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Overlapping Divergences and the S-Matrix

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By extending considerations given by Dyson, general rules are obtained for isolating divergent parts from integrals corresponding to overlapping graphs, and a proof is obtained for the appearance of an extra factor Z_1^{-1} from "*b* divergences." In the last section the possibility of renormalization for scalar meson-nucleon interactions is demonstrated.

I. INTRODUCTION

In his treatment of spinor electrodynamics Dyson[1] has defined operators Λ_μ, Σ^*, and Π^* corresponding to the three types of primitive divergent graphs in the theory. In the calculation of the contribution to Γ_μ arising from a reducible vertex part V_R it is possible to break V_R down unambiguously into an irreducible vertex part plus various inserted self-energy (*S*) and vertex (*V*) parts. The divergences introduced by the latter can be removed in a well-defined manner because any two of the insertions made in V_R are either completely non-overlapping or else are so arranged that one is completely contained in the other. This procedure fails, however, in the calculation of the contributions to Σ^* or Π^* from reducible self-energy graphs. Considering, for example, the electron self-energy, there is just one irreducible graph W_1 (Fig. 1). *V* parts inserted at one of the two end vertices *a* or *b* appear simultaneously as vertex insertions at the other vertex. Correspondingly, the contribution to Σ^* arising from a reducible part W_R is, in general, an integral which involves divergences corresponding to each of the ways in which W_R might have been built up by insertion of *V* parts at either or both vertices of W_1. Dyson has called these "*b*-divergences" and their expected effect is appearance of an extra factor Z_1^{-1} in his Eqs. (88) and (89). In order to demonstrate the possibility of renormalization, it is vital that this factor should appear; and it is the purpose of this paper to attempt a formal proof. The considerations presented here throw some light on the prospects of renormalization for scalar electrodynamics.

II. SEPARATION OF OVERLAPPING DIVERGENCES

Dyson[2] (unpublished) has defined a formal mathematical procedure for the isolation of the divergent part from an integral representing overlapping divergent graphs. This procedure is illustrated most readily by an example.

Figure 2 represents an electron self-energy graph which could be obtained by the insertion of a *V* part at vertex *a* or *b* of Fig. 1. Here

$$\Sigma(W_2, p)=e^4\int\int dt_1dt_2F_\mu(p, t_1)G_{\nu\mu}(p, t_1, t_2)H_\nu(p, t_2), \quad (1)$$

where

$$F_\mu(p, t_1)=\frac{1}{t_1^2}\gamma_\mu\frac{i\gamma(p-t_1)-\kappa}{(p-t_1)^2+\kappa^2},$$

$$G_{\nu\mu}(p, t_1, t_2)=\gamma_\nu\frac{i\gamma(p-t_1-t_2)-\kappa}{(p-t_1-t_2)^2+\kappa^2}\gamma_\mu,$$

$$H_\nu(p, t_2)=\frac{i\gamma(p-t_2)-\kappa}{(p-t_2)^2+\kappa^2}\gamma_\nu\frac{1}{t_2^2}.$$

Not only is the double integral over t_1t_2 linearly divergent, but it also diverges logarithmically if the integration is performed over either t_1 or t_2, while the other variable is held fixed. In order to isolate these divergences, we use (here and in the subsequent work) the invariant separation procedure outlined in Sec. VI of D II.

Rewrite Eq. (1) as follows:

$$\begin{aligned}\Sigma(W_2, p)=e^4\int\int dt_1dt_2[&F_\mu(p, t_1)G_{\nu\mu}(p, t_1, t_2)H_\nu(p, t_2)\\ &-F_\mu(p_0, t_1)G_{\nu\mu}(p_0, t_1, 0)H_\nu(p, t_2)\\ &-F_\mu(p, t_1)G_{\nu\mu}(p_0, 0, t_2)H_\nu(p_0, t_2)]\\ &+e^4\left(\int dt_1F_\mu(p_0, t_1)G_{\nu\mu}(p_0, t_1, 0)\right)\\ &\times\left(\int dt_2H_\nu(p, t_2)\right)+e^4\left(\int dt_1F_\mu(p, t_1)\right)\\ &\times\left(\int dt_2G_{\nu\mu}(p_0, 0, t_2)H_\nu(p_0, t_2)\right), \quad (2)\end{aligned}$$

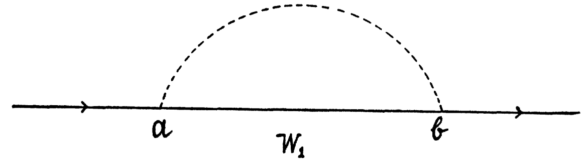


FIG. 1.

[1] F. J. Dyson, Phys. Rev. 75, 1736 (1949), referred to as D II, in this paper.

[2] The only published calculation for an integral corresponding to an overlapping (fourth order) graph is that of R. Jost and J. M. Luttinger, Helv. Phys. Acta 23, 201 (1949), who have followed Dyson's procedure.

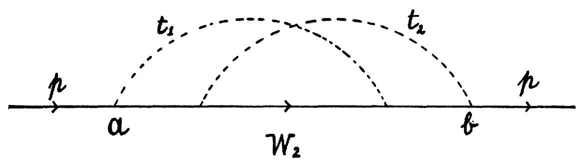


FIG. 2.

where $p_0^2+\kappa^2=0$, $i\gamma p_0+\kappa=0$. If

$$\int dt_1 F_\mu(p_0, t_1) G_{\nu\mu}(p_0, t_1, 0) = L_a \gamma_\nu,$$

$$\int dt_2 G_{\nu\mu}(p_0, 0, t_2) H_\nu(p_0, t_2) = L_b \gamma_\mu,$$

then we write

$$\Sigma(W_2, p) = e^4 \int\int dt_1 dt_2 [R(p, t_1, t_2)] + e^4 L_a \int \gamma_\nu H_\nu(p, t_2) dt_2 + e^4 L_b \int F_\mu(p, t_1) \gamma_\mu dt_1, \quad (3)$$

where $R(p, t_1, t_2)$ is the expression within square brackets in Eq. (2).

The terms within the square bracket are such that if t_2 is held fixed, the integration over t_1 is convergent if the first and the second terms are combined—this second term being obtained by putting in those factors of $F_\mu(p, t_1)G_{\nu\mu}(p, t_1, t_2)H_\nu(p, t_2)$ which contain t_1, all momenta other than t_1 equal to their free particle values—while, if t_1 is held fixed, the first and the third terms in the square bracket combine to give a convergent t_2 integration.

Algebraically, it is impossible to find a single term which if subtracted might make the t_1 and t_2 integrations convergent simultaneously. Even as it stands, the expression within the square brackets, is, as a whole, convergent neither over t_1 nor over t_2, nor has this subtraction made any difference in its degree of divergence when the double integration is performed. However, when t_2 is held fixed and the integration over t_1 is performed, the third term in the square bracket

$$-\int\int dt_1 dt_2 F_\mu(p, t_1) G_{\nu\mu}(p_0, 0, t_2) H_\nu(p_0, t_2)$$

can be written

$$-\left\langle \int\int dt_1 dt_2 [F_\mu(p, t_1) - F_\mu(p_0, t_1) - (p-p_0)(\partial/\partial p) F_\mu(p_0, t_1)](G_{\nu\mu}(p_0, 0, t_2) H_\nu(p_0, t_2)) \right\rangle - \left\langle \int dt_1 (F_\mu(p_0, t_1) + (p-p_0)(\partial/\partial p) F_\mu(p_0, t_1)) \right\rangle \times \left\langle \int dt_2 G_{\nu\mu}(p_0, 0, t_2) H_\nu(p_0, t_2) \right\rangle.$$

The integrand in the first angular bracket, $\langle\ \rangle$, above is absolutely convergent as far as the t_1 integration is concerned, while the second angular bracket, $\langle\ \rangle\langle\ \rangle$, is a product of a divergent constant L_b with a term of the form $A+B(\gamma p - i\kappa)$. Hence, with t_2 fixed, the integral of $R(p, t_1, t_2)$ over t_1 consists of an absolutely convergent part together with a divergent part, the latter being unambiguous and fully determinate. Since a similar result holds for t_2, we can say that the expression in the square bracket is "convergent" in a special sense over both the t_1 and t_2 integrations. It is to be emphasized that this has been made possible because, in the second (and also the third) term in the square bracket, the overlap factor G no longer appears as a function of two variables and so the integrand $F_\mu(p_0, t_1)G_{\nu\mu}(p_0, t_1, 0)H_\nu(p, t_2)$ splits into a product of two functions, each a function of a single variable.

