eter and thus without changing θ, we have been able to minimize the effect of the systematic error in θ. We actually compute $E_x=\Delta Q=Q-Q^*$ from

$$\Delta Q=\frac{M_3+M_2}{M_3}(E_2-E_2')-\frac{M_3-M_1}{M_3}(E_1-E_1')$$
$$-2\cos\theta\frac{(M_1M_2)^{\frac{1}{2}}}{M_3}[(E_1E_2)^{\frac{1}{2}}-(E_1'E_2')^{\frac{1}{2}}], \quad \text{(C3)}$$

where the unprimed quantities are determined in $Li^7(p,p)Li^7$ or $B^{10}(p,\alpha)Be^7$ and the primed ones in $Li^7(p,p')Li^{7*}$ or $B^{10}(p,\alpha')Be^{7*}$. It is clear that the systematic error in θ can even be reduced to zero by appropriate choices (not independent) of the E's such that $(E_1E_2)^{\frac{1}{2}}-(E_1'E_2')^{\frac{1}{2}}=0$. Also, it can be shown that if the scales for E_1 and E_2 are calibrated against the same primary standard, $\delta\Delta Q=(\Delta Q/E_1)\delta E_1=(\Delta Q/E_2)\delta E_2$. In the Be^{7*} case, we choose $E_2'\sim E_2$ in order to minimize the large errors arising from uncertainty in the energy losses of the outgoing alpha-particles. This also reduced the error from θ to 30 percent of its effect on the main Q-value. In the Li^{7*} case, for intensity reasons, we choose $E_1=E_1'$. It should be emphasized that the systematic error in the location of E_{2B} is large only in determining Q^* and so this error is not reduced in the calculation of ΔQ. The various errors and the final values are discussed and tabulated in the text (Tables III and V).

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Spin-Orbit Coupling in Li^7 and Be^7

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The magnetic spin-orbit coupling of the p-shell nucleons is shown to have the correct sign and about half the magnitude to account for the 45-kev difference of the first excitation energies, 479 kev in Li^7 and 434 kev in Be^7, if these are interpreted as doublet splittings arising mainly from a spin-orbit coupling term of the same form as the Thomas term but stronger and presumably of mesonic origin, such as seems to be responsible for the appearance of (jj) coupling in heavy nuclei. Be^7 is expected to be slightly larger than Li^7 because of the added coulomb repulsion, so that the average value of the main term is expected to be smaller for Be^7, and it is estimated that this coulomb expansion explains another one-fourth to the splitting difference, leaving about one-forth unaccounted for. These estimates are made both very simply by means of the droplet model and more reliably by minimizing the energy obtained with exchange interactions and three-dimensional isotropic oscillator wave functions. Energies attributed to spin-orbit coupling in other nuclei indicate that the main term should make the Li^7 and Be^7 doublet splittings about 50 percent larger than observed. This suggests that second-order perturbations may reduce both by nearly the same amount, leaving their estimated difference significant. The possibility of more drastic perturbations and their relation to the quadrupole-moment problem are discussed. The agreement of the estimated splitting difference in sign and order of magnitude is consistent with other evidence favoring the (LS)-coupling interpretation that the two low states form a ^2P.

I. INTRODUCTION

ONE very interesting feature of the 434-kev state in Be^7 is that its excitation energy is approximately[1] but not exactly equal to that of the 479-kev state in the mirror nucleus Li^7. One, of course, assumes that they are mirror states,[2] or that they differ only in isotopic spin and coulomb energy, and the current experimental situation[3] favors the conclusion that they have nuclear spin $I=\frac{1}{2}$ in spite of the intensity-ratio difficulty in the thermal reaction $B^{10}(n,\alpha)Li^{7*}$, Li^7. This permits one to assume[4] that this state, together with the ground state, forms a ^2P. The magnitude of the 479-kev doublet splitting has long presented a problem of interpretation in terms of nuclear spin-orbit coupling, and now the 45-kev difference between the two doublet splittings enriches the opportunity for testing simultaneously the interactions responsible for spin-orbit coupling, the theory of nuclear structure, and the ^2P assignment. The original interpretation[4] of this splitting was given in terms of the picture of a nucleus consisting of nucleons as its fundamental constituents, with their binding forces treated phenomenologically, and with no attention paid to the source of these forces. The relativistic kinematic effect known as the Thomas precession was seen as the primary source of the splitting, modified by the magnetic effect. This simple theory automatically gives the correct sign of spin-orbit coupling to account for the nuclear spins of the ground states of most fairly simple nuclei (whereas subsequent theories of spin-orbit coupling unfortunately leave an arbitrary choice of sign). For this reason it remained attractive even after it became increasingly apparent that it was inadequate to account for the magnitude of the 479-kev splitting in Li^7, because of the possibility that a narrower doublet might be hidden in the rather broad ground-state group of the range measurements, until rather recent magnetic analysis[5] showed the ground state to be single within a few kev. The recent success of the (jj) coupling shell model[6] indicates the presence of strong spin-orbit coupling in heavy nuclei

[1] Brown, Snyder, Fowler, and Lauritsen, Phys. Rev. **82**, 159 (1951).
[2] As is verified by the approximately equal gamma-ray intensities in the mirror reactions.
[3] B. Rose and A. R. W. Wilson, Phys. Rev. **78**, 68 (1950); B. T. Feld, Phys. Rev. **75**, 1618 (1949); S. Devons, Proc. Phys. Soc. (London) **62A**, 580 (1949); D. R. Inglis, Phys. Rev. **81**, 914 (1951).
[4] D. R. Inglis, Phys. Rev. **51**, 783 (1936).
[5] Buechner, Strait, Stergiopoulos, and Sperduto, Phys. Rev. **74**, 1569 (1948).
[6] M. G. Mayer, Phys. Rev. **78**, 16, 22 (1950); D. Kurath, Phys. Rev. **80**, 98 (1950); Haxel, Jensen, and Suess, Naturwiss. **36**, 155 (1949).

of the same sign and order of magnitude as required to account for the 479 kev excitation in Li^7 as a ^2P splitting. It is impressive that in quite another branch of nuclear physics, the high energy nucleon–nucleon scattering, strong spin-orbit coupling of a compatible order of magnitude has recently appeared[7] as a possible means of preserving the symmetry of the interactions.

