Electric Excitation and Disintegration of Nuclei. I. Excitation and Disintegration of Nuclei by the Coulomb Field of Positive Particles*

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General formulas for the cross section for the 2^l-pole excitation (or disintegration) of a nucleus by the electric field of a non-relativistic positively charged projectile have been obtained by using the Born approximation; the finite size of the nucleus has been taken into account. For electric dipole and quadrupole excitation a more accurate evaluation has been made in which the initial and final states of the charged projectile are described by the exact coulomb field wave functions.

The cross section for electric 2^l-pole excitation of a nucleus by the field of a charged projectile is proportional to the corresponding photo-excitation cross section; the proportionality factor may be interpreted as the number of virtual electric 2^l-pole quanta in the field of the charged projectile. In the electric dipole case, the cross section for the inelastic scattering of a charged projectile is shown to be proportional to the cross section for the production of continuous x-rays by deflection of the projectile in a coulomb field. Thus a relation is established between the virtual quanta representing the coulomb field of the projectile and the real quanta of the x-ray spectrum corresponding to the scattering process.

The theory of electric transitions has been applied, in some detail, in a discussion of two cases in which it seems likely that these transitions play a significant role. These two cases are (1) the inelastic scattering of deuterons and (2) the electric break-up of the deuteron when it "collides" with a target nucleus.

I. INTRODUCTION

WHEN target nuclei are bombarded with positively charged projectiles, nuclear excitation or disintegration may take place by either of two competing processes: (1) A direct nuclear interaction may take place in which the projectile enters the nucleus, thereby forming an excited compound nucleus which decays by the emission of a γ-ray or a particle; or (2) the target nucleus may be excited or disintegrated by the electromagnetic field of the charged projectile. Under conditions most favorable for electromagnetic excitation, the cross section for process (2) should be given roughly by the product of the fine structure constant ($e^2/\hbar c=1/137$) and the photo-excitation cross section.

Thus, generally speaking, we may expect the excitation by direct nuclear interaction to be more probable than excitation by electromagnetic interaction. However, we may expect transitions induced by electromagnetic interaction to play a significant role if either of the two following conditions is fulfilled: (1) The transition energy is small compared to the coulomb barrier energy. In this case the excitation energy can be supplied by electric interaction even though the projectile misses the nucleus by several nuclear diameters. (2) Because of special conditions the probability of excitation by direct nuclear interaction is small.

The smallness of the cross section for the "electric excitation" of nuclei by positively charge projectiles has made experimental detection difficult, with the result that although the electric excitation process has been looked for in the past[1] only recently has rather clear-cut evidence for the occurrence of this process been obtained. This evidence has been obtained in the inelastic scattering of deuterons.[2,3] In this case electric transitions may be expected to play the major role because once a deuteron enters the nucleus, the probability that a deuteron will be emitted is very small; and the much more probable event in which a single nucleon is evaporated from the excited compound nucleus does not yield an inelastically scattered deuteron. We shall show that the angular distribution of the inelastically scattered deuterons obtained experimentally agrees with that to be expected on the basis of electric transitions. A second example of a reaction involving deuterons in which electric transitions may play a significant role is the stripping process. This process, in which one of the nucleons is stripped off the deuteron as it passes the nucleus, may be due to electric interaction or to direct nuclear encounter of one of the nucleons of the deuteron with the target nucleus.

The excitation of a nucleus by the electromagnetic field of a charged projectile is very closely related to the corresponding photo-excitation. This relation is seen readily if one uses the quantum-mechanical analog to the semiclassical method of virtual quanta.[4] The cross section, σ_e, for excitation by the electromagnetic field of a charged projectile can be written in terms of the photo-excitation cross section, $\sigma^{(\gamma)}$, as follows:

$$\sigma_e(E_i, E)=N(E_i, E)\sigma^{(\gamma)}(E), \tag{1}$$

where E is the excitation energy, E_i the energy of the incident projectile, and $N(E, E_i)$ is the number of virtual quanta (per unit energy) of energy E in the electromagnetic field of the charged projectile which are available for producing the transition. If the final energy level lies in an energy continuum (as in the case, for example, when a disintegration is produced), the cross section for the transition is obtained by integrating

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[1] Lark-Horovitz, Risser, and Smith, Phys. Rev. 55, 878 (1939). E. H. Rhoderick, Nature 163, 848 (1949). H. V. Halban, private communication (unpublished).

[2] Greenlees, Kempton, and Rhoderick, Nature 164, 663 (1949).

[3] J. R. Holt and C. T. Young, Nature 164, 1000 (1949).

[4] The quantum mechanical analog of the method of virtual quanta has been used by E. Guth and C. J. Mullin [Phys. Rev. 76, 234 (1949)] to describe the disintegration of nuclei by electrons in the Born approximation. A general discussion of the method, also in the Born approximation, has been given by M. Lax and H. Feshbach (unpublished).

over all permissible transition energies:

$$\sigma_e = \int N\sigma^{(\gamma)} dE. \tag{2}$$

The number of quanta, $N(E, E_i)$, is given by a matrix element involving only the projectile's coordinates and, therefore, is independent of specific assumptions made in choosing a model for the nucleus.