In order to obtain the absolutely convergent (and physically significant) part of $\Sigma(W_2, p)$, we now perform the final separation. Thus,

$$\Sigma_c(W_2, p) = e^4 \int\int dt_1 dt_2 [R(p, t_1, t_2) - R(p_0, t_1, t_2) - (p-p_0)(\partial/\partial p) R(p_0, t_1, t_2)]. \quad (4)$$

If Eq. (4) is written out in full, we notice the basic point of this procedure. Whereas, in Eq. (4), we have subtracted terms with a view to securing convergence over the double integration, we have also succeeded, by the same step, in making $\Sigma_c(W_2, p)$ convergent for each of the two t_1 and t_2 integrations. The part which we designated above as the unambiguous and fully determinate divergent part of the t_1 integration, has in fact canceled out.[2a] The first subtraction in Eq. (2) made the expression "convergent" in our special sense over and t_2, the final subtraction in Eq. (4) made it absolutely convergent over all the t_1, t_2 and $t_1 t_2$ integrations. With the final step we have succeeded in isolating from the integral, divergences corresponding to the inserted V parts as well as the divergence corresponding to the over-all S part.

III. GENERAL RULES FOR THE SEPARATION OF DIVERGENCES

The procedure of Sec. II for the separation of divergent parts can be generalized to apply to any divergent n-fold integral howsoever complicated by overlapping or non-overlapping insertions. Consider a graph having n basic[3] momentum-vectors t_i associated with internal lines, such that the momentum vectors of the remaining lines are expressed in terms of these n vectors. The contribution to the corresponding matrix element is an integral, I, over n variables.

In general the integral will consist of a product of functions of each variable t_i (and possibly of momenta corresponding to external lines) with overlap functions,

[2a] This has happened because the integrals multiplying L_a and L_b in Eq. (3) are themselves self-energy integrals.

[3] P. T. Matthews, Phil. Mag. 41, 185 (1950).

i.e., functions of two or more variables, t_i. If a set of variables $t_a t_b \cdots t_m$ is held fixed, the integration over the remaining variables will be called a "subintegration." To estimate the convergence of each subintegration we employ the considerations given in Sec. V of D II and merely count the powers of the relevant t in the numerator and the denominator of the integrand. An n-fold integral, I, will be absolutely convergent if, besides the final integration, the subintegrations over all of the smaller sets of variables t_i, $t_i t_j$, $t_i t_j t_k$, $\cdots$, are also absolutely convergent for all possible choices of the basic variables. If by merely counting the powers of t_i we find that an integral is itself convergent, while any one of the subintegrations is not, the integral will be called "superficially convergent."

In general in an n-fold integral over $t_1 t_2 \cdots t_n$ a subintegration over $t_1 t_2 \cdots t_r$ can be made convergent only if we subtract from the integrand terms in which those factors in the integrand which are functions of t_{r+1}, $\cdots t_n$ only are left unchanged, while in the remaining factors the external momenta (which now include t_{r+1}, $\cdots t_n$) are given their free-particle values. Thus, to fulfill the condition of convergence over all subintegrations, we must subtract from the integral divergent terms corresponding to each subintegration, the degree of divergence of the integrand for a particular subintegration, in general, being unaffected by subtractions corresponding to other subintegrations.

The considerations of Sec. II can now be generalized. We define true divergence over a subintegration $t_1 t_2 \cdots t_r$ as the divergent part finally to be subtracted corresponding to this subintegration, in order to make the subintegration absolutely convergent, after the terms involving divergences corresponding to t_i, $t_i t_j$, $t_i t_j t_k$, $\cdots$, $t_1 t_2 \cdots t_{r-1}$, $\cdots$ subintegrations have already been separated. Anticipating the procedure to be formulated, we shall find that the above-mentioned divergences, corresponding to these latter integrations over smaller number of variables are themselves their true divergences. Thus, for example, to obtain the true divergence over $t_1 t_2$ subintegration we hold, first, all variables except t_1 fixed and subtract from the integrand divergent terms corresponding to t_1, leaving the rest of the integrand unchanged; similarly, holding all variables except t_2 fixed, we subtract divergent terms corresponding to t_2. Finally, holding all variables except $t_1 t_2$ fixed in this *new* integrand, we subtract divergent terms corresponding to the double integration. These last constitute the true divergence over $t_1 t_2$ subintegration. The integrand after the separation of all these divergent parts is absolutely convergent as far as $t_1 t_2$ subintegration is concerned.

We can now give, in analogy with Sec. II, unambiguous rules for the separation of divergences from an n-fold integral. Before making the final subtraction corresponding to the n-fold integration which will make the remaining integral an absolutely convergent one, we subtract from the integrand divergent terms corresponding to each of the subintegrations t_i, $t_i t_j$, $t_i t_j t_k$, $\cdots$, $t_1 t_2 \cdots t_{n-1}$. The divergent term to be separated corresponding to the $t_i t_j$ subintegration would, for example, consist of an n-fold integral over $t_1 t_2 \cdots t_n$, in which the part of the integrand containing t_i or t_j is such as to give, on integration, the true divergence corresponding to $t_i t_j$ multiplied by those factors in the original integrand which contain neither t_i nor t_j and have thus been left unchanged. In this expression, the integration over all variables other than $t_i t_j$ will correspond to a graph which we shall call the "reduced graph," and the corresponding integral the "reduced integral." Thus, in Sec. II, $\int dt_2 \gamma_\nu H_\nu(p, t_2)$ and $\int dt_1 F_\mu(p, t_1)\gamma_\mu$ are reduced integrals.

Our final result can be stated thus. The entire divergence to be separated from an n-fold integral is equal to the true divergence over the t_1 integration multiplied by the reduced integral over $t_2 \cdots t_n$ plus the true divergence over the t_2 integration multiplied by the reduced integral over $t_1 t_3 \cdots t_n$ and so on, together with the true divergence over the $t_1 t_2$ integration multiplied by the reduced integral over $t_3 t_4 \cdots t_n$ and so on, together with similar sets of terms ending finally with the true divergence over the final $t_1 t_2 \cdots t_n$ integration. After all these divergent terms have been isolated, the remainder is an absolutely convergent integral.

The maximum possible number of terms which may need to be isolated is $2^n - 1$. Of course, not all of the subintegrations will be divergent, a considerable number being "superficially" convergent, so that no corresponding separations will need to be made.

It may be possible (and indeed as we shall see later, it is possible in some special cases) to secure the actual convergence of certain subintegrations before the true divergence of the n-fold integration is separated. However, that this last separation will always leave behind an absolutely convergent integral as remainder, would follow from a generalization of considerations given in Sec. II.[3a]

A few remarks on superficial convergence will be relevant at this stage.

(a) In spinor electrodynamics all graphs with four or more external lines (except graphs corresponding to scattering of light by light) are at least superficially convergent. The "scattering of light by light" itself is not a genuine primitive divergent on account of the gauge invariance of the theory. It should, therefore, always be possible to find other graphs which when combined with the graph in question should give a convergent result. We shall therefore treat the integrals corresponding to the "scattering of light by light" too, as being at least formally superficially convergent. Opening two or more lines in a connected self-energy graph always leads to superficial convergence in the connected[3b] part of the remainder and so does the

[3a] A general proof of this will be published elsewhere.

[3b] Any part of a connected graph is joined to the rest by at least two lines.

opening of one or more lines in a vertex part, unless, of course, so many lines are opened that the only connected graph left is a V part. These considerations will immediately give the number of superficially convergent integrations, when dealing with S or V parts.

