_\mu\gamma_\mu+im}{(q_i+u_i-i\Delta_i)^2+m^2}\right]\times\prod_p\left[\frac{1}{(-q_i-u_i+i\Delta_i)^2+\lambda^2}\right]\times\prod_{a=n+1}^{n+g}\left[\frac{k_{a\mu}\gamma_\mu+im}{k_a{}^2+m^2}\right]\prod_{a=n+g+1}^{b}\left[\frac{1}{k_a{}^2+\lambda^2}\right]. \tag{54}$$

The factors correspond, in an obvious way, to the factors in Eq. (53). We now assert that this function, Φ, depends only on G and is essentially the same for all doubled graphs, G_T, which are derived from G. To

prove this we write Eq. (54) in the form,

$$\Phi=\prod_r\left[\frac{R_{r\mu}\gamma_\mu+im}{R_r^2+m^2}\right]\prod_r\left[\frac{1}{R_r^2+\lambda^2}\right], \tag{55}$$

where there is one factor, and one 4-vector, R_r, corresponding to each internal line, λ_r, of G whether doubled or undoubled. The two products in Eq. (55) together run over the values $r=1, \cdots, b$ in some order. The R_r consist of the vectors k_a, $a=n+1, \cdots, b$, together with n vectors $\eta_i(-q_i-u_i+i\Delta_i)$, $i=1, \cdots, n$. Here, η_i is ±1 according to whether the arrow in λ_i is in the direction x_iy_i or y_ix_i. Let ϵ_{rj} be defined to be $+1$ if λ_r is a line incident at x_j with the arrow pointing away from x_j, -1 if λ_r is incident at x_j with the arrow towards x_j, and zero if λ_r is not incident at x_j. Then, in consequence of the structure of G_T, we have the identity

$$\sum_{j=1}^{n}\epsilon_{jl}\eta_j\theta_T(x_j, x_i)=\delta_{il}, \tag{56}$$

which is only an alternative statement of Eq. (38). By the definition of R_r, with Eqs. (40) and (56),

$$\begin{aligned}\textstyle\sum_r \epsilon_{rl}R_r&=\textstyle\sum_a \epsilon_{al}k_a+\sum_j \epsilon_{jl}\eta_j(-q_j-u_j+i\Delta_j)\\ &=-t_l+i\Gamma_l, \quad l=1, \cdots, n.\end{aligned} \tag{57}$$

These n linear relations between the R_r are equivalent to the n relations (40) expressing the q_j in terms of the k_a. But (57) is independent of T and depends only on G. Therefore, for all T, the function (54) is identical with (55), with the R_r related by (57). The functions Φ for different T differ only in that different sets of f of the R_r are chosen as the basic vectors k_a in terms of which Φ is expressed, the remaining R_r being eliminated by means of (57). Let L be the number of T which are compatible with G. Then these different T correspond precisely to the L possible ways of choosing a set of f of the R_r which are linearly independent.

Let the $3b$ space-components of the R_r be given any arbitrary real values consistent with (57). We consider (55) as an analytic function of the b time-components $\zeta_r=R_{r0}$, subject to the relations

$$\sum_r \epsilon_{rl}\zeta_r=-t_{l0}+i\Gamma_{l0}, \tag{58}$$

the ζ_r being complex variables. The denominator of (55) is

$$D=\prod_r[(\alpha_r-\zeta_r)(\alpha_r+\zeta_r)], \tag{59}$$

where

$$\left.\begin{aligned}\alpha_r&=(R_{r1}^2+R_{r2}^2+R_{r3}^2+m^2)^{\frac{1}{2}},\\ \alpha_r&=(R_{r1}^2+R_{r2}^2+R_{r3}^2+\lambda^2)^{\frac{1}{2}},\end{aligned}\right\} \tag{60}$$

for factors in the first and second product, respectively, in Eq. (55). D is a product of $2b$ factors which are non-homogeneous linear forms in the f complex variables k_{a0}, when Φ is represented by Eq. (54).

