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Angular and Radial Distributions of Particles in Cascade Showers*

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This paper is concerned with the calculation of the angular and radial distributions of electrons and photons at the maximum of a cascade shower. In Sec. II(A) we calculate the moments of the distribution functions in a shower neglecting collision loss of electrons, i.e., for energies E much greater than the critical energy ϵ. In Sec. II(B) we derive expressions for the moments when collision loss is not negligible. These expressions are in the form of a series in ϵ/E, and are valid down to a few times the critical energy. In Sec. III we use the moments to calculate the distribution functions for energies down to a few times the critical energy.

I. INTRODUCTION

IN interpreting experiments on large air showers, one often must know something about the angular and radial distributions of the electrons and photons in them. A considerable amount of work has been done on this subject. Some of this work has been concerned with finding the mean squares of the quantities of interest rather than the distribution functions themselves. Roberg and Nordheim,[1] for example, have calculated quite accurately the mean square angles and displacements integrated over the shower for electrons and photons down to rather low energies. Belenky[2] has done the same for electrons, although somewhat less accurately. Also, there have been several calculations of the mean squares at the shower maximum for very high energies[3] and some calculation of the mean squares as a function of depth.[4]

There have also been attempts to calculate the distribution functions at the maximum. Belenky[5] has calculated the angular distribution of electrons under[6] Approximation A and in the same approximation Molière[7] has calculated the radial distributions of both electrons and photons and the angular distributions of electrons.

Although it is quite difficult to calculate the distribution functions, it is easy to calculate their moments accurately in Approximation A. If one does this, and compares the moments derived from Molière's functions with the accurate ones, it appears that Molière's high energy functions are in error, particularly for the large values of the argument. This is shown in Sec. II. Now, knowing the moments of the distribution functions one might be tempted to see how much he can deduce from them about the nature of the function. Of course, there are recondite theorems which state that under certain conditions (which are satisfied for the functions we consider) the functions are uniquely determined from the moments. These theorems are useless for our purposes, since to use them one must have fairly simple analytic expressions for the general moment. These are not available. It appears, however, for the functions we consider—which we can assume on physical grounds to be monotonically decreasing and "smooth"—that one can deduce the function over most of its range with considerable accuracy from a knowledge of only the first few moments. Only the behavior at very small arguments is not determined. We have no rigorous proof for this statement, but several test examples which we have tried successfully have given us considerable confidence that it is true. These are discussed in Sec. III. In Sec. II(A) we have calculated the moments under Approximation A, and compared our results with the moments derived from Molière's functions. In Sec. II(B) we show how one can obtain expressions for the moments in the form of "asymptotic" expansions[8] which hold to two or three times the critical energy. Finally, in Sec. III(A) and III(B) we calculate the actual distribution functions for energies down to about twice the critical energy using the moments found in Sec. II.

The basic assumptions of our calculations are that the scattering angles are small and that the asymptotic expressions for radiation and pair production are valid. The calculations hold for any element for which, at the energy considered, these assumptions are valid.

II. MOMENTS OF THE DISTRIBUTION FUNCTIONS IN SHOWERS

(A) Approximation A

In this section we derive expressions for the moments of the angular and radial distribution functions in a

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[1] J. Roberg and L. Nordheim, Phys. Rev. **75**, 444 (1949).
[2] S. Belenky, J. Phys. (U.S.S.R.) **8**, 9 (1944).
[3] Nordheim, Osborne, and Blatt, Proceedings of the Echo Lake Cosmic Ray Symposium, p. 273; L. Janossy, *Cosmic Radiation* (Oxford University Press, 1948); G. Molière, *Cosmic Radiation*, edited by W. Heisenberg (Dover Publications, New York, 1946); and Z. Physik **125**, 250 (1948).
[4] A. Borsellino, Nuovo cimento **6**, 543 (1949) and **7**, 638 (1950). John Blatt (private communication).
[5] S. Belenky, J. Phys. (U.S.S.R.) **8**, 347 (1944).
[6] Following B. Rossi and K. Greisen, Revs. Modern Phys. **13**, 240 (1941), we call Approximation A that in which ionization loss of electrons is neglected.
[7] G. Molière, *Cosmic Radiation*, edited by W. Heisenberg (Dover Publications, New York, 1946); Phys. Rev. **77**, 715 (1950).
[8] We have not been able to prove that the series we derive are asymptotic in the strict sense, although they appear to be useful for computation; by analogy to the German term *halb-konvergent* they might better be called *half-asymptotic*. We shall, however, simply call them "*asymptotic*."

large shower for energies much greater than the critical energy.[9] We shall consider the distribution functions which have been integrated over the length of the shower. These will also be the distribution functions at the shower maximum, since the dE/E^2 energy spectrum of particles at the maximum is the same as the energy spectrum integrated over the length. Also, it will be convenient to assume that the showers we consider are initiated by a single electron of energy E_0; but the results are really independent of this particular boundary condition so long as the initiating particles have energies much larger than the energies of the electrons or photons in which we are interested.