If an internal line t in a graph contains a self-energy insertion with associated momenta $t_1, t_2, \cdots t_r$, then the subintegrations over tt_i, tt_it_j, $\cdots$, $tt_1t_2\cdots t_{r-1}$, $\cdots$ (but not $tt_1t_2\cdots t_r$) are superficially convergent. t, being an internal line, itself belongs to a loop, which at the very worst consists of two electron lines, with but two external photon lines, the whole forming as far as $tt_1\cdots t_r$ integration is concerned, a photon self-energy graph. Opening one single line belonging to the self-energy insertion inside t gives at worst a scattering of light by light graph. Thus the subintegrations tt_i, tt_it_j, $\cdots$, $tt_1\cdots t_{r-2}$, $\cdots$ are certainly superficially convergent, while with the convention adopted above for the scattering of light by light graphs, the subintegrations over the last set $tt_1t_2\cdots t_{r-1}$, $\cdots$ are also superficially convergent.

(b) If two sets of variables $(t_i\cdots t_j)$ and $(t_a\cdots t_b)$ are such that the corresponding graphs do not overlap and if the integration over each set has been made convergent, then the product of these convergent integrands is also convergent over the subintegrations t_i, t_a, t_it_a, $\cdots$, $t_i\cdots t_jt_a\cdots t_b$. For integration of functions with no overlap terms certain new features arise. Taking a simple example, the integral $\int\int dt_1dt_2F(p, t_1)G(p, t_2)$ has no overlap term, and thus, if both t_1 and t_2 subintegrations are divergent (in this example logarithmically),

$$\int\int dt_1dt_2[F(p, t_1)-F(p_0, t_1)][G(p, t_2)-G(p_0, t_2)]$$

is absolutely convergent. We notice that the divergent part separated is

$$\int\int dt_1dt_2[F(p, t_1)G(p_0, t_2) + F(p_0, t_1)G(p, t_2)-F(p_0, t_1)G(p_0, t_2)].$$

Although it loses its precise significance, we may still formally retain the expression "true divergence" over t_1 for $\int dt_1F(p_0, t_1)$, over t_2 for $\int dt_2G(p_0, t_2)$, and over t_1t_2 for $-\int\int dt_1dt_2F(p_0, t_1)G(p_0, t_2)$. The negative sign before the true divergence from t_1t_2 integration, when overlap does not occur, makes the use of our general rule inconvenient for non-overlapping graphs.

The above remarks do show, however, that for non-overlapping graphs it is immaterial in what order divergences are removed, while for divergences arising from graphs one of which is completely contained in the other, Dyson's prescription of successive removal starting from the innermost insertion is a particular case of the general rule given above. We shall illustrate this rather more fully by considering self-energy insertions.

(c) Suppose a line t, in a graph I, contains a self-energy insertion with momenta $t_1, t_2, \cdots t_r$. It will be convenient to remove divergent parts from the integral corresponding to subintegrations t_1, $\cdots t_1t_2$, $\cdots$, $t_1t_2\cdots t_r$. After the "true divergence" over $t_1t_2\cdots t_r$ has been removed, the subintegration over this set is left absolutely convergent. The momenta associated with the self-energy insertion in line t define (except for t), no overlap function with any other momentum variable t_a in I. According to Remark (b), if the integrand is made convergent over t_a, its convergence over t_at_1, $\cdots t_at_1t_2\cdots t_r$, is already assured. Remark (a) shows that the same result holds for the only variable (t) which does have overlap functions with these variables. At this stage the only integrations in which $t_1, t_2, \cdots t_r$ appear explicitly are the $tt_1t_2\cdots t_r$ subintegration, and other subintegrations in which these variables appear as a group. But from considerations given by Dyson we know already that, for questions of convergence, this group will behave precisely as though the line t had no insertion inside it. This shows that it is immaterial whether we retain the variables $t_1, t_2, \cdots t_r$ or perform the integration over them at this stage; the latter procedure implies replacing $S_F(t)$ by $S_c(t)S_F(t)/2\pi$, which is precisely the result one would get if one followed the procedure of successive removal as described in D II. By using results on superficially convergent subintegrations derived by applying Remark (a), a similar proof could be constructed to show the validity of the procedure of successive removal for divergence arising from V parts one of which is completely contained in the other.

IV. APPLICATION TO ELECTRON SELF-ENERGY GRAPHS

We shall now apply the above considerations to the case of electron self-energy graphs. According to the Dyson-Feynman procedure for obtaining the effective radiative corrections to any physical process, one draws the relevant irreducible graphs and then replaces S_F by S_{F1}', D_F by D_{F1}', and γ_μ by $\Gamma_{\mu 1}$ in the corresponding integrals. One already uses the same procedure for calculating $\Gamma_{\mu 1}$ itself (Sec. VII, D II); one draws all of the irreducible vertex graphs and then makes the above substitutions. We attempt to obtain a similar set of self-energy graphs in which these substitutions can be made unambiguously—only in this case this set of graphs will not be irreducible. For this purpose we establish the following categories of self-energy graphs.

Category [1] consists of the sole irreducible self-energy graph in Fig. 1.

Category [2] consists of all graphs formed by inserting[4] at the end vertex b in Fig. 1 all irreducible V parts of any order in e. The first end vertex of each category $[n]$ will be called a and the last b, although sometimes, for clarity, we use $a_{[n]}$ and $b_{[n]}$.

To obtain the graphs in category [3] insert at $b_{[2]}$ all

[4] End vertices of self-energy graphs, vertex graphs, or Compton graphs, are the vertices where the external electron line enters or leaves the graph.

irreducible vertex parts, and so on for all subsequent categories.

Continuing the above procedure we obtain an infinite number of categories, each (except [1]) containing an infinite number of graphs. Figure 2, for example, belongs to category [2]. All graphs in $[n]$ can, in fact, be built up in precisely n different ways from the graph in Fig. 1 by the insertion of vertex parts at $a_{[1]}$ or $b_{[1]}$ or both. Further, all graphs in $[n]$ could equally well have been built by insertions at $a_{[n-1]}$ rather than at $b_{[n-1]}$. This complete symmetry will be found to be important for the subsequent proof.

A graph belonging to $[n]$ will contain precisely $(n-2)$ photon lines such that if these are opened (i.e., their associated integration variables held fixed), the graph splits into two vertex parts. Given a graph in $[n]$ we denote by L_b^1 the divergent constant arising from the irreducible V part which was inserted at $b_{[n-1]}$ in order to obtain this particular graph in $[n]$, by L_b^2 the "true" divergent constant arising from the reducible part which was inserted at $b_{[n-2]}$ to obtain this graph in $[n]$, and so on. This last V part is reducible because its end vertex has as an insertion the V part with the divergent constant L_b^1. Similar definitions apply for L_a^1, L_a^2, $\cdots$. We shall now prove the following lemmas.

Lemma 1.—Denote by $[n]$ the integral corresponding to a particular graph in the category $[n]$. We have the following result.

$$[n]=\langle L_a^1[n-1]+L_a^2[n-2]+L_a^3[n-3]+\cdots+L_a^{n-1}[1]\rangle +\langle L_b^1[n-1]+L_b^2[n-2]+L_b^3[n-3]+\cdots+L_b^{n-1}[1]\rangle -\left\langle \begin{matrix} L_a^1L_b^1[n-2]+L_a^1L_b^2[n-3]+L_a^1L_b^3[n-4]+\cdots \\ +L_a^2L_b^1[n-3]+L_a^2L_b^2[n-4]+\cdots \end{matrix} \right\rangle +\Sigma_{[n]}^*, \quad (5)$$

where

$$\Sigma_{[n]}^*=A_{[n]}+B_{[n]}(\gamma p-i\kappa)+\Sigma_{c[n]}^*.$$

A and B characterize the true divergence from $[n]$, and $[n-r]$ stands for an integral corresponding to a certain graph in category $[n-r]$ obtained from $[n]$ by successive removal of V parts from a or b as the case may be.

Proof.—It is convenient to associate the basic variables with the photon lines in the graph. Suppose $[n]$ has k basic variables of which $n-2$ are critical. The photon line t_1 starts from $a_{[n]}$, while t_k ends at $b_{[n]}$.