Suppose now that $\lambda\neq0$, so that by Eq. (60) $\alpha_r\neq0$. Then the factors of D have two properties which are essential to the discussion: (i) there exists a set of f linearly independent factors, and (ii) it is impossible for any set of $(f+1)$ factors to vanish simultaneously for any values of the variables. The proof of (ii) is as follows. If there is a set of $(f+1)$ factors vanishing simultaneously, at least one of these factors, say, F_1, is linearly dependent on a subset $F_2, F_3, \cdots, F_m$ of the others, where the $F_2, \cdots, F_m$ are themselves linearly independent. We can choose a representation of the R_r in terms of basic vectors k_a, such that $F_2, \cdots, F_m$ are of the form

$$(\alpha_a\pm k_{a0}), \tag{61}$$

with $(m-1)$ different values of a. Then F_1 is of the form

$$F_1=\alpha_1\pm(-q_{j0}-u_{j0}+i\Delta_{j0}), \tag{62}$$

where q_{j0}, according to (40), is a linear combination with real coefficients of the k_{a0} occurring in (61). The α_r are real, and therefore when all factors (61) vanish, q_{j0} is real. But u_{j0} is real, and Δ_{j0} has a positive real part, and so (62) and (61) cannot vanish simultaneously.

In consequence of properties (i) and (ii) of D, there exists an expansion of Φ in partial fractions

$$\Phi=\textstyle\sum_U[g(U)/(\zeta_\mu\pm\alpha_\mu)(\zeta_\nu\pm\alpha_\nu)\cdots(\zeta_\rho\pm\alpha_\rho)]. \tag{63}$$

Here the summation is over the $(L2^f)$ possible ways U of choosing f linearly independent factors from D. Each U involves a choice of f linearly independent variables $\zeta_\mu, \zeta_\nu, \cdots, \zeta_\rho$, from the ζ_r, forming the k_{a0} corresponding to a particular T, and a choice of f independent signs ±1. The coefficients $g(U)$ are uniquely determined and depend on the space components of the R_r but not on the ζ_r. We omit the proof that properties (i) and (ii) imply (63), since it involves only elementary but rather tedious algebra.

We see from Eq. (63) that Φ is not at all a general rational function of f complex variables, but has a very simple structure. It has precisely $L2^f$ simple poles, or points at which f linear factors in its denominator vanish. To each term in Eq. (63) corresponds a pole which we call the pole U. The function is completely determined by Eq. (63), given the positions of the poles and the constants $g(U)$, which are the residues at the poles.

In the above analysis, the fact that $\lambda\neq0$ was used only in order to ensure that the α_r given by Eq. (60) should not vanish. The discussion is still valid for $\lambda=0$, except when

$$R_{r1}=R_{r2}=R_{r3}=0 \tag{64}$$

for some value of r. We shall finally integrate Φ over the space components of the R_r, and we do not need even to define Φ at the isolated points where Eq. (64) holds. However, the behavior of the coefficients $g(U)$ in the neighborhood of such points must be gentle

enough to make the integrals well defined. In fact, the $g(U)$ become infinite at most like

$$(R_{r1}^2+R_{r2}^2+R_{r3}^2)^{-\frac{1}{2}},$$

and so the integrals over the space-components are easily convergent in the neighborhood of the singularities. This allows us to use Eq. (63) for $\lambda=0$ without danger of inconsistency.

The introduction of the Γ_i factors in Eq. (33) has entirely eliminated, from the present analysis, the two kinds of unessential but troublesome divergences which arise in the study of the S-matrix,[15] namely, accidental divergences due to vanishing energy denominators in perturbation theory and infrared divergences. The disappearance of these divergences is due directly to the fact that more than f factors in the denominator of Φ can never vanish simultaneously, which in turn is a consequence of the exponential convergence of Eq. (33). In particular, if we take $\lambda=0$ and then make all $\Gamma_i\to 0$, the integrals of $g(U)$ over the space-components of the R_r become logarithmically divergent at the points (64), and these are typical infrared divergences.