We specify the lateral position and direction of an electron or photon in the shower by the coordinates x and y in a plane perpendicular to the shower axis and angles θ_x and θ_y in two perpendicular planes whose intersection is parallel to the shower axis. We call $\mathbf{r}$ the vector (x, y) and $\boldsymbol{\theta}$ the vector (θ_x, θ_y). We denote by $\pi(E, \theta_x, \theta_y, x, y)$ or, more briefly, $\pi(E, \boldsymbol{\theta}, \mathbf{r})$ the number of particles of energy E at the point (x, y) in $dxdy$ traveling at an angle (θ_x, θ_y) in $d\theta_x d\theta_y$ and by $\gamma(E, \theta_x, \theta_y, x, y)$ the analogous quantity for photons. Then, the diffusion equations which describe the propagation and scattering in the shower are[10]

$$\delta(E_0-E)\delta(x)\delta(y)\delta(\theta_x)\delta(\theta_y)=L_1(\pi, \gamma)+\frac{E_s^2}{4E^2}\left(\frac{\partial^2\pi}{\partial\theta_x^2}+\frac{\partial^2\pi}{\partial\theta_y^2}\right)-\theta_x\frac{\partial\pi}{\partial x}-\theta_y\frac{\partial\pi}{\partial y}, \quad (1a)$$

$$0=L_2(\pi, \gamma)-\theta_x(\partial\gamma/\partial x)-\theta_y(\partial\gamma/\partial y). \quad (1b)$$

Using the notation of Rossi and Greisen,[11] we find the integral operators L_1 and L_2 to be

$$L_1(\pi, \gamma)=2\int_0^1 \gamma(E/u, \boldsymbol{\theta}, \mathbf{r})\psi(u)du/u - \int_0^1 [\pi(E, \boldsymbol{\theta}, \mathbf{r})-(1/1-v)\pi(E/1-v, \boldsymbol{\theta}, \mathbf{r})]\phi(v)dv + \epsilon\partial\pi(E, \boldsymbol{\theta}, \mathbf{r})/\partial E, \quad (2a)$$

$$L_2(\pi, \gamma)=\int_0^1 \pi(W/v, \boldsymbol{\theta}, \mathbf{r})\phi(v)(dv/v)-\sigma\gamma(W, \boldsymbol{\theta}, \mathbf{r}). \quad (2b)$$

In this section we will set ϵ equal to zero. This is Approximation A of Rossi and Greisen. Defining $\pi_{mn}(E)$ as

$$\pi_{mn}(E)=\int\int\int\int_{-\infty}^{+\infty} \pi(E, \theta_x, \theta_y, x, y) \times(x^m\theta_x^n+y^m\theta_y^n)d\theta_x d\theta_y dxdy, \quad (3)$$

we have

$$(x^m\theta_x^n+y^m\theta_y^n)_{\substack{\text{average for}\\ \text{electrons}}}=\pi_{mn}/\pi_{00}\equiv\langle\pi_{mn}\rangle, \quad (4)$$

where π_{00} is just *twice* the electron track length; i.e.,

$$\pi_{00}=2\cdot Z_\pi(E_0, E)=2\cdot(0.437E_0/E^2). \quad (5)$$

$\langle\gamma_{mn}(W)\rangle$ is defined analogously. Now we get a recursion relation for π_{mn} and γ_{mn}. We multiply Eqs. (1a, b) by $(x^m\theta_x^n+y^m\theta_y^n)$ and integrate over x, y, θ_x, θ_y. The terms containing derivatives with respect to the spatial variables are then transformed by integration by parts and we find that our equations become

$$0=L_1(\pi_{mn}(E), \gamma_{mn}(W))+(E_s^2/4E^2)n(n-1)\pi_{m, n-2}(E)+m\pi_{m-1, n+1}(E), \quad (6a)$$

$$0=L_2(\alpha_{mn}(E), \gamma_{mn}(W))+m\gamma_{m-1, n+1}(E). \quad (6b)$$

Equations (6a, b) have a solution of the following form:

$$\pi_{mn}(E)=\alpha_{mn}E_s^{m+n}/E^{m+n+2}, \quad (7a)$$

$$\gamma_{mn}(W)=\beta_{mn}E_s^{m+n}/W^{m+n+2}, \quad (7b)$$

where α_{mn} and β_{mn} are independent of E and W. If we substitute from Eqs. (7a, b) into Eqs. (6a, b), we are led to the following equations, for $E\neq E_0$:

$$0=-A(m+n+1)\alpha_{mn}+B(m+n+1)\beta_{mn}+\tfrac{1}{4}E_s^2 n(n-1)\alpha_{m, n-2}+m\alpha_{m-1, n+1}, \quad (8a)$$

$$0=C(m+n+1)\alpha_{mn}-\sigma\beta_{mn}+m\beta_{m-1, n+1}. \quad (8b)$$

The functions A, B, C appearing here are the same as those in ordinary shower theory and are defined by Eq. (2.17) of Rossi and Greisen's article.

Since π_{00} and γ_{00} are known, one can solve Eqs. (8a, b) by successively putting: $m=0$, $n=2$; $m=1$, $n=1$; $m=2$, $n=0$; $m=4$, $n=0$; $m=3$, $n=1$, etc. We have calculated α_{mn} and β_{mn} for m, n up to $m+n=10$, and then used Eq. (4) to find $\langle\pi_{mn}\rangle$ and $\langle\gamma_{mn}\rangle$ over the same range. The moments for which $m+n$ is odd vanish by symmetry.

The infinite sequence of moments thus obtainable determines in principle the distribution functions which we desire. We shall see in the next sections what can be deduced concerning the distribution functions from the partial sequence actually calculated.