For the proof we shall systematize our procedure as follows: we hold each one of the variables t_i initially fixed, in turn, and remove all of the divergent terms corresponding to all possible subintegrations over the remaining $(n-1)$ variables. Of course, in following this procedure we shall have to guard against subtracting a divergent term more than once. However, holding t_k fixed we make the removal unambiguously corresponding to the subintegrations $t_1, \cdots, t_{k-1}, t_1t_2, \cdots, t_1t_{k-1}, t_2t_3, \cdots, t_2t_{k-1}, \cdots, t_1t_2t_3, \cdots, t_1t_2\cdots t_{k-1}$. This immediately gives us the terms in the first bracket of Eq. (5) by our general rule. The constants L_a', L_a^2, $\cdots$, etc., come from the successive reduction of a reducible vertex parts. It is to be emphasized that the first line in Eq. (5) is the result of the removal of divergences, corresponding to all the above subintegrations, due regard being paid to those subintegrations which are superficially convergent [see Remark (a)]. Now holding t_1 fixed we remove divergent terms corresponding to all the possible subintegrations over subsets of variables $t_2, t_3, \cdots t_k$. A number of these terms (those, for example, corresponding to subintegrations over $t_2, t_3, \cdots t_{k-1}, t_2t_3, \cdots t_2t_{k-1}, \cdots$, etc.) have already been removed. However, the subintegrations from these subsets which still remain, i.e., over $t_k, t_kt_{k-1}, \cdots t_kt_2, \cdots t_kt_{k-1}t_{k-2}, \cdots$ (barring those which are superficially convergent), cover the successive divergent parts of the (reducible) vertex graph one obtains by opening the line t_1 and are just sufficient to give us the second bracket in Eq. (5). The maximum number of subintegrations corresponding to which divergent terms still remain to be separated is 2^{k-2} at this stage. These terms must be separated by holding each of the remaining $k-2$ basic variables initially fixed. Of these only $(n-2)$ are critical. As noted earlier, opening a critical line splits the graph into two disconnected segments with no overlap term between them. As an example, let us assume t_2 to be the first critical variable, occurring from the left. Holding it fixed, we notice that the only divergent terms yet to be separated are the true divergences corresponding to the subintegrations $t_1t_k, t_1t_kt_{k-1}, t_1t_kt_{k-2}, \cdots$, those corresponding to $t_1, t_k, t_kt_{k-1}, t_kt_{k-2}, \cdots$, etc., already having been removed. Since these are the true divergences for integrations over nonoverlapping parts, we shall get a negative sign before the separated terms as noted in Remark (b). This leads to the isolation of divergent terms of the form $-L_a^1L_b^1[n-2]$. Starting from end a and opening all the critical lines, we shall get all the subtracted terms on the right-hand side of the Eq. (5). It remains to prove now that holding each one of the remaining $(k-n)$ lines fixed gives no further divergent terms to be separated. We notice that every other line, if opened, converts the graph into a C part,[5] which we know to be at least superficially convergent in spinor electrodynamics. If the C part obtained by opening a line is reducible, the only insertions its irreducible skeleton has are vertex parts, inserted at its end vertices. But the divergent terms corresponding to the subintegrations over the variables associated with these vertex insertions have already been removed by the successive removal performed by holding t_1 and t_k fixed initially. Thus, these C parts give no further contribution to Eq. (5). This establishes the lemma.

If we had started with basic variables associated with electron rather than with photon lines, opening a basic line would convert the graph into an M part[6]—again at least superficially convergent —and similar considerations would apply.

Lemma 2.—This is a restatement of Remark (c) for the particular case of graphs in these categories. The divergent separations after inserting S parts in all the lines or V parts in all the *internal* vertices (all vertices except a and b) of $[n]$ are given by the divergent parts corresponding to the subintegrations over the internal variables of the insertions [as in Remark (c)], together with the divergent terms to be separated from $[n]$, where $S_F(t)$ is replaced by $S_c(t)S_F(t)$, $D_F(t)$ by $D_F(t)D_c(t)$, and γ_μ by $\Lambda_{\mu c}(t, t')$. As a concrete example, we see that if a photon self-energy graph is inserted in a line t of $[n]$, such that $D_F(t)\Pi^*(t)=C+D_c(t)$, then the resulting integral is equal to $C[n]+[n]^\times$, where $[n]^\times$ is the integral obtained by replacing $D_F(t)$ by $D_c(t)D_F(t)$ in the integrand. The proof is exactly similar to that of Remark (c).

If we now desire to construct all self-energy graphs up to a given order e^{2k+2}, say, we draw all of the graphs in the various categories defined above up to this order. No graph belonging to category $[k+2]$ or higher will be needed, the lowest order graph belonging to the category $[k+2]$ being of the order e^{2k+4}. Taking each one of the graphs drawn above, V and S parts will be inserted in its internal vertices and all its lines. *No insertions are ever to be made in the end vertices of the graphs of any category.* If the graph in which insertions are being made is of order $2r$, these insertions need only consist of graphs up to the order $(2k-2r+2)$.

It is easy to verify now that the above procedure gives accurately all self-energy graphs up to order $(2k+2)$ and that no graph appears more than once. Among them we shall, of course, find all the graphs which could have been obtained by insertions of vertex parts up to order $2k$ at the vertices a (or b) of the fundamental self-energy graph in Fig. 1. To verify the appearance of all vertex parts, up to the given order, we notice, for example, that by the very principle of their construction the graphs of

[5] A Compton graph—a graph with 2 external photon and 2 external electron lines.

[6] Møller graph—a graph with four external electron lines.

category [3] represent the insertion of vertex parts in the end vertices of graphs of category [2], while the insertions in their own end vertices are given by the graphs of the higher categories.

Correspondingly, adopting the procedure in Sec. VII of D II of dropping the divergent constants when they appear, if we assume that S_{F1}', D_{F1}', $\Gamma_{\mu 1}$ are defined up to order e^{2k} by, respectively, S_F plus S_F multiplied by a finite sum of products of $S(W,t)$, D_F plus D_F multiplied by a finite sum of products of $D(W,t)$, and γ_μ plus a finite sum of $\Lambda_{\mu c}(t,t')$, we can obtain S_{F1}' up to the order e^{2k+2} by drawing all of the graphs in the various categories, up to the order e^{2k+2}, and then substituting in all lines S_{F1}', D_{F1}' (*defined as above*) instead of S_F and D_F, and $\Gamma_{\mu 1}$ in all internal vertices instead of γ_μ. The γ_μ occurring at the two end vertices are always to be left unchanged. Once again, in a graph of order $2r$ belonging to any one of the categories, it will only be necessary to substitute S_{F1}', etc., up to the order $(2k-2r+2)$.

Lemma 3.—If by the above changes in all of the lines and internal vertices of an integral arising from a graph in category $[n]$ we obtain an integral $[n]^\times$, then

$$\begin{aligned}[n]^\times = & L_a^{1\times}[n-1]^\times + L_a^{2\times}[n-2]^\times + L_a^{3\times}[n-3]^\times + \cdots \\ & + L_b^{1\times}[n-1]^\times + L_b^{2\times}[n-2]^\times + L_b^{3\times}[n-3]^\times + \cdots \\ & - (L_a^{1\times}L_b^{1\times}[n-1]^\times + L_a^{1\times}L_b^{2\times}[n-3]^\times + \cdots) + \Sigma^*{}_{[n]^\times},\end{aligned}$$

where $L_a^{1\times}$ stands for the "true" divergent constant from the V part introduced at $a_{[n-1]}$ in $[n-1]^\times$ in order to obtain $[n]^\times$. The definition of the other constants is similar. Noticing that $S_{F1}'(t)$ and $D_{F1}'(t)$, and $\Gamma_{\mu 1}$, as defined above, behave precisely as do $S_F(t)$, $D_F(t)$, and γ_μ as far as the power of variables involved and the question of convergence are concerned, the proof follows from Lemma 1.