Both the Thomas term and the magnetic term in the spin-orbit coupling satisfy, and indeed arise from, the requirements of relativistic invariance, as does the spin itself (apart from the rather abitrarily introduced anomalous nucleon gyromagnetic ratios), and Furry[8] has shown how they are related to the introduction of electromagnetic or scalar potentials in the Dirac equation for a single particle. They are not the only Lorentz-invariant types of spin-orbit coupling: Breit[9–11] has shown that, with ordinary non-exchange interactions (or with Heisenberg interactions that are also spin-independent in the isotopic-spin language), invariance to order v^2/c^2 is obtained with a spin-orbit coupling between two nucleons of the form $Q_{kl}+Q_{lk}$ with

$$Q_{kl}=(\hbar/4M^2c^2)[-a\boldsymbol{\sigma}_k+(1-a)\boldsymbol{\sigma}_l]\cdot\mathbf{p}_k\times\boldsymbol{\nabla}_kJ_{kl},\quad(1)$$

where a is arbitrary. The special case $a=-1$ gives the spin-orbit coupling for Dirac particles interacting through an electromagnetic field, ∇_kJ_{kl} then corresponding to the (charge times the) electric field, the time component of a four-vector, which transforms into a magnetic field plus an electric field in the coordinate system of a moving particle (of which the magnetic field gives the magnetic term and the electric field contributes to the Thomas term in the spin-orbit coupling). The term containing $2\sigma_l$ is the purely magnetic term for the particle whose motion is not represented, corresponding to having the other particle circulate around it. The term containing σ_k is the magnetic term reduced by a factor $\frac{1}{2}$ by the Thomas term for the moving particle k, corresponding to the Thomas precession[12] introduced by the acceleration arising from ∇_kJ_{kl}. When arbitrarily multiplied by $g/2$ (where g is the gyromagnetic ratio, which is 2 for a Dirac particle) to account for the anomalous magnetic moment, the magnetic term for a nucleon (k) due to its own motion is

$$H_m=(\hbar g/4M^2c^2)\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\times\nabla_kJ_{kl}$$
$$=-(\hbar^2ge/2M^2c^2)(1/r)(dU/dr)\mathbf{l}\cdot\mathbf{s}.\quad(1a)$$

Here the last member is written for the case of a central electric potential energy $J_{kl}\rightarrow eU(r_k)=eU(r)$, as though the other particle were effectively at rest as in an average over a closed shell, the angular momentum is $\mathbf{l}\hbar=\mathbf{r}\times\mathbf{p}_k$ and the spin $s=\sigma/2$. This term arises directly from the charge and magnetic moments of the nucleons involved, which we consider to be intrinsic properties of the nucleons in addition to any other effects related to the exchange nature of the forces, so that we assume this term to be present in any case. The Thomas term is

$$H_T=-(\hbar/4Mc^2)\boldsymbol{\sigma}_k\cdot\mathbf{v}_k\times\nabla_kJ_{kl}$$
$$=(\hbar^2/2M^2c^2)(1/r)(dV/dr)\mathbf{l}\cdot\mathbf{s},\quad(1b)$$

where $v_k=p_k/M$ and V is a central non-electromagnetic potential which simply transforms into itself and enters the Dirac equation of a single particle as a scalar[8] added to the mass term. Although this is too small to account for the large spin-orbit coupling appearing in nuclei, it is of interest here because of its form. Breit also showed that there are still other possibilities with a space-exchange (Majorana) interaction, for example, and that with "reasonable" choice of the arbitrary constant (that is, with the constant analogous to a having order of magnitude unity) one can obtain[10] a sufficiently large spin-orbit coupling. The effect of the exchange nature of nuclear forces, which was thus treated phenomenologically with this promising result, has been studied more recently from the point of view of the meson origin of nuclear forces. Dancoff[13] has shown by a perturbation theory treating the nucleons relativistically that a scalar coupling to a field of neutral vector mesons gives only a term which is essentially the same as the Thomas term, too small, but of determinate and correct sign (as may be associated with the concept that it corresponds to the centripetal acceleration of mesons as the Thomas term does to that of the nucleon, the nucleon with spin up being part of the time a nucleon with spin down and a meson with spin up). Møller and Rosenfeld (reference 11, Sec. 17.43) have shown that a mixed vector and tensor coupling may, with appropriate choice of the arbitrary constant, give rise to a coupling of the form (1b) but with $\pm1/\mu$, the reciprocal of the meson mass, in place of $1/M$ in the second member of (1b), which makes it sufficiently large and leaves its sign arbitrary. In a later discussion of essentially the same theory, Gaus[14] has brought out more clearly the simplification that can be introduced in the spin-orbit coupling problem by treating the saturation of forces as arising from a saturation of the meson field. In an evaluation of the expression $\Delta V/\Delta r$ at the edge of the nucleus, that is, dV/dr in (1b), this allows one to identify the energy increment ΔV for the meson with the familiar

[7] K. M. Case and A. Pais, Phys. Rev. **80**, 203 (1950). The terminology of this very important paper, unfortunately, adds to the current confusion, with rebirth of interest, in the use of the word "coupling" as taken from atomic physics: it erroneously uses the term "*LS* coupling" as an abbreviation for "spin-orbit coupling (energy)" whereas "(*LS*) coupling" has traditionally meant "Russell-Saunders coupling (scheme)" in which L and S are approximately diagonal.

[8] W. Furry, Phys. Rev. **51**, 784 (1936).

[9] G. Breit, Phys. Rev. **51**, 248 (1937).

[10] G. Breit and G. R. Stehn, Phys. Rev. **53**, 459 (1938).

[11] L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, Amsterdam, 1948).

[12] S. Dancoff and D. R. Inglis, Phys. Rev. **51**, 784 (1936). Errata. The following nearly obvious misprints in this pedagogical note have not been previously recorded: In the equation for dx and dt the numerator, and in the equations for u_x and u_y the denominator should have + in place of −. In the first equation of the second column, δ_T should be ω_T.

[13] S. Dancoff, Phys. Rev. **78**, 382 (1950).

[14] H. Gaus, Z. Naturforsch. **4A**, 721 (1949).

ΔV for a nucleon. This is a theory which applies perhaps better to heavy nuclei than to light nuclei because it assumes a full realization of the saturation property of nuclear forces. A still newer consideration of the problem by Heisenberg[15] discusses the interaction between individual nucleons in terms of an analogy with the coupling of electrons to the photon field, purposefully avoiding any further specification of the nature of the meson field than is necessary for the discussion of this analogy in general terms. Because it treats the interaction of individual nucleons, this formulation seems to be better suited to the discussion of Li^7 and Be^7, where the saturation of the forces is not fully realized. His argument consists simply in pointing out that the coupling of two electrons to the photon field leads not only to the coulomb force but also to a spin-orbit coupling and a spin-spin coupling as well. The analogous coupling of the nucleons to the meson field is known to give rise to a Yukawa force, and in the analysis of the properties of the deuteron, to a spin-spin, or tensor force between nucleons. By analogy it may be expected to give rise to a spin-orbit coupling as well. By a careful comparison of the constants involved in the two cases, including dimensional arguments, Heisenberg makes it plausible that the spin-orbit coupling for a circulating nucleon (k), which in analogy to the spin-orbit coupling for an electron $[(M/m)^2$ times (1) with $a=-1]$ is taken to have the form (1b), has the magnitude

$$H_d=\pm(\hbar/\mu c^2)\boldsymbol{\sigma}_k\cdot\mathbf{v}_k\times\nabla_k J_{kl} \rightarrow -(2\hbar^2/\mu Mc^2)(1/r)(dV/dr)(\mathbf{l}\cdot\mathbf{s}), \quad (1c)$$

with the meson mass μ in the denominator. The negative sign of the last member is determined empirically from the Li^7 ground state.