Here we shall use these moments to check Molière's and Belenky's calculations of the distribution functions. Molière has derived expressions for the radial distribution functions, integrated over all angles, for both electrons and photons and the angular distribution function, integrated over all lateral displacements, for electrons.[7] Belenky has calculated the angular distribution of electrons.[5] To assess the correctness of Molière's and Belenky's calculation of these functions we have calculated[12] $\langle r^n\rangle_{\text{Av}}$ and $\langle\theta^n\rangle_{\text{Av}}$ from their distributions and compared these moments with those derived from

[9] The essential results of Sec. II(A) appear in an unpublished thesis by one of us (L. Eyges, Dissertation, Cornell University, 1948). These results were derived independently by Professor John Blatt and incorporated into his lecture notes. We are very grateful to Professor Blatt for allowing us to see copies of these notes. We have profited from several illuminating observations in them, as well as from suggestions for notation.

[10] L. Landau, J. Phys. (U.S.S.R.) 3, 237 (1940).

[11] B. Rossi and K. Greisen, Revs. Modern Phys. 13, 240 (1941).

[12] $r^2=x^2+y^2$ and $\theta^2=\theta_x^2+\theta_y^2$.

TABLE I. Exact moments in Approximation A and comparison with those derived from the distribution functions calculated by Molière and Belenky.[a]

n	Electrons: $\langle r^n\rangle_{Av}\cdot(E/E_s)^n$ Molière	Exact[a]	Electrons: $\langle\theta^n\rangle_{Av}\cdot(E/E_s)^n$ Molière	Belenky	Exact	Photons: $\langle r^n\rangle_{Av}\cdot(W/E_s)^n$ Molière	Exact	Photons: $\langle r^n\rangle_{Av}\cdot(W/E_s)^n$ Exact
0	1	1	1	1	1	1	1	1
2	0.830	0.725	0.602	0.655	0.570	1.02	1.13	0.176
4	6.40	7.24	1.72	1.43	0.959	16.0	26.4	0.178
6	1.06×10^2	4.95×10^2	30.4	6.56	3.10	5.45×10^2	3.34×10^3	0.415
8	2.76×10^3	1.38×10^5	1.12×10^3	51.6	16.1	3.19×10^4	1.36×10^6	1.69
10	1.03×10^5	1.03×10^8	5.42×10^4	620	121	2.85×10^6	1.34×10^9	10.4

[a] Similar results for the exact radial moments of electrons have been obtained by Nordheim, Osborne, and Blatt, Proceedings of the Echo Lake Cosmic Ray Symposium, December, 1949.

the above calculations, using, e.g., for electrons

$$\langle\pi_{m0}\rangle=\langle x^m+y^m\rangle_{Av}=2\langle x^m\rangle_{Av}=2\langle r^m\rangle_{Av}\langle\cos^m\phi\rangle_{Av} \quad (9a)$$

$$\langle\pi_{0n}\rangle=\langle\theta_x{}^n\rangle_{Av}+\langle\theta_y{}^n\rangle_{Av}=2\langle\theta_x{}^n\rangle_{Av}=2\langle\theta^n\rangle_{Av}\langle\cos^n\phi\rangle_{Av}. \quad (9b)$$

The results are given in Table I.

(B) Asymptotic Expansions for the Moments

We now consider the problem of finding the moments for energies where collision loss of electrons is not negligible, i.e., when we retain the term $\epsilon\partial\pi/\partial E$ in Eq. (2a). First, consider the equations for the track lengths, neglecting scattering. We call the electron track length $Z_\pi(E_0, E)$ and the photon track length $Z_\gamma(E_0, W)$. They satisfy the following equations:

$$\delta(E_0-E)=L_1(Z_\pi, Z_\gamma), \quad (10a)$$

$$0=L_2(Z_\pi, Z_\gamma). \quad (10b)$$

How to obtain an asymptotic solution for these equations which is valid down to energies a few times the critical energy is well known.[13] One assumes that Z_π and Z_γ have their high energy forms, modified by a correction factor in the form of a series in ϵ/E and ϵ/W; i.e.,

$$Z_\pi(E_0, E)=\frac{0.437E_0}{E^2}\sum_{n=0}^{\infty}a_n\left(\frac{\epsilon}{E}\right)^n, \quad (11a)$$

$$Z_\gamma(E_0, W)=\frac{0.437E_0}{\sigma W^2}\sum_{n=0}^{\infty}b_n\left(\frac{\epsilon}{E}\right)^n. \quad (11b)$$

If one substitutes these expressions into Eqs. (2a, b), one gets the following infinite set of equations:

$$\left.\begin{array}{l}[b_nB(n+1)/\sigma]-a_nA(n+1)\\ \qquad\qquad=(n+1)a_{n-1}\\ b_n=C(n+1)a_n\end{array}\right\}n=1, 2\cdots. \quad \begin{array}{r}(12a)\\(12b)\end{array}$$

These equations can be solved for a_n and b_n. Setting $a_0=1$, one gets for the electron track length:

$$Z_\pi(E_0, E)=\frac{0.437E_0}{E^2}\left(1-1.638\frac{\epsilon}{E}+2.799\frac{\epsilon^2}{E^2}-5.312\frac{\epsilon^3}{E^3}+11.18\frac{\epsilon^4}{E^4}+\cdots\right). \quad (13)$$

[13] Reference 11, p. 293.