Lemma 3 immediately gives the following set of equations

$$\begin{aligned}[1]^\times &= \Sigma^*{}_{[1]^\times} \\ [2]^\times &= (L_a^{1\times}+L_b^{1\times})[1]^\times + \Sigma^*{}_{[2]^\times} \\ [3]^\times &= (L_a^{1\times}+L_b^{1\times})[2]^\times + (L_a^{2\times}+L_b^{2\times})[1]^\times \\ & \qquad - L_a^{1\times}L_b^{1\times}[1]^\times + \Sigma^*{}_{[3]^\times} \\ & \cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots \end{aligned} \quad (6)$$

If in each equation the $L^\times$'s are once again understood as defined up to the requisite order and a summation is carried out over all graphs, we obtain, by summing up Eqs. (6),

$$([1]^\times + [2]^\times + \cdots + [k+1]^\times)(1-2L+L^2) = \Sigma^*{}_{[1]^\times} + \Sigma^*{}_{[2]^\times} + \cdots + \Sigma^*{}_{[k+1]^\times}, \quad (7)$$

where, following Dyson's definition of L,

$$L = L_a^{1\times} + L_a^{2\times} + L_a^{3\times} + \cdots = L_b^{1\times} + L_b^{2\times} + L_b^{3\times} + \cdots. \quad (8)$$

It is to be understood that the terms in the factor $(1-2L+L^2)$ on the left-hand side of Eq. (7) in fact only appear to the order needed. The symbol $[r]^\times$ now stands not only for an individual graph but for all of the graphs in the category $[r]$ with the insertions which have been needed to build up S_{F1}', to the required order. The right-hand side, consisting of the "true" divergent and the absolutely convergent parts of all the self-energy graphs up to the given order is, in Dyson's terminology, precisely equal to $A(e) + (1/2\pi)B(e)S_F^{-1} + (1/2\pi)S_c(e)S_F^{-1}$. Equation (7) is the result of substituting $S_{F1}'(e)$, $D_{F1}'(e)$ in all the lines and $\Gamma_{\mu 1}(e)$ in all but two vertices of the various graphs in our categories. Thus $[1]^\times + [2]^\times + \cdots$ will be called Σ_1^* (corresponding, but not equal to Σ_1^* in D II), and we obtain

$$S_F\Sigma_1^*(e) = -2\pi i\delta\kappa S_F + (1-L)^{-2}[A(e)S_F + (1/2\pi)B(e) + (1/2\pi)S_c(e)]. \quad (9)$$

Equation (9) replaces Dyson's Eq. (88). Here the term involving $\delta\kappa$ has been added to take account of the self-energy graph consisting of a point.

To proceed with the rest of the proof for the possibility of renormalization, we follow Dyson's arguments closely. Since no overlaps occur for V parts, their structure gives immediately a proof of Eq. (96) in D II; i.e., $Z_1 = 1 - L(e_1)$. We notice now that a self-energy graph of order $2r$, belonging to any of the categories $[n]$, contains r photon line, $2r-1$ electron lines, and $2r$ vertices. A substitution of $\Gamma_\mu = Z_1^{-1}\Gamma_{\mu 1}(e_1)$, $S_F' = Z_2 S_{F1}'(e_1)$, $D_F' = Z_3 D_{F1}'(e_1)$, made in the internal vertices, and all the lines gives an e factor, $e^{2r}Z_1^{-2r+2}Z_2^{2r-1}Z_3^r$. An extra factor Z_1^{-2} appears from Eq (9), and one thus obtains a possibility of absorption of these factors in charge renormalization if we choose $Z_2 = 1 + B(e_1)/2\pi$. It may be emphasized, once again, that in this manner of proof all constants and operators, and the charge renormalization itself are defined up to the requisite orders. A similar proof can be constructed for photon self-energy graphs, where, once again, insertions at one end vertex appear simultaneously as V-insertions at the other end vertex. It will be found to be expedient, in that case, to associate basic variables with electron lines in order to be able to remove practically all possible divergences with a single choice of basic variables.

(A) Pseudoscalar and Scalar Meson-Nucleon Interactions

The considerations given so far apply with minor modifications to exhibit the possibility of renormalization for the scalar theories of nuclear interaction. The possible primitive divergents for the interaction of scalar and pseudoscalar mesons with nucleons are the same as for spinor-electrodynamics.[3] But, whereas in electrodynamics graphs representing scattering of light by light do not in fact constitute a genuine primitive divergent, the corresponding graphs in meson theories, with four external meson lines are definitely logarithmically divergent. Further, in electrodynamics, graphs with three external photon lines could be paired with others obtained by reversing the direction of internal electron lines and were thus shown to vanish by an application of Furry's theorem (D II, Sec. IV). Since "charge-conjugation" does not lead to a reversal of sign in the case of scalar theories, graphs with three external meson lines present a new type of primitive divergent.

However, if Furry's theorem does not apply, we can still exclude graphs with odd numbers of external meson lines if the mesons are charged, by charge conservation. Also, for neutral pseudoscalar mesons, Dyson has shown, by using reflection properties of the relevant wave functions, that such graphs are in fact not divergent.[7]

In the sequel we shall consider pseudoscalar (charged or neutral) and scalar (charged) theories. Thus, besides the primitive divergents in electrodynamics the only new feature would be the divergent graphs corresponding to the Møller scattering of two mesons (M-parts). Following a suggestion first given by Matthews, it will be proved that the divergences arising from these M parts can be canceled consistently by adding an appropriate δ-type interaction between mesons to the lagrangian. We consider in detail only the case of pseudoscalar neutral meson, the remaining cases being almost identical.

Following Matthews we write the interaction hamiltonian as

$$H_1(x)=if\bar{\psi}(x)\gamma_5\psi(x)\phi(x)-\hbar c\delta\kappa_0\bar{\psi}(x)\psi(x)-\tfrac{1}{2}\delta\kappa^2\phi^2(x)-\delta\lambda\phi^4(x).$$

The corresponding graphs will have, besides the vertices with two nucleon and one meson line incident, also vertices with two nucleon, two meson, or four meson lines incident. These last will be called 4-vertices, while graphs formed entirely from 3-vertices will be called the "original graphs."

The following remark will apply to all except meson self-energy graphs, which will be considered later.

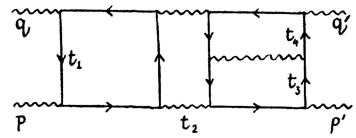
Corresponding to any original graph (formed entirely from 3-vertices), which itself represents a Møller scattering of mesons, or contains M parts inside itself, there exist in the theory graphs formed by replacing the M parts by 4-vertices *in all possible ways.* It is easy to verify that for a field theory with ϕ^4 as the only interaction term there are two primitive divergent graphs, graphs with two external meson lines (meson self-energy graphs), and graphs with four external meson lines (M parts).

We shall call a graph "simple" if it does not contain an M part inside itself, or is a Møller graph which cannot be formed by joining two or more M parts. We shall retain the definition of reducibility of a graph as adopted in electrodynamics. Thus, a graph may not be "simple" but may still be irreducible if it satisfies the criteria of irreducibility as in electrodynamics.

To choose the constant $\delta\lambda$ we proceed in three steps.

(1) Consider the category of all original M parts which are both "simple" and irreducible. If $\delta\lambda$ is chosen

FIG. 3.



equal to the sum of their true divergences (the true divergence of an M part consists of a constant), the term $-\delta\lambda\phi^4$ suffices to cancel all divergences arising from the above graphs. $\delta\lambda$ is thus a power series in f.

(2) Now consider the category of all original irreducible M parts which are not simple. Before removing the true divergence from the over-all M part, divergent terms have to be removed corresponding to the M parts inside the graph in accordance with the procedure given in Sec. III. We illustrate the points involved with a simple example. Figure 3 represents an (irreducible) M part obtained by joining two simple M parts. The corresponding integral is given by[8]

$$f^{10}\int dt_1dt_2dt_3dt_4[\text{Trace }\gamma_5S(p+t_1)\gamma_5S(p+t_1-t_2)\times\gamma_5S(t_1-q)\gamma_5S(t_1)]\Delta(t_2)\Delta(p+q-t_2)\times[\text{Trace }\gamma_5S(p'+t_3)\gamma_5S(t_3)\gamma_5S(t_4)\gamma_5S(t_4-q')\times\gamma_5S(p'+t_4-t_2)\gamma_5S(p'+t_3-t_2)]\Delta(t_4-t_3).$$

The integral is logarithmically divergent with respect to the t_1, t_3t_4, and $t_1t_3t_4$ subintegrations, besides being logarithmically divergent with respect to the final 4-fold integration. The remaining subintegrations are at least superficially convergent.