One may prefer to look on the existence of a large spin-orbit coupling merely as an empirical fact.[5-7] From invariance considerations we have interactions of the general form of (1) available, and with $a=\frac{1}{2}$, which corresponds to a scalar or non-electromagnetic J_{kl}, it includes a factor $\mathbf{L}\cdot(\mathbf{s}_i+\mathbf{s}_k)$, where in the case of just two nucleons $\mathbf{L}$ is their angular momentum. This it what is needed in the high energy scattering problem,[7] an interaction in a triplet state and none in a singlet state so the Pauli principle may make a large difference between like-nucleon and unlike-nucleon scattering. Such an interaction (or (1) with any $a\neq 1$), although it contains two spins, still reduces to a simple spin-orbit coupling of the form (1b) when applied to a single nucleon (k) circulating under the influence of other nucleons (l) in closed shells or pairs (since in the sum over l the opposing σ_l cancel) and this is consistent with the needs of the (jj) coupling shell model.

In discussing the magnitude of the first excitation energies of Li^7 and Be^7, we wish to test the possibility of attributing the spin-orbit coupling to a main term H_d (which may be considered to include H_T), of the order of magnitude given by (1c), whose exact magnitude we determine only empirically in terms of the average excitation energy of the two nuclei, and the magnetic term H_m which contains no arbitrary parameter. The main term, H_d, even though it does not have any direct connection with the coulomb energy, can be expected to contribute more to the doublet splitting in Li^7 than in Be^7 because of the larger size of Be^7 arising from its additional coulomb repulsion. The magnetic term, though very much smaller, contributes more directly to the difference of excitation energies, being positive for Li^7 and negative for Be^7 because of the opposite signs of the proton and neutron magnetic moments. We shall show that the two effects together have the right order of magnitude as well as the right sign to account for the 45 kev Li^7-Be^7 difference.

Evaluation of the energy resulting from the assumed interactions involves use of a nuclear model, so we are forced to test the adequacy of the interactions and the model at the same time. Because it may help to show how much uncertainty arises from the model and for the sake of a presentation in steps of decreasing simplicity, we shall discuss the question separately in terms of two models. The first is the very simple uniform-density or droplet model of the nucleus which suffices for puposes of preliminary orientation, even though seven particles are too few for the valid application of statistical methods. The second is a simple variation calculation based on the use of a phenomenological exchange interaction and oscillator wave functions, which presumably more adequately represents the contention between potential and kinetic terms in determining the relevant nuclear properties, but which also falls far short of a complete treatment of the seven-body problem.

II. COULOMB EXPANSION

In calculating the effect on H_d of the added coulomb expansion in Be^7 one considers the balance between kinetic and potential energies which determines the stiffness against expansion, or the compressibility of the nuclei. In either model one obtains, as functions of a nuclear size parameter such as R, the mean kinetic energy $T(R)$, the mean potential energy $V(R)$ arising from the specific nuclear interactions, and the average coulomb energy for Li^7, $U_{\mathrm{Li}}(R)$, and for Be^7, $U_{\mathrm{Be}}(R)=U_{\mathrm{Li}}(R)+E_c(R)$, E_c being the *coulomb* difference. The main term H_d in the spin-orbit coupling contributes an averaged value $E_d(R)$ to the *doublet* splitting, and at the equilibrium value R_0 is assumed to be $E_d(R_0)\approx 450$ kev. Of this, according to the Lande interval rule, $-(1/3)E_d$ belongs to the energy of the ground state and $(2/3)E_d$ to the excited state. The zeroth-order energy for the ground state of Li^7 is taken to be $E_0(R)=T+V+U_{\mathrm{Li}}$ (here $-E_d/3$ is neglected). The equilibrium value R_0 for Li^7 is found from the minimization process $E'(R_0)=0$. The equilibrium values for Li^{7*}, Be^7, and Be^{7*} may be called R_1, R_2 and R_3, respectively. For the excited state Be^{7*}, for example, the minimizing

[15] W. Heisenberg (unpublished lectures).

value R_3 is found from the Taylor expansion

$$E_{\mathrm{Be}}^*(R)=E_0+\tfrac{1}{2}(R-R_0)^2E_0''+E_c(R_0)+(R-R_0)E_c'+E_d(R_0)+(R-R_0)E_d'+\cdots \quad (2)$$

as

$$R_3-R_0=-(E_c'+E_d')/E_0'' \quad (3)$$

and thus one has the equilibrium value $E_{\mathrm{Be}}^*(R_3)$. Here E_0'' represents the compressibility. Similarly, one has $E_{\mathrm{Be}}(R_2)$ and $E_{\mathrm{Li}}^*(R_1)$ at their individual minima by omitting E_d' or E_c', respectively. Then the difference in doublet splittings of Li^7 and Be^7 is

$$\begin{aligned}E_{\mathrm{dif}}&=[E_{\mathrm{Li}}^*(R_1)-E_0(R_0)]-[E_{\mathrm{Be}}^*(R_3)-E_{\mathrm{Be}}(R_2)]\\&=E_c'E_d'/E_0''.\end{aligned} \quad (4)$$

The result is the same as would be obtained by considering only the effect of the coulomb expansion caused by E_c' on the evaluation of the spin-orbit term E_d, the effect[16] on E_c of the expansion brought about by E_d' having disappeared in taking differences.