It is possible to take forms somewhat different from Eqs. (11a, b) for Z_π and Z_γ. Following Rossi and Greisen, e.g., one can assume

$$Z_\pi(E_0, E)=0.437E_0/E^2\left(1+\alpha_1\frac{\epsilon}{E}+\alpha_2\frac{\epsilon^2}{E^2}+\cdots\right)^2. \quad (14)$$

Using the binomial expansion, Eq. (14) can be brought into the form (11a) and the relationship between a_n and α_n can be determined. Doing this, one gets the well-known expressions of Rossi and Greisen; namely, Eq. (2.96) of their paper. Similar results can be obtained for Z_γ.

We would like to emphasize that this last form for the track length is arbitrary in that one could assume series expansions in ϵ/E and ϵ/W, raised to *any* power, and then determine the coefficients in the same manner as above. The essential behavior of the series is not changed by writing it in a form other than that of Eq. (13). Thus, both Eq. (13) and Eq. (2.96) of Rossi and Greisen, seem to be valid down to $\epsilon/E\sim\frac{1}{3}$, and break down for higher values. It is true that for numerical computation one sometimes needs fewer terms in the series when it is in the latter form. On the other hand, the coefficients in Eq. (13) increase more slowly and in any computation, if one continues either of the series up to the point where the terms begin to increase, they give the same answer.

Now let us turn to the problem of calculating the moments when energy loss is taken into account. One transforms Eq. (1a, b) by integration by parts and defines π_{mn} and γ_{mn} as before. Equation (2a, b) still holds with the understanding that in the operator $L_1(\pi_{mn}, \gamma_{mn})$ one retains the term $\partial\pi_{mn}/\partial E$. One can find a formal solution of these equations by setting

$$\pi_{mn}(E)=\frac{\alpha_{mn}E_s{}^{m+n}}{E^{m+n+2}}\sum_{l=0}^{\infty}a_{mn}{}^{(l)}\left(\frac{\epsilon}{E}\right)^l, \quad (15a)$$

$$\gamma_{mn}(W)=\frac{\beta_{mn}E_s{}^{m+n}}{W^{m+n+2}}\sum_{l=0}^{\infty}b_{mn}{}^{(l)}\left(\frac{\epsilon}{W}\right)^l. \quad (15b)$$

On putting these expressions into Eqs. (6a, b) and equating to zero various powers of ϵ/E and ϵ/W, one

TABLE II. Electron moments including correction for ionization loss.

$$(\langle r^m\rangle_{\text{Av}})_\epsilon=\frac{[\langle r^m\rangle_{\text{Av}}]_{\epsilon=0}}{[1+\sum_{l=1}^\infty \alpha_{m0}^{(l)}(\epsilon/E)^l]^m} \qquad (\langle \theta^n\rangle_{\text{Av}})_\epsilon=\frac{[\langle \theta^n\rangle_{\text{Av}}]_{\epsilon=0}}{[1+\sum_{l=1}^\infty \alpha_{0n}^{(l)}(\epsilon/E)^l]^n}$$

	$\alpha_{m0}^{(l)}$				$\alpha_{0n}^{(l)}$		
$l \backslash m$	2	4	6	$l \backslash n$	2	4	6
1	1.73	1.60	1.52	1	0.813	0.915	1.02
2	−1.16	−2.02	−2.53	2	−0.747	−0.950	−1.17
3	2.57	5.42	6.62	3	1.46	2.09	2.88
4	−7.60	−19.6	−30.8	4	−3.73	−6.05	−9.31

TABLE III. Photon moments including correction for ionization loss.

$$(\langle r^m\rangle_{\text{Av}})_\epsilon=\frac{[\langle r^m\rangle_{\text{Av}}]_{\epsilon=0}}{[1+\sum_{l=1}^\infty \beta_{m0}^{(l)}(\epsilon/W)^l]^m} \qquad (\langle \theta^n\rangle_{\text{Av}})_\epsilon=\frac{[\langle \theta^n\rangle_{\text{Av}}]_{\epsilon=0}}{[1+\sum_{l=1}^\infty \beta_{0n}^{(l)}(\epsilon/W)^l]^n}$$

	$\beta_{m0}^{(l)}$				$\beta_{0n}^{(l)}$		
$l \backslash m$	2	4	6	$l \backslash n$	2	4	6
1	1.32	1.16	1.21	1	0.836	0.916	1.01
2	−0.766	−1.10	−1.95	2	−0.556	−0.755	−0.952
3	1.54	2.52	2.66	3	0.949	1.50	2.23
4	−3.80	−7.71	−13.0	4	−2.15	−3.98	−6.70

gets the following set of equations, for $l=0, 1, 2\cdots$:

$$\begin{aligned}\alpha_{mn}a_{mn}^{(l)}A(m+n+l+1)-\beta_{mn}b_{mn}^{(l)}B(m+n+l+1)&\\+(m+n+l+1)\alpha_{mn}a_{mn}^{(l-1)}&\\=\tfrac{1}{4}E_s^2n(n-1)\alpha_{m,\,n-2}a^{(l)}{}_{m,\,n-2}&\\+m\alpha_{m-1,\,n+1}a^{(l)}{}_{m-1,\,n+1};&\quad(16\text{a})\end{aligned}$$

$$\begin{aligned}\alpha_{mn}a_{mn}^{(l)}C(m+n+l+1)-\sigma\beta_{mn}b_{mn}^{(l)}&\\+m\beta_{m-1,\,n+1}b^{(l)}{}_{m-1,\,n+1}=0.&\quad(16\text{b})\end{aligned}$$