The divergent terms to be separated before the true divergence over $t_1t_2t_3t_4$ is separated, in order to leave behind an absolutely convergent integral are

$$(\delta\lambda)_1f^6\int dt_2dt_3dt_4\Delta(t_2)\Delta(p+q-t_2)\times[\text{Trace }\gamma_5S(p'+t_3)\gamma_5S(t_3)\gamma_5S(t_4)\gamma_5S(t_4-q')\times\gamma_5S(p'+t_4-t_2)\gamma_5S(p'+t_3-t_2)]\Delta(t_4-t_3)$$
$$+(\delta\lambda)_2f^4\int dt_1dt_2[\text{Trace }\gamma_5S(p+t_1)\gamma_5S(p+t_1-t_2)\times\gamma_5S(t_1-q)\gamma_5S(t_1)]\Delta(t_2)\Delta(p+q-t_2)$$
$$-(\delta\lambda)_1(\delta\lambda)_2\int dt_2\Delta(t_2)\Delta(p+q-t_2),$$

where

$$(\delta\lambda)_1=f^4\int dt_1[\text{Trace }\gamma_5S(p_0+t_1)\gamma_5S(p_0+t_1-t_{20})\times\gamma_5S(t_1-q_0)\gamma_5S(t_1)]$$

$$(\delta\lambda)_2=f^6\int dt_3dt_4[\text{Trace }\gamma_5S(p_0'+t_3)\gamma_5S(t_3)\times\gamma_5S(t_4)\gamma_5S(t_4-q_0')\gamma_5S(p_0'+t_4-t_{20})\times\gamma_5S(p_0'+t_3-t_{20})]\Delta(t_4-t_3)$$

and

$$p_0=q_0=p_0'=q_0'=t_{20},\quad p_0^2+\kappa^2=0$$

[7] *Note added in proof.*—Dyson has, in fact, proved that the contribution $F(p_1, p_2, p_3)$ from any graph with just three meson lines (p_1, p_2, p_3) in neutral pseudoscalar theory vanishes identically. Let $\bar{p}$ be a vector obtained from p by space reflection. Since $\phi(p_1)\phi(p_2)\phi(p_3)F(p_1, p_2, p_3)$ is a scalar, $F(p_1, p_2, p_3)$ is a pseudoscalar function and therefore $F(p)=-F(\bar{p})$. (This can be demonstrated by a more explicit mathematical proof.) But $F(p)$ is invariant under a proper Lorentz transformation; for example, under a space rotation, through 180° about an axis in line perpendicular to p_1 and p_2, which transforms p_1 to $\bar{p}_1$ and p_2 to $\bar{p}_2$, and since $p_1+p_2+p_3=0$, also transforms p_3 to $\bar{p}_3$. Therefore, $F(p)=F(\bar{p})$. Hence, $F(p)$ must vanish identically.

The author is indebted to Mr. F. J. Dyson for kindly communicating the proof.

[8] S and Δ stand for Feynman's functions.

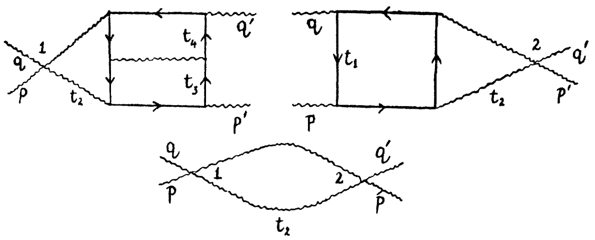


FIG. 4.

$(\delta\lambda)_1$, is the true divergence over t_1, $(\delta\lambda)_2$ over t_3t_4, and $-(\delta\lambda)_1\times(\delta\lambda)_2$ over $t_1t_3t_4$. The negative sign before the latter follows from the considerations about non-overlapping graphs given in Remarks in Sec. III.

Now we notice that on account of the presence of $-\delta\lambda\phi^4$ term in the hamiltonian with $\delta\lambda$ (for the present) chosen as the sum of true divergences from all irreducible simple M parts, there are three other graphs of order f^{10} in the theory, *viz.*, the graphs in Fig. 4. Since $\delta\lambda$ contains among other terms $(\delta\lambda)_1$ and $(\delta\lambda)_2$, we may assume that in the graphs drawn in Fig. 4 $-(\delta\lambda)_1\phi^4$ acts at the 4-vertex marked 1, and $-(\delta\lambda)_2\phi^4$ at the 4-vertex marked 2. Quite obviously the sum of contributions from these four graphs suffices to cancel all the divergent terms exhibited so far. The true divergence from the $t_1t_2t_3t_4$ integration $(\delta\lambda)_3$ which still remains to be separated from (1) to leave behind a convergent integral will now be added $\delta\lambda$ to already formed and we secure the cancellation of all divergences if for this process the graph consisting of a single 4-vertex $-(\delta\lambda)_3\phi^4$ is also taken into account. Thus if we consider all (irreducible) M parts which are not simple, only the true divergence over the final integration need be added to the $\delta\lambda$ term formed by adding the true divergences from all irreducible simple M parts. The result is of general validity and holds for the general case of an M part formed by joining any number of simple M parts.

It may be noticed that the operation of taking the trace does not interfere with the separation of divergences corresponding to an M part which is contained inside another. The reason is that nucleon lines run in loops while (in the case of original graphs) meson lines do not run continuously, thus making it possible to localize M parts unambiguously in this theory.

(3) $\delta\lambda$ so far has been obtained as a power series in f, and is equal to the sum of true divergences from all irreducible M parts. Considering reducible graphs now,

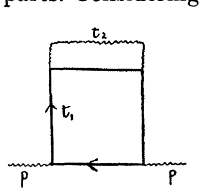


FIG. 5.

we have to consider the effect of inserting S and V parts in all the irreducible M parts. Since $S_F'=Z_2S_{F1}'(f_1)$, $\Delta_F'=Z_3\Delta_{F1}'(f_1)$, and $\Gamma^5=Z_1^{-1}\Gamma_1^5(f_1)$, we replace $S_F(p)$ by $S_{F1}'(p)$, $\Delta_F(p)$ by $\Delta_{F1}'(p)$, and γ_5 by $\Gamma_1^5(p,p')$ in all the lines and the vertices of the original irreducible M parts. An irreducible M part of order $2r$ contains $2r$ 3-vertices, $2r$ nucleon lines, and $r-2$ meson lines. The above substitutions give $f^{2r}Z_1^{-2r}Z_2^{2r}Z_3^{r-2}=Z_3^{-2}f_1^{2r}$ as the f-factor outside the integral corresponding to this irreducible M part, while the integral itself can be written as the sum of divergent and convergent terms, both expressed as power series in f_1. Extending the arguments given above for irreducible graphs we readily see that finally $\delta\lambda(f_1)=Z_3^{-2}\times M_d(f_1)$, where $M_d(f_1)$ is a power series in f_1 (starting with a term f_1^4) each of whose terms corresponds to the true divergence of some original M part.

So far we have proved that with the above choice of $\delta\lambda$ not only can all divergent terms be separated from the integral corresponding to the original M parts to be canceled, but further that the new M graphs introduced into the theory because of the term $-\delta\lambda\phi^4$ in the hamiltonian are also accounted for.

Similar arguments prove that if we are considering graphs corresponding to any other process (except meson self-energy graphs) which contain M parts inside them, a combination of the original graphs for the process, with graphs obtained by replacing the M parts by 4-vertices *in all possible ways* will always lead to a cancellation of all divergences arising from the contained M parts, in the original graphs. The rest of the proof for the finiteness of the S-matrix follows D II.

(B) Meson Self-Energy Graphs

We now consider the case of meson self-energy graphs. By joining up an incoming meson line to an outgoing meson line in any M part we obtain a meson self-energy graph. Conversely, in a meson self-energy graph, if any meson line is opened, we obtain an M part. Just as it was impossible to make the substitution $\Gamma_{\mu 1}$ for γ_μ at the end vertices of a self-energy graph in electrodynamics unambiguously, similarly, it is impossible to make an unambiguous substitution Δ_{F1}' for Δ_F (or S_{F1}' for S_F) if we desire to obtain a higher order meson self-energy graph from one of lower order.