III. THE DROPLET MODEL

In the uniform-droplet model the kinetic energy is

$$\begin{aligned}T=k/R^2&=(3/10)(9\pi)^{\frac{2}{3}}(137^2/1837)A\ \mathrm{mc}^2\\&=28.5A\ \mathrm{mc}^2\end{aligned} \quad (5)$$

with the equilibrium value of the nuclear radius

$$R=R_0=A^{\frac{1}{3}}e^2/2\ \mathrm{mc}^2. \quad (6)$$

We may assume that the only lack of saturation of the nuclear forces is found on the surface and thus that the change of potential energy on expansion of the nucleus is the change of the surface energy, which is proportional to the surface area,

$$E_s=sR^2. \quad (7)$$

The nuclear equilibrium size is determined by the vanishing of the derivative

$$d(E_s+\bar{T})/dR=2(E_s-\bar{T})/R=0. \quad (8)$$

We may take $E_0=\bar{T}+E_s$, neglecting the coulomb term, $\bar{U}_{\mathrm{Li}}$, and at the equilibrium radius have

$$E_0'=2(E_s-\bar{T})/R=0, \quad (9)$$

and

$$E_0''=2(E_s+3\bar{T})/R^2=8\bar{T}/R^2. \quad (10)$$

The main term in the doublet splitting, E_d, contains a factor $\langle(1/r)(dV/dr)\rangle_{\mathrm{Av}}$ according to (1c), and the method of averaging used by Gaus[14] is appropriate for the droplet model: The probability that the nucleon is in the surface layer of thickness Δr over which the potential drop ΔV takes place is proportional to the relative volumes, $3\Delta r/R$, so $E_d(R)\sim R^{-2}$, or $E_d'=-2E_d/R$ with $E_d=450$ kev$=0.9$ mc^2. The coulomb energy of a uniformly charged sphere is

$$\bar{U}=C/R=(6/5)Z(Z-1)A^{-\frac{1}{3}}\ \mathrm{mc}^2; \quad (11)$$

the latter for the usual equilibrium radius (6), and thus the coulomb difference E_c has the equilibrium value $(36/5)7^{-\frac{1}{3}}$ mc^2, and $E_c'=-E_c/R$. From (4) we then have the coulomb-expansion part of the $\mathrm{Li}^7-\mathrm{Be}^7$ difference in excitation energy

$$E_{\mathrm{dif}}=E_cE_d/4\bar{T}=(9/5)(0.9/28.5)7^{-4/3}\,\mathrm{mc}^2=2.2\,\mathrm{kev}. \quad (12)$$

This is very small compared to the observed difference 45 kev, but this simplified estimate is unreliable because of the use of the surface energy, as is discussed further below.

The magnetic term in the spin-orbit coupling, H_m, contains a factor $\mathbf{l}\cdot\mathbf{s}$ the evaluation of which depends on the angular properties of the ^2P states or the way in which the individual-nucleon vectors $\mathbf{l}$ and $\mathbf{s}$ are compounded to make the nuclear vectors $\mathbf{L}$ and $\mathbf{S}$. The simple spin-independent symmetric hamiltonian discussed by Feenberg and Wigner[17] gives a well-isolated ^2P having maximum symmetry, an appropriate combination of $^1S^2p$ and $^1D^2p$ in which the neutrons, in the case of Li^7, are in singlet states and do not contribute to $\mathbf{S}$, although the three p nucleons contribute equally to $\mathbf{L}$. It appears that the only simple way one can obtain a ^2P as well isolated from other multiplets, as required by experiment, is to have an approximation to this state of maximum symmetry. For this ^2P we have for the expectation values,

$$\mathbf{l}\cdot\mathbf{s}=(1/3)\mathbf{L}\cdot\mathbf{S} \quad (13)$$

because the proton which has the spin carries only one-third of the orbital angular momentum.[10] $\mathbf{L}\cdot\mathbf{S}$ is of course -1 for the $^2P_{\frac{1}{2}}$ and $\frac{1}{2}$ for the $^2P_{\frac{3}{2}}$, the difference being $-(L+\frac{1}{2})=-\frac{3}{2}$. The difference E_m of the proper values of the magnetic energy H_m then contains a factor $-(\frac{3}{2})/3=-\frac{1}{2}$ in place of $\mathbf{l}\cdot\mathbf{s}$ in (1a), for the ^2P of maximum symmetry.

The odd neutron in Be^7 in the droplet model moves in the parabolic electric potential of the uniform spherical distribution of charge Ze representing the four protons,

$$U=(3-(r/R)^2)Ze/2R \quad (14)$$

for which the factor $(1/r)(dU/dr)$ appearing in H_m has the value $-Ze/R^3$. The magnetic contribution to the doublet splitting is then

$$E_m=(4/A)(137/1837)^2Zg\ \mathrm{mc}^2=0.0061Z(g/2)\ \mathrm{mc}^2, \quad (15)$$

where $(g/2)$ is -1.91 for the neutron in Be^7 and 2.78 for the proton in Li^7. For Li^7 only the other protons contribute to U, and Z is to be replaced by $Z-1$. Thus, we find $E_d=17.5$ kev for Li^7 and -24 kev for Be^7. The difference 41 kev (when augmented by 2 kev from (12)) agrees exactly with the experimental difference 45 ± 4

[16] E. Feenberg, Revs. Modern Phys. **19**, 239 (1947), in the neighborhood of Eq. (20), in a rather similar treatment of nuclear compressibility, discusses the effect of the state of excitation in a different manner.

[17] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937); E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937).

kev, which of course should not have been expected of so rough a model, and indeed the agreement is not so good with the more refined oscillator estimate.

IV. THE CENTRAL MODEL WITH OSCILLATOR FUNCTIONS

The treatment of the problem in terms of oscillator wave functions permits the following improvements over the crudity of the droplet model: (1) introduction of a reasonable radial and angular density dependence, (2) allowance for exchange, and (3) determination of nuclear size from binding energy and the range of forces given by scattering, rather than by the rough assumption that the density is the same as in heavy nuclei. The compressibility is again given as the result of a competition between kinetic and potential energies, but these quantities are presumably expressed more nearly correctly in this more refined model.

The single-nucleon oscillator wave functions are[18, 19]

$$\begin{aligned}\psi_s&=s=\pi^{-\frac{3}{4}}\exp(-\rho^2/2),\\ \psi_{p,x}&=p_x=2^{\frac{1}{2}}\pi^{-\frac{3}{4}}\xi\exp(-\rho^2/2),\end{aligned}\tag{16}$$

where $\rho^2=\sigma\alpha r^2$, $\xi^2=\sigma\alpha x^2$. Here α is the inverse-square range parameter which appears in the simple Majorana interaction $-BP^q\exp(-\alpha r^2)$ which is introduced into the calculation, and σ gives the size of the wave function relative to this range.