In these equations $a_{mn}^{(0)}=b_{mn}^{(0)}=1$ and $a_{mn}^{(-1)}=0$. For $l=0$, Eqs. (16a, b) are identical with Eqs. (8a, b). Thus, the quantities α_{mn} and β_{mn} are known from the work in Sec. II(A). The quantities $a_{mn}^{(l)}$ and $b_{mn}^{(l)}$ can then be determined successively in the following sequence:

$$\begin{aligned}&a_{02}^{(1)},\ b_{02}^{(1)};\ a_{02}^{(2)},\ b_{02}^{(2)};\ \cdots\to a_{11}^{(1)},\ b_{11}^{(1)};\\&\qquad a_{11}^{(2)},\ b_{11}^{(2)}\cdots\to a_{20}^{(1)},\ b_{20}^{(1)};\\&\qquad a_{20}^{(2)},\ b_{20}^{(2)}\cdots\to a_{04}^{(1)},\ b_{04}^{(1)};\\&\qquad\qquad a_{04}^{(2)},\ b_{04}^{(2)}\cdots,\ \text{etc.}\end{aligned}$$

As before, we are not directly interested in π_{mn} and γ_{mn} but in these quantities divided by π_{00} and γ_{00}, respectively. If we then formally carry out this division using $\pi_{00}=2Z_\pi$, $\gamma_{00}=2Z_\gamma$ as given by Eq. (13), we get expressions for π_{mn} and γ_{mn} again in the form of the high energy expressions multiplied by a series in powers of ϵ/E or ϵ/W. For convenience in computation we can convert these series to the form

$$\langle\pi_{mn}(E)\rangle_\epsilon=\langle\pi_{mn}\rangle_{\epsilon=0}\Big/\left[\sum_{l=0}^{\infty}\alpha_{mn}^{(l)}\left(\frac{\epsilon}{E}\right)^l\right]^{m+n},$$

$$\langle\gamma_{mn}(W)\rangle_\epsilon=\langle\gamma_{mn}(W)\rangle_{\epsilon=0}\Big/\left[\sum_{l=0}^{\infty}\beta_{mn}^{(l)}\left(\frac{\epsilon}{W}\right)^l\right]^{m+n}.$$

We will not present here our numerical results for all the quantities $\alpha_{mn}^{(l)}$ and $\beta_{mn}^{(l)}$, since they are probably of no great interest. The quantities of real interest are $\langle r^n(E)\rangle_{\text{Av}}$ and $\langle\theta^n(W)\rangle_{\text{Av}}$. In the Tables II and III we present our results for these quantities.

It is hard to estimate the range of validity of the series in the denominators of the expressions in Tables II and III. First, we have derived them purely formally, and in the process have divided dubiously convergent series into one another. Also, it is clear that for small values of E/ϵ and W/ϵ they diverge rather violently, particularly for the higher radial moments. Nevertheless, it is probably satisfactory to compute with them, provided one terminates the series when the terms start to increase. The reason we believe this to be so is that the series for the track lengths in Eq. (13) seems to show the same dubious convergence, but they have been checked and found to be quite accurate for E/ϵ greater than two or three. As a further check we have compared our results for the mean squares with the fairly accurate calculations of Roberg and Nordheim and have found good agreement down to about five times the critical energy, and even at twice the critical energy our results do not differ from theirs by more than 20 or 30 percent.

It is worth noting that our results for the moments of the angular distribution are valid down to somewhat lower energies than for the radial distribution. This is also true for the higher order moments; from Tables II and III we see that the series for the higher radial moments converges more poorly than for the angular moments of the same order.

III. THE DISTRIBUTION FUNCTIONING

(A) Approximation A

We now turn to the problem of calculating the actual distribution functions under Approximation A, using the moments found in Sec. II(A). We shall concern ourselves with the angular distributions integrated over all displacements; from symmetry this distributions are a function only of $\theta=(\theta_x^2+\theta_y^2)^{\frac{1}{2}}$. Similarly, the radial distributions are a function only of $r=(x^2+y^2)^{\frac{1}{2}}$. Moreover, from the structure of the equations, the distributions depend on (E, r) and (W, r) through the combinations Er/E_s and Wr/E_s. We can denote both of these quantities by x without confusion. The angular distributions depend on $E\theta/E_s$ and $W\theta/E_s$, both of which we call y; r is measured in radiation units and θ in radians. We shall call $P_r(Er/E_s)\equiv P_r(x)$ the radial distribution of electrons and $P_\theta(E\theta/E_s)\equiv P_\theta(y)$ the angular distribution of electrons. Similarly, we call $Q_r(Wr/E_s)\equiv Q_r(x)$ the radial distribution of quanta and $Q_\theta(W\theta/E_s)$ the angular distribution. The distribution functions are defined so that $P_r(Er/E_s)rdr$ is proportional to the number of electrons of energy E in the annular ring between r and $r+dr$, and $P_\theta(E\theta/E_s)\theta d\theta$ is proportional to the number of electrons of energy E in the solid angle between θ and $\theta+d\theta$. We will choose normalization so that $\int_0^\infty P_r(x)xdx=1$ and $\int_0^\infty P_\theta(y)ydy=1$ and similarly for the photons.

We assume that knowledge of the first few moments of a "reasonable" function essentially determines the function over a limited range. This assumption is based on the results of several "experiments" in which we tried to reconstruct known functions from a knowledge of the first four even moments alone. The functions chosen for this test were roughly of the same form as the expected distribution function. After trying various analytical schemes[14] we found that the most convenient method for reconstructing the function was simply to graph an arbitrary function, calculate its moments numerically, alter the function as indicated by the discrepancies from the correct moments, etc.