Returning to Eq. (53), carrying out the integrations over the k_{a0}, and remembering the properties of the function $C(k_{n+1}, \cdots, k_b)$ stated after Eq. (36), we have

$$M_\Gamma(S, G_T)=\int d_3R \sum_U C(U)g(U). \tag{65}$$

Here the integration is over the space components of the R_r subject to Eq. (57). The summation is over the 2^f poles U of Φ at which the δ-functions in Eq. (53) have singularities, which are just the poles corresponding to the given G_T. The coefficients $C(U)$ are derived from the 2^f values of the function $C(k_{n+1}, \cdots, k_b)$ and are numbers independent of all the variables.

Finally, we sum Eq. (65) over the G_T derived from a given G, and we find

$$M_\Gamma(S_G)=A\int d_3R \sum_U C(U)g(U), \tag{66}$$

where the summation is now over all the poles of Φ, the $C(U)$ are as before pure numbers, and A is a multiplying factor which is independent of m, λ, and the Γ_i; A incorporates the various factors which were dropped in the preceding work. By Eq. (63), we may write instead of Eq. (66),

$$M_\Gamma(S_G)=A'\int d_3R \int d\zeta_r \Phi(R_r), \tag{67}$$

where $\int d_3R$ is as before a real integral, while $\int d\zeta_r$ is some contour integral in the space of the complex ζ_r variables. The form of this contour integral will be determined in the following section.

[15] See Sec. V of reference (B).

VI. DETERMINATION OF THE CONTOUR

Let m and λ now be considered to be very large numbers, greater than any combinations of the p, Γ_i, and t_i which occur in the theory. In this case we make an alternative analysis of $\mathbf{j}_\mu(p)$ based on Eq. (12) instead of on Eq. (11). Consider the contribution J_{qp} to $\mathbf{j}_\mu(p)$ derived from the term (q, p) in the double sum on the right of Eq. (12). Let J_{qp} be expanded as a sum of normal products according to the rules of Sec. II.

The coefficient of Eq. (31) in the expansion of J_{qp} is as before a sum of contributions $M(\bar{S})$ from different integrals of the form (32) with $n=q+p$, where $\bar{S}$ is now an expression like

$$\bar{S}=\bar{P}(H_1(x_1), \cdots, H_1(x_q))P(H_1(x_{q+1}), \cdots, H_1(x_n), j_\mu(x)) \tag{68}$$

instead of Eq. (20). We consider instead of $M(\bar{S})$ the corresponding coefficient $M_\Gamma(\bar{S})$ in the expansion of Eq. (33). But now, since there are no θ-functions in Eq. (68), the Γ_{i0} must all have real part zero in order that Eq. (33) should not diverge exponentially.

Let $M_\Gamma(\bar{S}_G)$ as before be a contribution to $M_\Gamma(\bar{S})$ arising from a connected graph G. Suppose first that $q>0$. In this case, G will contain at least one internal line $x'x''$, where x' belongs to the $\bar{P}$-product and x'' to the P-product in Eq. (68). Let C be the set of all internal lines of G with this property. To each line of C will correspond in $M_\Gamma(\bar{S}_G)$ a factor

$$S^+(x'-x''), \quad S^-(x''-x'), \quad \text{or} \quad D^+(x'-x''), \tag{69}$$

according to Eqs. (14), (15), (16). In the momentum representations these factors contain only fourier components of the form $\exp[ik\cdot(x'-x'')]$ with $k_0>0$ and either

$$k_1^2+k_2^2+k_3^2-k_0^2+m^2=0 \quad \text{or} \quad k_1^2+k_2^2+k_3^2-k_0^2+\lambda^2=0.$$