However, the procedure for isolating divergences from M parts, outlined in Sec. III, leads to an essential simplification. The principles involved can be illustrated once again by considering a simple example.

Figure 5 represents a meson self-energy graph, in one of the nucleon lines of which a nucleon self-energy graph is inserted. The corresponding integral is

$$\Pi(p)=f^4\int dt_1dt_2[\text{Trace }\gamma_5S(t_1-p)\gamma_5S(t_1)\times\gamma_5S(t_1-t_2)\gamma_5S(t_1)]\Delta(t_2).$$

The integral is logarithmically divergent for the t_1 subintegration, linearly divergent for the t_2 subintegration, and quadratically divergent for the t_1t_2 integration. The divergent terms to be separated in order to leave behind an absolutely convergent integral are the following:

$$f^4\int dt_1dt_2\,\text{Trace}\,[\gamma_5S(t_1-p_0)\gamma_5S(t_1)\times\gamma_5S(t_1-t_{20})\gamma_5S(t_1)]\Delta(t_2)+f^4\int dt_1dt_2\times\text{Trace}\,[\gamma_5S(t_1-p)\gamma_5S(t_1)\{\gamma_5S(t_{10}-t_2)\gamma_5+\langle(t_1-t_{10})\partial/\partial t_1\rangle_{t_1=t_{10}}(\gamma_5S(t_1-t_2)\gamma_5)\}S(t_1)\Delta(t_2)]$$
$$+\left\{\left\{f^4\int dt_1dt_2\,\text{Trace}\,[\gamma_5S(t_1-p_0)\gamma_5S(t_1)\times\gamma_5S(t_1-t_2)\gamma_5S(t_1)\Delta(t_2)-\gamma_5S(t_1-p_0)\gamma_5S(t_1)\times\gamma_5S(t_1-t_{20})\gamma_5S(t_1)\Delta(t_2)-\gamma_5S(t_1-p_0)\gamma_5S(t_1)\times\{\gamma_5S(t_{10}-t_2)\gamma_5+\langle(t_1-t_{10})\partial/\partial t_1\rangle_{t_1=t_{10}}\times(\gamma_5S(t_1-t_2)\gamma_5)\}S(t_1)\Delta(t_2)]\right\}\right\}$$
$$+f^4\int dt_1dt_2\{(p-p_0)_\mu(\partial/\partial p_\mu)+\tfrac{1}{2}(p-p_0)_\mu(p-p_0)_\nu(\partial/\partial p_\mu)(\partial/\partial p_\nu)\}^9{}_{p=p_0}\times\text{Trace}\,[\gamma_5S(t_1-p)\gamma_5S(t_1)\gamma_5S(t_1-t_2)\times\gamma_5S(t_1)\Delta(t_2)-\gamma_5S(t_1-p)\gamma_5S(t_1)\times\{\gamma_5S(t_{10}-t_2)\gamma_5+\langle(t_1-t_{10})\partial/\partial t_1\rangle_{t_1=t_{10}}\times(\gamma_5S(t_1-t_2)\gamma_5)\}S(t_1)\Delta(t_2)].\quad(10)$$

where

$$i\gamma t_{10}+K_0=t_{10}^2+K_0^2=0$$

and

$$t_{20}^2+K^2=p_0^2+K^2=0.$$

Let

$$f^4\int dt_1\,\text{Trace}\,[\gamma_5S(t_1-p_0)\gamma_5S(t_1)\times\gamma_5S(t_1-t_{20})\gamma_5S(t_1)]=\delta\lambda,$$

$$f^2\int dt_2,\quad\gamma_5S(t_{10}-t_2)\gamma_5\Delta(t_2)=A,$$

and let the expression in double brackets, $\{\{\ \ \}\}$, in Eq. (10) be equal to A'. Then among others the following are divergent terms separated:

$$\delta\lambda\int dt_2\Delta(t_2)+Af^2\int dt_1\times\text{Trace}\,[\gamma_5S(t_1-p)\gamma_5S(t_1)S(t_1)\gamma_5]+A'.\quad(11)$$

The integral with Af^2 as coefficient will be canceled by being combined with the integral arising from the nucleon mass renormalization term $-\delta\kappa_0\bar{\psi}\psi$ in the hamiltonian, while A' is similarly canceled by a proper choice of $\delta\kappa^2$ in the term $-\delta\kappa^2\phi^2$. To cancel $\delta\lambda\int dt_2\Delta(t_2)$ we can combine it with a meson self-energy graph obtained by replacing the M part by a 4-vertex as we have so far been doing; but now we notice that

$$A'=A''-\delta\lambda\int dt_2\Delta(t_2),$$

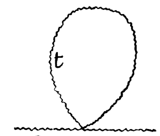


FIG. 6.

where A'' is that value of the meson mass renormalization constant which it would have if M parts were not divergent in the theory; i.e.

$$A''=f^4\int dt_1dt_2\,\text{Trace}\,[\gamma_5S(t_1-p_0)\gamma_5S(t_1)\times\gamma_5S(t_1-t_2)\gamma_5S(t_1)\Delta(t_2)-\gamma_5S(t_1-p_0)\gamma_5S(t_1)\times\{\gamma_5S(t_{10}-t_2)\gamma_5+\langle(t_1-t_{10})\partial/\partial t_1\rangle_{t_1=t_{10}}\times(\gamma_5S(t_1-t_2)\gamma_5)\}S(t_1)\Delta(t_2)],$$

so that (11) can be written as

$$A''+Af^2\int dt_1\,\text{Trace}\,[\gamma_5S(t_1-p)\gamma_5S(t_1)S(t_1)].$$

Further, since $\delta\lambda$ is not a function of p, the divergent term separated corresponding to t_1 gives no contribution to the charge renormalization constant set out in the last integral in Eq. (10) as part of the coefficient of $(p-p_0)_\mu(p-p_0)_\nu$. The result is that the divergent terms to be separated in order to obtain an absolutely convergent integral from a meson self-energy graph are precisely the same as if M-parts were not divergent in the theory, this result holding for only those M parts which are obtained by opening a single meson line in a meson self-energy graph. A further consequence of this is that the self-energy graph in Fig. 6, unlike other graphs containing 4-vertices, has to be considered separately and is not like others a compensatory graph.

The reason why the result shown to hold true for the particular example can be generalized is this. Let us first consider a meson self-energy graph belonging to any category $[n]$. The meson and nucleon lines or the vertices in $[n]$ do not contain any S or V insertions so far. In order to remove the divergence from the M part formed by opening a meson line *we must associate one*

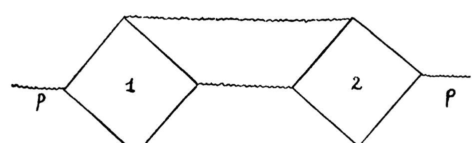


FIG. 7.

[9] The divergent terms shown here corresponding to t_1t_2 integration are of the form $A'+B(p^2+k^2)+C[p_0(p-p_0)]^2$ where C can be shown to be a finite constant. $C[p_0(p-p_0)]^2$ is thus properly to be included in the convergent part of Π^*.

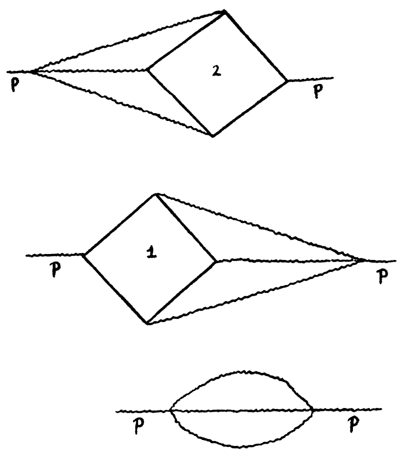


FIG. 8.

of the basic variables, say, t_1, with the particular meson line being opened. The true divergence corresponding to the subintegration $t_2t_3\cdots t_k$ is a constant ($\delta\lambda$) and the reduced integral is necessarily $\int\Delta_F(t_1)d^4t_1$. Thus, when the final true divergence over $t_1t_2\cdots t_k$ is separated, $\Delta_F(t_1)$ not containing the external momentum p, contributes $-\delta\lambda\int dt_1\Delta_F(t_1)$ to A' and makes no contribution to the charge-renormalization constant at all, precisely as in the particular example. The effect is the same as if M parts were not divergent in the theory, and the mass renormalization constant were not A' but A''.