The simplest wave function we can write for the Li^7 nucleus is then the product

$$\langle s^+s^-p_x^+p_x^-\rangle_\nu\langle s^+s^-p_x^+\rangle_\pi,\tag{17}$$

where the superscripts refer to spin direction and the subscripts ν and π to neutron and proton. Such a product is to be thought of as antisymmetric in exchange of like nucleons, and the effect of the space-exchange operator of the Majorana interaction is to introduce a negative exchange term for each like-spin like-nucleon pair. The sum of the interactions within the s shell and between the s and p shells can then be written, in the notation of reference 19,

$$\begin{aligned}-B(6f_{0000}+12f_{1001}-3f_{1010})(\sigma/\tau)^{\frac{3}{2}}\\ =-3B(\sigma/\tau)^{\frac{3}{2}}(1+5/\tau).\end{aligned}\tag{18}$$

Here $\tau=\sigma+2$. This part is valid for any state of the configuration s^4p^3, not just for (17). The interaction in the p shell for the simplified wave function (17) is

$$-3f_{1111}B(\sigma/\tau)^{\frac{3}{2}}=-3B(\sigma/\tau)^{\frac{3}{2}}(1-2/\tau+3/\tau^2).\tag{19}$$

A more complete calculation[17] which takes into account the angle and symmetry relations of the p-nucleons in the ^2P of maximum symmetry gives, as an improved evaluation of the Majorana interaction between them,

$$\begin{aligned}3L+2K&=-B(\sigma/\tau)^{\frac{3}{2}}(3f_{1111}+2f_{1100}{}^2)\\ &=-3B(\sigma/\tau)^{\frac{3}{2}}(1-2/\tau+11/3\tau^2).\end{aligned}\tag{20}$$

[18] W. Heisenberg, Z. Physik **96**, 473 (1935).
[19] D. R. Inglis, Phys. Rev. **51**, 531 (1937).

TABLE I. Coulomb energies.

	Li^7	Be^7	
s-shell	2	2	(23)
s–p	13/4	13/2	
p-shell	0	17/10	
U	5.25	10.2	

This is seen to differ very little from (19). (It appears that the failure to symmetrize the space wave function of the two neutrons is approximately compensated by the excessive angular concentration of the p-nucleons in (17).) The potential energy $V(\sigma)$ is the sum of (19) and (20). The kinetic energy corresponding to (17) is $(3+21/2)\alpha\sigma/2$, the zero-point energy of 21 one-dimensional harmonic oscillators, plus one unit of excitation each for three of them. This includes a small erroneous contribution from the zero-point fluctuations of the center of mass in (17), and the corrected value[20] is slightly lower

$$\bar{T}=6\alpha\sigma\tag{21}$$

n units $\hbar^2m/M$, or mc^2 including the unit $mMc^2/\hbar^2$ of α. The coulomb energy is, of course, larger for Be^7 than for Li^7. In calculating the coulomb interaction within the s-shell and the direct interaction between s and p shells, the integrations are elementary radial ones making use of the constant potential within a thin spherical shell. The exchange part of the s–p interaction and both parts of the interaction within the p shell involve angular integrals over the expansion which appears in the theory of atomic spectra,

$$1/r=\sum_{k,m}(k+\tfrac{1}{2})^{-1}\Theta_k{}^m(\theta)\Theta_k{}^m(\theta')r^k{}_{\text{smaller}}/r^{k+1}{}_{\text{larger}},\tag{22}$$

the Θ's being normalized to unity. The interaction within the p-shell depends, of course, on the nature of the ground-state ^2P, whether it is purely the $^2P_{\frac{3}{2}}$ state of maximum symmetry[17] or whether it contains a large admixture of higher states in this representation. In seeking to attribute the first excitation energy to spin-orbit coupling we are in a sense trying to get along with a simple ^2P since large admixtures of higher states would contribute independently to the separation, as has been proposed as an alternative mechanism by Feingold and Wigner[21] (discussed in (*d*) below). For the simple ^2P the coulomb interaction between the two p-protons in Be^7 has been given[17] as $L_c+(2/3)K_c$, where L_c can be written briefly as $(zz|e^2/r|zz)=(49/30)(\sigma\alpha/2\pi)^{\frac{1}{2}}e^2$ and K_c as the exchange integral $(yz|e^2/r|zy)=(1/15)\times(\sigma\alpha/2\pi)^{\frac{1}{2}}e^2$. The exchange contribution is positive because of the high space symmetry in the two protons and small because of cancellation of positive and negative parts of the overlapping function. In units $(\sigma\alpha/2\pi)^{\frac{1}{2}}e^2$, the various coulomb energies are given in

[20] D. R. Inglis, Phys. Rev. **53**, 880 (1938), reference 17.
[21] A. M. Feingold and E. Wigner, Phys. Rev. **79**, 221 (1950).

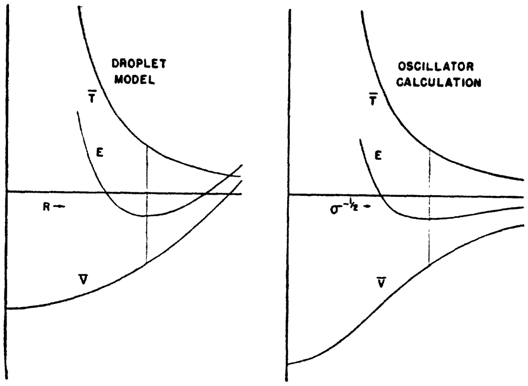

FIG. 1. The artificial treatment of the potential energy $\bar{V}$ in the droplet model makes the curvature of the energy E at equilibrium greater than in the oscillator calculation, and the nucleus stiffer.

Table I. For[22] $\alpha=22mMc^2/\hbar^2$, these totals are $U_{\mathrm{Li}}=3.08\sigma^{\frac{1}{2}}mc^2$ and $U_{\mathrm{Be}}=5.98\sigma^{\frac{1}{2}}mc^2$. The difference of these is the coulomb difference, $E_c=2.9\sigma^{\frac{1}{2}}mc^2$.

It is important that the arbitrary constants be determined in such a way as to give as good an account as possible of the competition between kinetic and potential energies in determining the nuclear compressibility. It is well known[17–19] that such a first-order calculation is quite inadequate to give a correct account of the binding energy of the nucleus, partly because the binding energy is a rather small difference between two larger quantities neither of which is estimated sufficiently accurately by this method, perhaps partly also because of inadequacy of the interactions. Yet the nuclear radius given by the calculation is very sensitive to the binding energy, a fault which may plausibly be remedied by increasing the nuclear interactions beyond their presumed actual strength by an amount just sufficient to give the correct binding energy. This procedure should give the size and stiffness of the nucleus as nearly correct as can be expected from any available simple approximation.