That is to say, only fourier components occur for which

$$k_0 \geqq \text{Min}(m, \lambda). \tag{70}$$

The internal lines of G not belonging to C will contribute to $M_\Gamma(\bar{S}_G)$ factors of the form $\phi(x'-x'')$, where x' and x'' both belong to the first product or both to the second product in Eq. (68). The external lines contribute a factor

$$\exp[-it_1\cdot x_1\cdots-it_n\cdot x_n].$$

Now the integral for $M_\Gamma(\bar{S}_G)$ is obtained by substituting all these factors for $\bar{S}$ in Eq. (33). We may take as independent variables $x_1, \cdots, x_q, \xi_{q+1}, \cdots, \xi_n, x$, where

$$\xi_i=x_i-x.$$

When the momentum representations of the factors (69) are used, $M_\Gamma(\bar{S}_G)$ becomes an integral over these variables and over a set of momentum variables k satisfying Eq. (70). Let the integration over x be carried out first. The factors $\phi(x'-x'')$ are independent of x.

The factors depending explicitly on x are

$$\exp[-i(\sum k+p+\sum_{q+1}^{n} t_j+i\sum_{1}^{q}\Gamma_j)\cdot x]. \quad (71)$$

When (71) is integrated over x, the result is a 4-dimensional δ-function of the vector multiplying x; but the fourth component of this vector is never zero, provided that

$$\mathrm{Min}(m,\lambda)>|p_0|+\textstyle\sum_1^n|t_{j0}|+\sum_1^n|\Gamma_{j0}|. \quad (72)$$

It is assumed that m and λ are large enough to satisfy (72). Therefore, the $M_\Gamma(\bar{S}_G)$ derived from Eq. (68) vanish identically whenever $q>0$ for every connected G.

It was proved in Sec. III that the normal constituents of $\mathbf{j}_\mu(p)$ arising from disconnected G are zero. Therefore, when m and λ satisfy (72), the normal constituents of $\mathbf{j}_\mu(p)$ can be calculated as follows. We use for $\mathbf{j}_\mu(x)$ instead of Eq. (12) the expression

$$\mathbf{j}_\mu{}^F(x)=\sum_{n=0}^{\infty}\left(-\frac{i}{\hbar c}\right)^n\frac{1}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1\cdots dx_n \times P(H^I(x_1),\cdots,H^I(x_n),j_\mu(x)), \quad (73)$$

and include contributions only from connected graphs.

The operator $\mathbf{j}_\mu{}^F(x)$ has been encountered before. It is the current operator in the "mixed representation" of reference (A), where initial states are specified at a remote time in the past, and final states at a remote time in the future. The equivalence of $\mathbf{j}_\mu(x)$ and $\mathbf{j}_\mu{}^F(x)$ for large m and λ is a consequence of the fact that there is not sufficient energy available in a measurement of $\mathbf{j}_\mu(p)$ for the real production of particles of large rest-mass.

For our present purposes, the physical interpretation of $\mathbf{j}_\mu{}^F(x)$ is unimportant. The important fact is that $\mathbf{j}_\mu{}^F(x)$ has the same structure as the S-matrix, involving chronological products only. We are able to apply to it the rules given in reference (B) for the calculation of the S-matrix, with only minor alterations. These rules lead to an alternative expression for the integral on the right of Eq. (67), namely,

$$M_\Gamma(S_G)=A'\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dk_{n+1}\cdots dk_b\Phi. \quad (74)$$

Here the integrals are along the real axis for all the $4f$ components of the vectors k_a, and Φ is given by Eq. (54), with the convention that the poles are moved an infinitesimal distance away from the real axis by replacing each m^2 and λ^2 by $(m^2-i\epsilon)$ and $(\lambda^2-i\epsilon)$, ϵ being a small positive number and the integrals being calculated in the limit $\epsilon\to+0$. This convention regarding the poles of Φ is due to Feynman,[5] and is the convention which is always correct for momentum representations of operators like (73) involving only chronological products.

It is possible now to transform the path of integration in Eq. (74) for each of the fourth components k_{a0} from the real axis to the imaginary axis, just as was done in Sec. V of reference (B). Provided that (72) is satisfied, no singularities will be encountered in transforming the path of integration through the first and third quadrants of the k_{a0} planes. There are no "displaced poles," because we assume the masses so large that Eq. (51) of reference (B) can never hold. Therefore, Eq. (67) holds, with the path of the integral $\int d\zeta_r$ defined as follows. Choosing any set of f variables ζ_r which are linearly independent, we integrate over each of these variables up the imaginary axis from $(-i\infty)$ to $(+i\infty)$. The ϵ-limiting process is no longer needed, because the path of integration is now separated from the poles.