If one can extrapolate conclusions from the examples we tried, it would appear that it is readily feasible to fit a function over most of its range to within a few percent, even when not much care is taken in fitting the highest moment.[15] One cannot determine the behavior at the origin, however, with any certainty. This fact is particularly bothersome because one often wants to know the cosmic-ray distribution functions near the origin. The reason that the behavior at the origin is not determinable is as follows. The integrand for the nth moment is of the form $x^{n+1}f(x)$, and for increasing n the maximum of this function moves farther and farther along the x-axis. Moreover, this integrand vanishes very strongly for small x, so the high moments are essentially independent of the behavior of the function at the origin. Thus, most of the information about the origin is contained in the second moment; but the integrand even of this vanishes very strongly at the origin. For example, if $f(x)$ has a $1/x$ singularity, the integrand behaves like x^2. We feel particularly keenly here the fact that we know only the even moments. Knowledge of f_1 would help considerably in the fitting the function for small x.

Consider now the radial distribution of electrons. As we have explained, our method of fitting functions by their moments does not give the behavior near the origin. For the higher moments Molière's function seems to be quite inaccurate; but the second moment, which depends most sensitively on the behavior near the origin, differs from the correct value by only 12 percent. It seems reasonable then to assume that Molière's function is essentially correct for small x, and to start calculations on this basis. Actually, we reversed the procedure and used the higher moments first; i.e., we found the form of the function for large x and worked down toward the origin. If one can base an estimate on the examples cited, our function should be quite accurate down to about an x of 0.4. Our function also joins smoothly to Molière's at this point. In Table IV we present our results.

In calculating the radial distribution of photons there is again the difficulty that the distribution function has a singularity at the origin. For this case also the second moment as calculated from Molière's distribution function is not far off. We have felt justified in assuming his distribution function to be correct up to $x=0.4$, and calculating the function for higher values from the moments. The results are given in column 3 of Table IV.

The calculations of the angular distribution of electrons is somewhat simpler than for the above two cases, since there is no singularity at the origin. Our results are given in the fourth column of Table IV.

The angular distribution of photons can, of course, be calculated by the same methods we have used for the other distributions. Alternatively, it is clear on physical grounds that it is determined once the angular distribution of electrons is known, since photons are not scattered, but inherit their angular distribution from parent electrons of higher energy. Mathematically,

TABLE IV. Distribution functions in Approximation A.

	$x=Er/E_s$ or Wr/E_s		$y=E\theta/E_s$ or $W\theta/E_s$	
x or y	$P_r(x)$[a]	$Q_r(x)$[b]	$P\theta(y)$	$Q\theta(y)$[c]
0			9.27	
0.1			7.13	
0.2	7.62	7.33	5.35	8.60
0.4	2.74	1.12	2.78	2.19
0.6	1.01	3.73×10^{-1}	1.52	7.68×10^{-1}
0.8	4.84×10^{-1}	1.96	8.20×10^{-1}	2.86
1.0	2.52	1.21	4.46	1.18
1.2	1.47	8.58×10^{-2}	2.32	4.56×10^{-2}
1.4	8.72×10^{-2}	6.04	1.18	1.96
1.6	5.36	4.55	6.04×10^{-2}	8.6×10^{-3}
1.8	3.49	3.48	2.90	3.82
2.0	2.26	2.67	1.46	1.72
2.2	1.53	2.08	7.13×10^{-3}	7.82×10^{-4}
2.4	1.02	1.62	3.56	3.58
2.6	7.20×10^{-3}	1.29	1.74	1.66
2.8	5.16	1.02	8.29×10^{-4}	7.79×10^{-5}
3.0	3.73	8.25×10^{-3}	3.92	3.59
3.5	1.74	4.70		
4.0	8.72×10^{-4}	2.73		
4.5	4.52	1.57		
5.0	2.51	9.70×10^{-4}		
5.5	1.46	6.04		
6.0	8.80×10^{-5}	3.82		
6.5		2.46		
7.0		1.60		

[a] For $0\leqq x\leqq 0.2$ we assume Molière's distribution function to be valid. In expanded form it is

$$P_r(x)=21.37x^{-1/3}-30.79+66.75x^{5/3}-66.99x^2+\cdots.$$

[b] For $0\leqq x\leqq 0.2$ we again use Molière's distribution function (renormalized); *viz.*,

$$Q_r(x)=\frac{31.94\exp[-2x/(0.1)^{\frac{1}{2}}]}{2x/(0.1)^{\frac{1}{2}}}+\frac{0.806\exp[-2x/(3.25)^{\frac{1}{2}}]}{3.25}.$$

[c] For $0\leqq y\leqq 0.2$

$$Q\theta(y)\approx(3.44/y)e^{-3.46y}.$$

[14] While this work was being prepared for publication, we received a copy of a paper by L. V. Spencer and U. Fano entitled "Penetration and diffusion of x-rays: VII. Calculation of space distributions by polynomial expansion." One of the points of their paper seems to be the same as that of this section; *viz.*, the first few moments of a reasonable function essentially determine the function over a limited range. Unfortunately, the neat method of polynomial expansions that they describe is not directly applicable to our problem, since our functions may be singular at the origin.