The actual specific nuclear interaction between two nucleons is taken to be specified by the parameters[22]

$$B\approx 50mc^2,\quad \alpha=22mMc^2/\hbar^2. \tag{24}$$

The first-order energy $E_0(\sigma)$, the sum (18)+(20)+(21)+(23), the latter for Li^7, is found to have a minimum at $\sigma_0=1.355$ and to give the binding energy 75.4 mc^2, thus matching sufficiently closely the experimental value 77 mc^2, if B in E_0 is increased to 104 mc^2. (So large an increase of B would not be required if a combination of exchange operators were used, so chosen as barely to satisfy the saturation requirement; but this added refinement would not greatly alter the way in which the potential energy varies with σ, in which the variation of the inevitable factor $(\sigma/\tau)^{\frac{1}{2}}$ is most important.)

[22] Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939); R. D. Hatcher and G. Breit, Phys. Rev. **75**, 1389 (1949). We have rounded out their numbers $A=51.44\ mc^2$ and $\alpha=21.59\ mMc^2/\hbar^2$.

The curvature of $E_0(\sigma)$ at the minimum is $E_0''(\sigma_0)=85.5\ mc^2$.

The oscillator wave functions are generated by a fictitious three-dimensional oscillator potential

$$V=(\hbar^2/2M)(\alpha\sigma)^2r^2$$

which in a perturbation theory similar to the Hartree procedure is taken to approximate the actual potential experienced by the nucleons throughout most of the range of their wave functions, so we may conveniently use this V in (1c). We then have[23]

$$\langle(1/r)(dV/dr)\rangle_{\mathrm{Av}}\sim\sigma^2\sim E_d=(\sigma/\sigma_0)^2\,0.9\ mc^2. \tag{25}$$

Thus, with $E_d'=1.8\sigma_0^{-1}\ mc^2$, $E_c'=1.45\sigma_0^{-\frac{1}{2}}\ mc^2$, and $E_0''=85.5\ mc^2$, we have from (4) the expansion contribution to the excitation difference

$$E_{\mathrm{dif}}=0.019\ mc^2=9.7\ \mathrm{kev}. \tag{26}$$

This is much larger than the value given by the cruder droplet model but still only about one-fourth of the observed difference of 45 kev. The reason for the discrepancy between the two models is displayed in Fig. 1. In using the droplet model we have gone beyond the usual comparison of situations in which the density remains constant, as in the elongation leading to fission, and have treated the saturation as so complete in the interior that the only lack of saturation is found in an idealized surface layer even when the density changes. This artificial treatment makes the potential energy V quadratic in the radius, whereas in the more realistic oscillator calculation it tails off to zero for large radii and indeed has negative curvature at the equilibrium radius. The positive curvature of V in the droplet model thus makes the curvature of the total energy at the equilibrium radius unduly large and makes the nucleus much too stiff.

This estimate is to be considered as an upper limit in the sense that it is based on the very favorable assumption of equal sizes of neutron and proton distributions: if the neutron in Be^7 were not expanded as much as its fellow protons, which are directly affected by the coulomb force, the effect would be smaller (or might even reverse its sign).

V. MAGNETIC SPIN-ORBIT COUPLING

In the usual approximate treatment of magnetic spin-orbit coupling, the electric potential $U(r)$ in (1a) is considered to be already averaged over the positions of the ("other") protons. In Li^7 the other protons are in the s-shell so U has spherical symmetry and the

[23] Keeping the constants of (1c) and (13), we have

$$E_d=(\hbar^4/2\mu Mc^2)(\alpha\sigma)^2=242(m/\mu)\sigma^2mc^2,$$

which is about 1.6 mc^2. This shows that Heisenberg's evaluation of the constant, with the present estimate of σ, gives too large a spin-orbit coupling by about 70 percent even with the factor $\frac{1}{3}$ from (13) included.

evaluation is straightforward:

$$\langle (1/r)(dU/dr)\rangle_{\text{Av}}$$

$$=2(\alpha\sigma)^{\frac{3}{2}}e\frac{\int_0^\infty \rho^4 d\rho \exp(-\rho^2)\int_0^\rho \rho^2 d\rho \exp(-\rho^2)}{\int_0^\infty \rho^4 d\rho \exp(-\rho^2)\int_0^\infty \rho^2 d\rho \exp(-\rho^2)}$$

$$=(4/3)(2\pi)^{-\frac{1}{2}}(\alpha\sigma)^{\frac{3}{2}}e. \quad (27)$$

With $\alpha=22\,mMc^2/\hbar^2$, this with (13) in (1a) gives the magnetic splitting for Li^7

$$E_{m,\,\text{Li}}=0.0047(g/2)\sigma^{\frac{3}{2}}mc^2=10.5\text{ kev}. \quad (28)$$

Here we have assumed that $\mathbf{l}\cdot\mathbf{s}$ contributes the factor $-\frac{1}{2}$ appropriate to a ${}^2P$ of maximum symmetry, and have put $g/2=2.78$, $\sigma=1.35$.

For completeness, one should also take into account the electrostatic Thomas term, as has been suggested privately by Professor Gregory Breit, but it merely has the effect of reducing (28) by a fraction $1/g$ of itself, which we neglect. It is absent in Be^7.

This calculation by use of (1a) ignores exchange. Actually, there is, in addition, a small exchange term of opposite sign corresponding to the reduced probability that the s and p protons with parallel spin find themselves close together, the nuclear wave function tending to be antisymmetrical in their space coordinates. This exchange term appears to be of the order of magnitude 10 percent of the direct term (for $a\approx\frac{1}{2}$ in (1)), so it is not very significant in the comparison with experiment in view of the other uncertainties in the calculation and is here neglected, but it is considered that its evaluation both in the s–p and the p-shell cases might merit further investigation at a later time.

In Be^7 the calculation of the magnetic spin-orbit coupling is essentially more complicated than in Li^7 because in this case there are four protons, rather than just two "other" protons, to give rise to the field at the neutron, and two of them are p-protons having an angular correlation with the neutron's position rather than the spherical symmetry which allows one in the s–p case to use the simplicity of the electrostatic potential of a spherical shell. We may think of the correlation as having two aspects (though it is not clear that they would be separately calculable): first, a geometrical "flattening" of the mass distribution corresponding to the composition of the three vectors $\mathbf{l}$ into $L=1$ and second, an increased probability that the p-neutron finds itself close to one of the p-protons because of the symmetry of the nuclear wave function in the space coordinates of these three nucleons. Both effects are expected to increase the calculated coupling. Neglecting them, we estimate the p-shell contribution to the spin-orbit coupling of Be^7, by use of (13) and (1a) with a spherically symmetric p-shell, to be

$$(\hbar^2 ge^2/2M^2c^2)(2/3)(2\pi)^{-\frac{1}{2}}(\alpha\sigma)^{\frac{3}{2}}(1/2)$$
$$=0.00235(g/2)\sigma^{\frac{3}{2}}mc^2=-3.6\text{ kev}, \quad (29)$$