Equation (67), with this special choice of contour, is valid under the two conditions: m and λ are sufficiently large, and each Γ_{i0} has real part zero.

VII. THE ANALYTIC CONTINUATION METHOD

We have found two alternative expressions for $M_\Gamma(S_G)$. One is the expression E_1 on the right of Eq. (66). The other is the expression E_2 on the right of Eq. (67), with the contour as defined in the preceding section. The regions within which these two expressions have been shown to be valid do not overlap. E_1 is valid in the region

$$\mathrm{Re}(\Gamma_{i0})>0, \quad (75)$$

for all values of m and λ including $\lambda=0$. E_2 is valid in the region

$$\mathrm{Re}(\Gamma_{i0})=0, \quad (76)$$

with m and λ restricted by (72).

It is clear, from the definition of $g(U)$ by Eq. (63), that E_1 is an analytic function of the Γ_{i0} throughout the region (75), for any given values of m and $\lambda\neq0$. Provided that the path of integration in E_2 is separated from the poles of Φ, E_2 is also an analytic function; therefore, E_2 is analytic for all complex values of the Γ_{i0} satisfying (72); that is to say, E_2 is an analytic function of the Γ_{i0} in a region including in its interior points satisfying (76). We use now a lemma from the theory of analytic functions.

Let $f_2(w)$ be a function of the complex variable w, analytic in a circle C. Let D be a diameter of C bounding the semicircle C_1. Let $f_1(w)$ be a function analytic in C_1, such that $f_1(w)\to f_2(w_0)$ when a point w in C_1 tends to a point w_0 on D. Then f_1 and f_2 are identical.

The proof of this lemma will not be given here. It follows easily from the Schwartz reflection principle.[16] To apply the lemma to our functions E_1 and E_2, take

$$\Gamma_{i0}=\mu_i w, \quad (77)$$

where the μ_i are positive real numbers, and write

[16] J. E. Littlewood, *Theory of Functions* (Oxford University Press, 1944), theorem 105, p. 109, and corollary, p. 130.

$E_1=f_1(w)$, $E_2=f_2(w)$. Then C_1 is a part of the region (75), and D a part of the region (76). The lemma states that $E_1=E_2$ for all Γ_{i0} given by Eq. (77) with $\mathrm{Re}(w)>0$; but, since both E_1 and E_2 are analytic functions of the μ_i, $E_1=E_2$ also for complex μ_i. The conclusion of the argument is that E_1 and E_2 are alternative representations of the same analytic function E, valid in different but overlapping domains.

Up to this point we have considered only the analyticity of E as a function of the Γ_{i0}; but E_1 is also an analytic function of m and λ in the region

$$\mathrm{Re}(\Gamma_{i0})>0, \quad m>0, \quad \lambda>0, \tag{78}$$

that is to say, for real values of m and λ and for complex values in the immediate neighborhood of the positive real axis, and E_2 is an analytic function of λ and m in the same sense, for all λ and m satisfying (72). Therefore, E is a unique analytic function of the variables $(m, \lambda, \Gamma_{i0})$ in the combined domain of all points satisfying *either* (78) *or* (72).

The procedure to be used for practical calculations is the following. Suppose that we wish to calculate a particular $M(S_G)$, the contribution from a given G to the coefficient of Eq. (31) in the normal product expansion of Eq. (32), with the physically occurring value of m and $\lambda=0$. We start by writing down the expression E_2 for $M_\Gamma(S_G)$ according to Eq. (67), with $\lambda\neq 0$. Assuming (72) to hold, the integrals in E_2 can very conveniently be evaluated analytically by means of the well-established techniques of Feynman.[5] The result of evaluating the integrals is the analytic function E. We continue this function through the region (78), keeping $\mathrm{Re}(\Gamma_{i0})>0$, and decreasing m and λ until m reaches its actual value and λ reaches a very small λ_0 less than any of the energies which are characteristic of the particular physical situation. Then we make all Γ_{i0} tend to zero, in a way which will be specified precisely in the next paper of this series. Finally, we may make $\lambda_0\to 0$ if we desire to reintroduce infrared divergences into our calculations, or we may leave λ_0 standing as a convenient cutoff parameter for these divergences.