[15] For further details see L. Eyges and S. Fernbach, UCRL Report No. 943, *Angular and Radial Distributions of Particles in Cascade Showers.*

TABLE V. Distribution functions for $E=10\epsilon$. Normalization of $P_r(x)$, $Q_r(x)$ is arbitrary. $\int_0^\infty P_\theta(y)ydy=1$. $x=10\epsilon r/E_s$, $y=10\epsilon\theta/E_s$.

x or y	$P_r(x)$	$Q_r(x)$	$P_\theta(y)$
0	...	...	10.0
0.1	...	...	7.42
0.2	9.0	7.33	5.51
0.3	4.5	2.50	4.24
0.4	2.45	1.12	3.08
0.5	1.35	6.36×10^{-1}	2.17
0.6	8.7×10^{-1}	3.73	1.56
0.7	5.7	2.63	1.11
0.8	3.85	1.96	7.95×10^{-1}
0.9	2.72	1.53	5.73
1.0	1.98	1.21	4.14
1.2	1.10	8.1×10^{-2}	2.01
1.4	6.5×10^{-2}	5.73	9.54×10^{-2}
1.6	3.85	4.13	4.29
1.8	2.35	3.13	1.96
2.0	1.53	2.33	8.80×10^{-3}
2.4	7.2×10^{-3}	1.37	1.64
2.8	3.43	8.22×10^{-3}	3.02×10^{-4}
3.2	1.75	5.28	5.40×10^{-5}
3.6	9.3×10^{-4}	3.32	
4.0	5.32	2.15	
4.5	2.77	1.30	
5.0	1.53	8.03×10^{-4}	
5.5		4.89	
6.0	5.04×10^{-5}	3.06	

this is clear from Eqs. (2b) and (3b), which give

$$Q_\theta(E\theta/E_s)=(1/\sigma)\int_0^1 P_\theta(E\theta/E_s v)\phi(v)dv/v.$$

If we take $\phi(v)=1/v$ and write $y=E\theta/E_s$, we have

$$Q_\theta(y)=(1/\sigma)\int_0^1 P_\theta(y/v)dv/v^2.$$

Now to a rough approximation $P_\theta(y)$ is just an exponential,

$$P_\theta(y)\approx 12\exp[-(12)^{\frac{1}{2}}y].$$

Therefore,

$$Q_\theta(y)\approx(12/\sigma)\int_0^1 \exp[-(12)^{\frac{1}{2}}y/v]\frac{dv}{v^2}\propto y^{-1}\exp[-(12)^{\frac{1}{2}}y]$$

is a rough approximation to the angular distribution of photons. We have improved on this approximation by the method of moments, assuming that the above expression for $Q_\theta(y)$ is approximately correct near the origin. Our results are given in the last column of Table IV.

It is interesting to compare the results of our calculations of the distribution functions with those of Molière. For the sake of brevity we shall simply describe the main features of this comparison. Consider first $P_\theta(y)$. Our calculations agree with Molière's to within a few percent up to about $y=1.7$. Beyond this point Molière's function becomes smaller than ours, by a factor 0.91 at $y=2$ and 0.36 at $y=2.5$. Around $y=3$ Molière's function becomes negative. Our calculations of $Q_r(x)$ also agree with Molière's to within a few percent up to $x=0.5$, where they begin to differ; but at no point up to $x=5$ do they differ by more than 25 percent. Our values for $P_r(x)$ show the greatest disagreement with those of Molière. There is good agreement up to about $x=0.6$, but at this point Molière's function begins to drop below ours and becomes lower by a factor 0.85 at $x=1.2$. Molière's curve then approaches ours, crosses at $x=1.8$ and becomes larger by a factor 2 at $x=3.5$. Then, it again approaches ours and crosses it at $x=5.5$.

(B) Effect of Collision Loss

Now we turn to the problem of calculating the distribution functions for energies where collision loss is not negligible, using the moments derived in Sec. II(A). There are no essential differences in this work from that of Sec. III(A); the main difficulty here is that for the lowest energies with which we deal the behavior of our series for the moments is rather dubious. For energies down to about five times the critical energy the latter difficulty is probably not very serious; our expressions for the moments are probably accurate within a few percent. Moreover, we are helped by the following fact: as one goes down in energy the distribution functions become steeper and their shape over the range of interest becomes less sensitive to the less accurate higher moments.

Also, when collision loss is included, we have less knowledge of the behavior of the functions at the origin. In Approximation A we could rely more or less on Molière's calculations; in the present case we must guess. The best guess seems to be that the singularities

TABLE VI. Distribution functions for $E=5\epsilon$. Normalization of $P_r(x)$, $Q_r(x)$, $Q_\theta(y)$ is arbitrary. $\int_0^\infty P_\theta(y)ydy=1$. $x=5\epsilon r/E_s$, $y=5\epsilon\theta/E_s$.

x or y	$P_r(x)$	$Q_r(x)$	$P_\theta(y)$	$Q_\theta(y)$
0	...	...	11.2	...
0.1	...	...	8.29	...
0.2	9.9	7.33	6.05	9.0
0.3	4.7	2.50	4.37	4.2
0.4	2.4	1.12	3.14	2.15
0.5	1.4	6.36×10^{-1}	2.24	1.20
0.6	8.0×10^{-1}	3.73	1.57	6.60×10^{-1}
0.7	4.8	2.63	1.10	3.70
0.8	3.3	1.96	7.7×10^{-1}	2.00
0.9	2.34	1.53	5.4	1.20
1.0	1.70	1.21	3.75	7.20×10^{-2}
1.2	9.00×10^{-2}	7.87×10^{-2}	1.74	2.80
1.4	5.1	5.43	7.7×10^{-2}	1.10
1.6	3.10	3.90	3.36	4.5×10^{-3}
1.8	1.90	2.83	1.40	1.7
2.0	1.20	2.06	5.6×10^{-3}	6.7×10^{-4}
2.4	5.3×10^{-3}	1.15	7.8×10^{-4}	8.0×10^{-5}
2.8	2.50	6.57×10^{-3}	1.12	7.2×10^{-6}
3.2	1.27	3.98		
3.6	6.70×10^{-4}	2.45		
4.0	3.70	1.57		
4.5	1.88	9.17×10^{-4}		
5.0	1.05	5.35		
5.5		3.13		
6.0	3.47×10^{-5}	1.87		