with the same substitutions again, except $\frac{1}{2}g=-1.91$. This together with the s–p contribution, which from (28) is just twice as great, gives for the magnetic splitting in Be^7, $E_{m,\,\text{Be}}=-10.8$ kev. In this treatment the Li^7 and Be^7 splittings are almost equal in absolute magnitude. This ratio is not the same as in the droplet model because in the oscillator treatment the added p-protons in Be^7 are only half as effective as the more centrally clustered s-protons, an effect reminiscent of the partial self-shielding of electron shells in atoms. The magnetic contribution to the Li^7-Be^7 difference as thus estimated is 21.3 kev, only about half of the estimate made with the rougher uniform-droplet model. Only the small contribution (29) would be expected to be increased by inclusion of exchange and angular correlations in the calculation, and this by probably not more than about 2 kev. Any such increase is at least partly compensated by the corresponding decrease in the s–p contribution. Even if exchange were taken into account, we would thus expect our best estimate of the Li^7-Be^7 difference based on these premises, with the expansion term included, to be rather close to 21.3 kev $+9.7$ kev$=31$ kev, to be compared with the experimental difference 45 kev$\pm$4 kev.

VI. DISCUSSION

The magnetic term estimated with the oscillator functions is thus too small to give agreement with the observed difference of 45 kev even when augmented by the expansion term (26). Among the possible shortcomings of the calculation or initial assumptions, which might be responsible for the discrepancy, are (*a*) the nuclear size parameter given by the fit to the binding energy may be too large; (*b*) the phenomenological treatment of the saturation properties of the interactions may make the nucleus too stiff; (*c*) the states discussed may be more complicated than the ${}^2P$ with maximum space symmetry having a factor $\frac{1}{3}$ in the spin-orbit coupling; (*d*) higher order perturbations may make a direct contribution to the difference of excitation energies by depressing one ${}^2P$ state more than the other. These we discuss in order.

(*a*) The nuclear size parameter may be adjudged also by its success in accounting for the coulomb energy difference between isobars, since one is prone to assume an otherwise symmetrical hamiltonian. In fact, this criterion has provided one of the principal justifications for taking the density of heavy nuclei to apply approximately also to light nuclei. The Be^7-Li^7 coulomb energy difference given by the oscillator calculation, from (23) with $\sigma=1.35$, is 1.72 Mev, only about 5 percent greater than the experimental value of 1.645 Mev, known from the $Li^7(p,n)Be^7$ threshold. This agreement is rather impressive as nuclear physics goes, since no special parameter was introduced to bring it about. An increase of nuclear size is required to bring about more exact agreement with the coulomb energy, whereas a smaller nucleus is required to make the magnetic spin-orbit coupling strong enough. A decrease of radius by about 19 (or at least 15 percent) is needed to bring about agreement with the experimental difference 45 kev (or at least 41 kev) and this would increase the calculated coulomb difference to 2.05 Mev (or at least 1.98 Mev). (The cruder droplet model, which gives large enough spin-orbit coupling, gives a colomb difference of 1.93 Mev.) It thus seems unlikely that the nuclei can be small enough to make a sufficiently strong magnetic spin-orbit coupling.

(*b*) The introduction of a Majorana interaction with a gaussian radial dependence must be recognized as a rather desperate attempt to represent a phenomenon about which we know very little, the nuclear saturation mechanism. We have seen in the discussion of Fig. 1 that the stiffness comes about as a balance between the curvature of the kinetic energy and of the potential energy when plotted as a function of nuclear size, $\sigma^{-\frac{1}{2}}$. If instead we consider $T(\sigma)$ and $V(\sigma)$, functions of σ itself, the curvature comes entirely from $V(\sigma)$. The factor $(\sigma/\tau)^{\frac{3}{2}}$ in (18), for example, is characteristic of the gaussian radial dependence introduced for mathematical convenience, and the other factor is characteristic also of the exchange operator. While these two assumptions give a rather good account of some significant nuclear phenomena, it is not inconceivable that the mesonic calculation which they are intended to approximate would in effect reduce the curvature of $V(\sigma)$ by about a factor two, which would make the expansion term about equal to the magnetic contribution and bring about agreement.

(*c*) Apart from the success of the simple exchange interactions[17] (along with the accepted sign of nucleon spin-orbit coupling)[4] in accounting for the spins of the p-shell nuclei, the main bits of evidence for the interpretation of the low ^2P in Li^7 as the simple one of maximum space symmetry are (1) the remarkable isolation of this doublet—the lack of other low excited states[24]—and (2) the need for the factor $\frac{1}{3}$ in the main term of the spin-orbit coupling in order to make the small splitting in Li^7 and Be^7 compatible with the large doublet splittings in heavy nuclei. The isolation of the ground doublet is a result of the symmetry with any attractive two-nucleon interactions. There are two other ^2P of lower symmetry in the ground configuration s^4p^3. In general, an admixture of higher multiplets would depress the two states of the ground doublet by different amounts and thus contribute directly to the splitting as discussed in (*d*), but certain interactions not containing spin-orbit coupling could admix the higher doublets ^2P without doing this. A strong admixture would lower the symmetry of the ground doublet and by altering the factor $\frac{1}{3}$ might increase the magnetic term enough to bring about agreement with the Li^7-Be^7 difference. Mixtures, in general, are expected to remove the isolation of the ground doublet by complicating the spectrum,[25] but such a special admixture might not—considered as a second-order perturbation it would depress the ground doublet, perhaps more than the next ^2F because there is only one ^2F in the configuration.

Interpretation of the ground doublet as a simple ^2P state of maximum symmetry requires that one ignore evidence for the positive quadrupole moment Q of Li^7 deduced from careful measurements of qQ by Kusch[26] on the basis of uncertain estimates of the molecular coupling constant q. If, instead, we should accept the positive quadrupole moment, it might be due to an admixture of states of maximum symmetry from other configurations,[27] in which case some modification of the present interpretation of the Li^7-Be^7 difference might apply, or it might be due to an admixture of states of other symmetries from the ground configuration,[28] in which case the two low states would be so complex that nothing could be said at present about the expected sign and order of magnitude of the Li^7-Be^7 difference. In either case, enormous admixtures of higher states are needed to obtain a positive quadrupole moment of reasonable magnitude (mostly ^2P from the configurations p^2p' and p^2f with only 12 percent left of the ground configuration p^3 is suggested in reference 27, and mostly states of lower symmetries with only 6 percent from the ^2P of maximum symmetry is suggested in reference 28), so large as to seem completely incompatible with any vestigial validity of a nuclear shell model with its single-nucleon first approximation which has been so useful, particularly in heavier nuclei. If the quadrupole moment really should be proved to be positive, perhaps the most reasonable evaluation of the situation would be as follows: Li^7 and Be^7, although particularly light, might be particularly complex (rather than simple as has been thought) because the field of the s-shell core is not of an appropriate shape to define the p-shell. The shell structure might thus develop again toward the end of the p-shell and be increasingly valid for heavier nuclei because of a tendency of the larger number of particles to approximate a nuclear fluid with the square-well potential of the droplet model. This could leave Li^7 and Be^7 with strong configuration interaction, perhaps still symmetrical between them, and with a ground state ^2P split at least partly by the same H_d responsible for (jj) coupling in heavy nuclei. The sign of the magnetic contribution to the Li^7-Be^7 difference would remain unchanged and it would in most

[24] W. W. Buechner and E. N. Strait, Phys. Rev. **76**, 1547 (1949); D. R. Inglis, Phys. Rev. **78**, 68 (1950); S. S. Hanna and R. W. Gelinas, Bull. Am. Phys. Soc. **26**, 1, C3 (1951); G. R. Keepin, Jr., Phys. Rev. **80**, 768 (1950).