In this paper, nothing has been said about the physically interesting ultraviolet divergences, the divergences at large momenta of the integrals (67). These are the divergences which are removed by mass and charge renormalization, as will be described in the following paper. Here we remark only that these divergent terms will be separated in a consistent way from the integrals E_2 before the integrals are evaluated analytically. The function E is therefore finite, and the ultraviolet divergences do not interfere with the process of analytic continuation.

The integrals E_2, when all Γ_{i0} are zero, are formally identical with the integrals occuring in the calculation of the S-matrix. If we maintain $\Gamma_{i0}=0$ and try to continue the function E_2 from large values of m and λ down to the values in which we are interested, we shall encounter values of m and λ at which E_2 is not analytic. These are the places where the S-matrix is nonanalytic because of the appearance of a threshold for some real process which is competing with the process under analysis. When we arrive at the actual values of m and λ, the integrals E_2 representing elements of the S-matrix differ from the values of E which we find by analytic continuation through the region (78), because of the effects of the competing processes. In practice, E will have a simpler analytic form than does E_2; this is the justification for the remarks made at the end of Sec. I.

VIII. SUMMARY OF RULES FOR THE CALCULATION OF HEISENBERG OPERATORS

A Heisenberg operator such as $\mathbf{j}_\mu(p)$ is expressed formally in terms of interaction representation operators by means of Eq. (11). By the theorem of Wick stated in Sec. II, $\mathbf{j}_\mu(p)$ may be expanded into a sum of normal products (31) multiplied by C-number coefficients M. According to Sec. IV, each M is a sum of contributions $M(S_G)$ corresponding to different terms S of the form (20) and to different connected graphs G. We calculate $M(S_G)$ as the limit when all $\Gamma_{i0}\to 0$ of the generalized coefficient $M_\Gamma(S_G)$ defined in Sec. IV.

To calculate $M_\Gamma(S_G)$, we consider m and λ to be large, satisfying (72). The operator $\mathbf{j}_\mu{}^F(p)$ is defined by Eq. (73). This operator having the analytic form of an S-matrix, we may calculate its decomposition into normal products by following the rules of the S-matrix analysis in (B). Let $M'(S_G)$ be the coefficient of Eq. (31) in the normal product expansion of $\mathbf{j}_\mu{}^F(p)$, arising from the term S and the graph G. Then $M'(S_G)$ is an integral in momentum-space of the form (67), with Φ given by Eq. (54) with all $\Delta_i=0$. From $M'(S_G)$ we obtain $M_\Gamma(S_G)$ simply by inserting the parameters Δ_i given by Eq. (40) into Φ as given by Eq. (54). The path of integration in Eq. (67) is defined in Sec. VI.

We evaluate the integrals (67) analytically, using the methods of Feynman, obtaining for $M_\Gamma(S_G)$ an expression which is an analytic function of the variables Γ_{i0}, m and λ in the region where either (72) or (78) is satisfied. By analytic continuation through the region (78), as described in Sec. VII, we derive $M_\Gamma(S_G)$ for the values of m and λ which are of physical interest. Hence, passing to the limit $\Gamma_{i0}\to 0$, and, if necessary, also letting $\lambda\to 0$, we finally obtain the coefficients $M(S_G)$ in the normal product expansion of $\mathbf{j}_\mu(p)$.

It should be observed that there is nothing in the methods of the present paper which is specific to quantum electrodynamics. Precisely the same rules can be used for the calculation of Heisenberg operators in any theory in which two or more quantized fields of any kind are in interaction.