at the origin is the same as for high energies. Thus, from Table II we see that the effect of collision loss on the moments is least for the lower order moments, i.e., for small distances and angles. It does not seem unreasonable then to guess that the behavior of the distribution functions for small values of their arguments is unchanged from that when collision loss is neglected. At the very least, it seems certain that, since $P_r(x)$, $Q_r(x)$, and $Q_\theta(y)$ are all singular at the origin in Approximation A, they will also be singular when collision loss is included. This is confirmed by our calculations. We cannot determine the *order* of the singularity by our method of moments, but can determine that the singularity exists. This shows up when one reconstructs the distribution functions for large values of the argument and then tries to continue the function in toward the origin. It turns out to be impossible to fit all of the moments with a function which is not singular. We have reconstructed $P_r(x)$, $Q_r(x)$ and $Q_\theta(y)$ for $E=10\epsilon$ and the first two of these functions for $E=5\epsilon$ in the manner indicated above, i.e., by starting with large values of the arguments and working toward the origin. If we can judge from the examples mentioned above, our functions should be quite accurate down to about x or y about 0.4 and not be off more than by about 50 percent down to 0.2. Our results are presented in Tables V and VI.

In Approximation A, $P_\theta(y)$ is finite at the origin. As far as one can tell from the moments, it is also finite when collision loss is taken into account. If one assumes this, it is possible to reconstruct the function down to $y=0$. Unfortunately, we cannot estimate the accuracy of our function for very small y. The results are given in Tables V and VI for $E=10\epsilon$ and $E=5\epsilon$.

For $E=10\epsilon$, $x=0.2$ corresponds to a distance of $r=0.2E_s/10\epsilon$ radiation lengths. For air at sea level, for which $\epsilon=88$ Mev and $r=300$ meters, this corresponds to a distance of 1.4 meters. For Pb for which $\epsilon=6.7$ Mev and $r=0.51$ cm, this corresponds to a distance of 0.3 mm. We should like to remind the reader that the assumptions implicit in our calculations of the distribution functions are that the scattering angles are small, the asymptotic expressions for pair production are valid, and that Compton effect is negligible. The distribution functions given in Table V and VI are thus more accurate for light elements than for heavy ones. For air, for example, $E=2\epsilon$ corresponds to 176 Mev, where the assumptions above are fairly well satisfied. For Pb, on the other hand, $E=2\epsilon$ corresponds to 13 Mev; at this energy the scattering angles cannot be considered small nor are the asymptotic cross sections valid.

TABLE VII. Distribution functions for $E=2\epsilon$. Normalization of $P_r(x)$, $Q_r(x)$, $Q_\theta(y)$ is arbitrary. $\int_0^\infty P_\theta(y)ydy=1$. $x=2\epsilon r/E_s$, $y=2\epsilon\theta/E_s$.

x or y	$P_r(x)$	$Q_r(x)$	$P_\theta(y)$	$Q_\theta(y)$
0	...	...	11.8	...
0.1	...	...	9.0	...
0.2	11.0	7.33	6.8	9.4
0.3	5.6	2.50	5.1	4.0
0.4	2.85	1.12	3.65	1.90
0.5	1.55	6.36×10^{-1}	2.6	9.5×10^{-1}
0.6	8.0×10^{-1}	3.73	1.80	5.2
0.7	4.35	2.63	1.18	2.87
0.8	2.30	1.96	7.1×10^{-1}	1.58
0.9	1.30	1.53	4.4	8.8×10^{-2}
1.0	8.5×10^{-2}	1.21	2.70	5.0
1.2	4.0	7.6×10^{-2}	1.08	1.5
1.4	2.18	5.04	3.9×10^{-2}	4.7×10^{-3}
1.6	1.28	3.48	1.42	1.38
1.8	8.0×10^{-3}	2.37	5.0×10^{-3}	4.1×10^{-4}
2.0	5.1	1.64	1.65	1.33
2.4	2.33	8.03×10^{-3}	1.70×10^{-4}	1.15×10^{-5}
2.8	1.14	4.10	2.10×10^{-5}	1.15×10^{-6}
3.2	5.85×10^{-4}	2.18		
3.6	3.14	1.24		
4.0	1.80	7.65×10^{-4}		
4.5	9.2×10^{-5}	4.28		
5.0	4.9	2.52		

We have also calculated the distribution functions for $E=2\epsilon$. For this case, the expressions for the radial moments are probably very inaccurate, and our radial distributions may very well be quite inaccurate. The numbers in Table VII thus represent more a guess at an extrapolation than a calculation. The angular distribution functions are probably somewhat more accurate than the radial functions since the expressions for the angular moments converge much better than for the radial moments. But even so, one cannot put much faith in even the angular functions for the reasons given in the last paragraph. The remarks made above concerning normalization and the behavior of the functions at the origin apply here also.