[25] H. H. Hummel and D. R. Inglis, Phys. Rev. **81**, 910 (1951).

[26] P. Kusch, Phys. Rev. **76**, 138 (1949).

[27] R. D. Present, Phys. Rev. **80**, 43 (1950).

[28] R. Avery and C. H. Blanchard, Phys. Rev. **78**, 704 (1950).

possibilities be larger because of an increase of the factor $\frac{1}{3}$ in (10). The present estimate of the expansion term could also retain some significance. Through the middle of the p-shell would be found nuclei of a complexity compatible with the rather dense spectra of low states observed, in which some of the agreement of nuclear spins with simple models might be fortuitous or might be significant in the manner discussed for Li^7. This interpretation based on the possibility of a positive Q seems so complicated, when compared to the empirical success of a much simpler theory, that one is reluctant to accept it until the determination of the relevant q can be based on more reliable molecular (or atomic) electron distributions than are available at present.

(*d*) As an alternative to a theory of spin-orbit coupling based on the introduction of a term such as (1) into the hamiltonian, Feingold and Wigner[21] have proposed that the tensor interaction responsible for the deuteron quadrupole moment has the effect of admixing excited states by a different amount in the two states of the ground doublet, depressing the ${}^2P_{\frac{3}{2}}$ more than the ${}^2P_{\frac{1}{2}}$ if excited configurations are taken into account, *vice versa* with the p^3 configuration only. In spite of the possible complexity of such effects, Feingold and Wigner suggest the preponderance of the matrix elements of single-nucleon excitations as a reason for considering this mainly a single-nucleon effect, so that it might not only apply to these light nuclei but also be a source of apparent (jj) coupling in heavy nuclei. With such a complicated mechanism one might expect that almost anything could happen, but the simplest guess concerning the Li^7-Be^7 difference gives the wrong sign. The more compact low states of Be^7 are raised more by the extra coulomb energy than are the highly excited states, so the "resonance denominators" of the second-order perturbation theory are smaller in Be^7 than in Li^7, which tends to make the splitting greater in Be^7. If this effect and possibly any contrary effects among the matrix elements could merely be shown to be small, we would again have the correct sign and order of magnitude given by the magnetic term, which is present also with this mechanism. The present investigation is an exploration, but is thus not necessarily a criterion, of the possibility (and hope) that a simpler mechanism is mainly responsible for the phenomena considered and that the second-order perturbations play only a secondary role. The judgment must be reached by comparison of successes and failures in a number of nuclei, of which Li^7 and Be^7 are among the lightest and perhaps among the simplest.

VII. COMPARISON WITH OTHER NUCLEI

That second-order effects are playing a considerable role is suggested by the failure to find a consistent indication of the magnitude of spin-orbit coupling among the various nuclei. Such a comparison involves averaging the expression $(1/r)(dV/dr)$ for various nuclei, and the method used by Gaus, which places the entire increment ΔV in a thin surface layer, gives a doublet splitting varying as $(l+\frac{1}{2})/R^2$, or $(l+\frac{1}{2})A^{-\frac{2}{3}}$, as we have seen. This neglects the tendency for nucleons with large angular momentum to be concentrated toward the outside, a tendency which is taken into account in an extreme fashion in the estimate which uses the centripetal-acceleration relation

$$(dV/dr)=l(l+1)\hbar^2/Mr^3. \tag{30}$$

This, when applied to the relevant wave functions with no radial nodes which correspond roughly to circular orbits with radii approximating the nuclear radius, gives a doublet splitting proportional to $l(l+\frac{1}{2})(l+1)A^{-4/3}$. The comparison of a few typical nuclei made in Table II is based on the assumption that the contribution of the main term H_a to Li^7 and Be^7 is 450 kev and that the single-nucleon splitting is three times this, 1.35 Mev. The single-nucleon splittings expected from both modes of variation are listed. If the true variation lies between the two estimates, then an increase of the doublet splittings by about 50 percent is needed to obtain agreement with the roughly known needs of He^5 and the heavy nuclei. The simplest way to obtain this would be to assume that second-order effects reduce the doublet splitting in Li^7 (this is the sign of contributions[21] from the ground configuration s^4p^3) by this amount without much affecting the Li^7-Be^7 difference, and that second-order effects are relatively unimportant in He^5 because of its paucity of excited states (stability of the alpha) and in heavy nuclei for some unknown reason (such a similarity to a degenerate electron gas in a metal) contributing to the success of the (jj) coupling shell model. This leaves the lack of states in N^{15} between the ground state and 5.3 Mev as an enigma, perhaps associated with a growing validity of the shell model as one gets up to this and heavier nuclei. It may be that almost-closed-shell nuclei are both exceptionally compact and have exceptionally large nucleon affinity (or, as in the discussion of Gaus, "meson affinity") ΔV, making an unusually large spin-orbit coupling which could also contribute to the sharpness of the "magic numbers."

TABLE II. Comparison of single-nucleon doublet splittings according to two simple estimates.

Nucleus	He^5	Li^7	N^{15}	Zr^{90}	Ce^{140}	Pb^{208}
	1	1	1	4	5	6
N				50	82	126
$E\sim(l+\frac{1}{2})A^{-2/3}$	1.68	(1.35)	0.8	1.1	1.0	0.9 Mev
$E\sim l(l+\frac{1}{2})(l+1)A^{-4/3}$	2.11	(1.35)	0.49	1.34	1.35	1.33 Mev
E "observed"[a]	3	(1.35)	5.3		2	2 Mev

[a] R. K. Adair, Phys. Rev. **81**, 310 (1950) (He^5); R. E. Malm and W. W. Buechner, Phys. Rev. **78**, 337 (1950) (N^{15}); see reference 6.

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