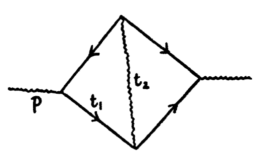


FIG. 9.

It must be emphasized that the above result holds only for those M parts obtained by opening one meson line. We shall call such M parts "final" M parts. Thus, for example, the self-energy graph shown in Fig. 7 containing the M parts with the loops marked 1 and 2 is necessarily to be combined with three other graphs in Fig. 8 just as in Sec. V(A).

So far we have restricted our considerations to graphs belonging to a category [n] in which S and V parts have not been inserted. To bring out the new features involved we consider another example. Figure 9 shows a meson self-energy graph belonging to [2] in which the M divergence is associated with the t_1 loop. In the line t_2 we insert the self-energy graph in Fig. 5, which is itself a modification of the graph in [1]. The resulting graph is shown in Fig. 10. M divergences are now associated with the subintegrations over t_3, t_1, t_1t_3, and $t_1t_2t_3$. Considering the line t_2 by itself, from the arguments given before it is obvious that we need not make an explicit separation of the M divergence corresponding to t_3. If now the "mass-renormalization constant" corresponding to the line p is considered, it will be seen that not only is it unnecessary to separate the divergence corresponding to $t_1t_2t_3$ subintegration (for which only one variable t_4 is held fixed), but the same applies for the t_1 and t_1t_3 subintegrations. The reasoning for all cases is similar. For example, the true divergence over t_1 is obtained by putting, among other variables, p equal to p_0 *wherever* it occurs in the integrand. When the mass renormalization constant corresponding to line p is being separated, its part corresponding to t_1 is the same as above, no further change being necessary, since p has already been replaced by p_0. Thus, $A'=A''-$true divergence over $t_1\times$the reduced integral over $t_2t_3t_4$ $-$true divergence over $t_1t_3\times$reduced integral over t_2t_4 $-$true divergence over $t_1t_2t_3\times$reduced integral over t_4. Similarly, we can see that the *charge* renormalization constant is precisely the same as if these M parts were not divergent in the theory. From our definition, that M corresponding to the subintegration t_3 was the "final" M part contained in the meson self-energy graph (of Fig. 5) inserted in line t_2, while t_1 was the "final" M part for the line p in Fig. 9. Because of the insertion in t_2, new M divergences t_1t_3, $t_1t_2t_3$ arise; since they can be associated with p, we shall call the divergences corresponding to t_1, t_1t_3, and $t_1t_2t_3$ also "final" divergences associated with the line p in Fig. 10. This is an extension of the definition of "final" M-divergence.[10] In general, the result can be stated thus: given a meson self-energy graph of category [n], the insertion of a meson self-energy graph with a number of final M divergences in any of its meson lines will increase the number (and complexity) of the M divergences in the entire graph. It will always be possible to associate these new divergences as the final M-divergences corresponding to some meson line, making it possible to prove that no explicit separation of corresponding M divergences is needed. We have the result that the mass and charge renormalization constants for meson self-energy graphs are precisely the same as if these "final" M parts were not divergent.

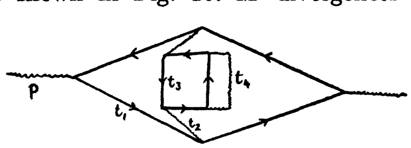


FIG. 10.

The same results (only simpler to prove) hold for insertions of S and V parts in the nucleon lines and vertices of graphs of categories [n].

The only meson self-energy graph which still needs to be considered is the graph in Fig. 6. All other meson self-energy graphs with 4-vertices act as compensatory graphs like those in Fig. 8. The corresponding integral is

$$-\delta\lambda(f_1)\int\Delta_F'(t)dt=-M_d(f_1)Z_3^{-1}\int\Delta_{F1}'(t)dt,$$

[10] Analytically a final M-divergence inside a meson self-energy graph may be defined such that its reduced integral is independent of the external momentum of the meson line.

so that finally we choose

$$\delta\kappa^2=-(1/\pi i)Z_3^{-1}(f_1)\left[A''(f_1)-M_d(f_1)\int\Delta_{F1}'(t,f_1)dt\right].$$

This completes the proof.

On account of the results above regarding "final" M-divergences, it will be found expedient to associate basic momenta with nucleon rather with meson lines in graphs belonging to any category $[n]$. This choice will remove all the significant divergences with a single choice of basic variables.

The case of the meson self-energy has been treated at length because it will serve as a prototype for renormalization in other theories such as scalar electrodynamics, where both C and M parts are divergent and it is not possible to make an unambiguous insertion of S parts in meson or photon lines.

The general rule for the separation of divergences given in Sec. III will also apply to other theories such as scalar electrodynamics; but the considerations given in the remarks will not, because both C and M parts are divergent in that theory, leading to overlaps of considerably greater complexity. It is, in fact, in scalar electrodynamics that the power of the general rule formulated in Sec. III exhibits itself. General considerations on that problem will be published shortly.

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Emission of Protons from the Compound Nucleus F^{18*}

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The emission of protons from the reaction $N^{14}(\alpha,p)O^{17}$ was studied by photographic emulsion technique. Two groups of protons were observed, of Q values of about -1.16 Mev and -2.0 Mev for $E_\alpha=3.6$ Mev and 4.2 Mev. The yield per million incident α-particles has been estimated at 0.348 and 0.409 for the two resonance levels, 3.6 Mev and 4.2 Mev, respectively. The angular distribution of protons in the CM system has a maximum at $\theta=90°$ for the two ground states, and, for the excited states, it is isotropic.

I. INTRODUCTION

The reaction $O^{16}(d,p)O^{17}$ and the reaction $N^{14}(\alpha,p)O^{17}$ both lead to the formation of the compound nucleus F^{18*}. It is known[1] that the former reaction yields two groups of protons. For $E_d=0.575$ Mev, which corresponds to an excitation of 8 Mev in F^{18*}, these two groups had Q-values of 1.75 Mev and 0.8 Mev, according to the observation of Pollard and Davison.[2] At this excitation, which is provided by α-particles of energy about 3.6 Mev, only one group of protons has been observed in previous experiments for the second reaction, although two groups might be expected, unless forbidden by selection rule. The presence of two groups of protons in the reaction $N^{14}(\alpha,p)O^{17}$ has, in fact, been demonstrated by Pollard and Davison,[2] but in this case, the energy of the α-particles was about 5.2 Mev.

The purpose of the present experiment was to investigate, by emulsion technique; (A) the group structure of the protons, at low bombarding energies of the α-particles, arising from the reaction $N^{14}(\alpha,p)O^{17}$; (B) the yield of the process; and (C) the angular distribution of the protons.

II. EXPERIMENTAL DETAILS

The α-particles from polonium were channeled through a circular slit, the arrangement of which is shown in Fig. 1. It consists of two circular brass disks D, fitted with guard rings R, cut at sharp angles. The strength of the source is 1 mc deposited at one end of a wire W and introduced into the slit through the guide G which holds the source centrally between the two disks. The position of the source 0 can be viewed through holes H drilled at the center of each disk. The two supports S and the guide channel G subtend an equal angle at the center. This facilitated the calculation of the solid angle through which α-particles emerged.

TABLE I. Proton groups from $N^{14}(\alpha,p)O^{17}$.

E_α (Mev)	No. of protons	R_μ	E_p (Mev)	Q (Mev)	Ratio (II)/(I)
3.6	(I) 1986	54.5	2.45	(a) -1.15 ± 0.04	0.12
	(II) 245	28.5	1.60	(b) -2.0 ± 0.04	
4.2	(I) 2321	75	3.02	(a) -1.18 ± 0.04	0.12
	(II) 291	46.2	2.20	(b) -2.0 ± 0.04	

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