In the past, recognition of electric excitation has been hampered by the lack of an adequate theory. Crude estimates of the cross section for electric excitation have been given by Landau[5] and Weisskopf.[6] An exact formula for the total cross section in a special case has been given by Guth[7] who employed Ehrenfest's theorem to reduce the rather complicated electric matrix element to the matrix element for the production of bremsstrahlung by a particle deflected by a coulomb field.

The principal objective of the present paper is to give a detailed treatment of the excitation and disintegration of nuclei by the electric field of nonrelativistic, positively charge particles. This treatment should facilitate comparison of theory with experiment. In Sec. II-A formulas for the cross sections for electric 2^l-pole excitation are developed using the Born approximation. Since the coulomb deflection of the projectile should decrease the overlap of the projectile's wave functions occuring in the electric matrix element. the Born approximation should yield an upper limit for the correct electric excitation cross section. In Sec. II-B, cross sections for electric dipole and quadrupole excitation are developed by using the exact coulomb wave functions to describe the initial and final states of the charged projectile. The quantum mechanical analog of the method of virtual quanta is applied to the electric excitation of nuclei by nonrelativistic positively charged projectiles in Sec. III of this paper; expressions for the numbers of electric dipole, quadrupole, and octupole quanta in the electric field of the charged projectile are derived. In the electric dipole case, the relationship is established between the virtual quanta representing the coulomb field of the charged projectile and the real quanta of the x-ray spectrum corresponding to the scattering process. In Sec. IV the theory of electric excitation is applied to the electric break-up of the deuteron in flight. In Sec. V the theory of electric transitions is applied in a detailed discussion of two cases in which it seems likely that these transitions play a significant role. These two cases are (1) the inelastic scattering of deuterons and (2) the electric break-up of the deuteron when it "collides" with a target nucleus.

[5] L. Landau, Physik. Z. U.S.S.R. **1**, 88 (1932).
[6] V. Weisskopf, Phys. Rev. **53**, 1018 (1938). Weisskopf assumes the projectile's wave functions to be constant over the region extending from the classical turning point to infinity.
[7] E. Guth, Phys. Rev. **68**, 280 (1945).

II. CROSS SECTIONS FOR ELECTRIC EXCITATION

In this section we shall develop the theory of the excitation of a nucleus by the coulomb field of an incident, nonrelativistic, charged projectile.

Cross sections for the excitation of a nucleus through the interaction of the electromagnetic field of relativistic electrons with the electric dipole, magnetic dipole, and the electric quadrupole moments of the nucleus have been obtained by Wick.[8] In the nonrelativistic limit Wick's results differ from ours because he neglected the finite size of the nucleus. We shall assume that all projectiles which penetrate into the nucleus give rise to nonelectric processes. Since we wish to exclude all nonelectric processes from our considerations, we shall set the interaction between the projectile and the nucleus equal to zero when the projectile is within a certain distance, r_0, from the center of the nucleus. This distance will be taken as the sum of the "radii" of the nucleus and the projectile. The extended range of nuclear forces also modifies the projectile's wave functions outside the nucleus. This latter effect plays a significant role only in the angular distributions of particles scattered through rather large angles and will be neglected here.

We neglect the nonzero extension of the charge distribution of the projectile. This procedure is correct when the projectiles are protons, but is an approximation when deuterons or α-particles are used. Thus the interaction between the nucleus of charge Ze and the projectile of charge ze is given by

$$\begin{aligned} V &= 0 \qquad & r \leqslant r_0, \\ V &= \sum_{\mu=1}^{Z} \frac{ze^2}{|\mathbf{r}-\mathbf{R}_\mu|} \qquad & r \geqslant r_0, \end{aligned} \tag{3}$$

where $\mathbf{r}$ is the coordinate of the projectile, and $\mathbf{R}_\mu$ is the coordinate of the μth of the Z protons in the nucleus. Making the multipole expansion:

$$V = \sum_{\mu=1}^{Z} \sum_{n=0}^{\infty} (ze^2/r)(R_\mu/r)^n P_n[\cos(\mathbf{r}, \mathbf{R}_\mu)] = \sum_l V_l, \tag{4}$$

where $(\mathbf{r}, \mathbf{R}_\mu)$ is the angle between the vectors $\mathbf{r}$ and $\mathbf{R}_\mu$. The $n=l$ term of this series gives the interaction between the projectile and the electric 2^l-pole moment of the nucleus and gives rise to electric 2^l-pole transitions. If we assume that the nuclear states are classified by parity and angular momentum, the terms in the series can be considered separately.

We shall work in the center-of-mass coordinate system. Using first-order perturbation theory, the cross section for an electric 2^l-pole transition is

$$d\sigma/d\Omega = [2\pi/\hbar(2j_A+1)] \sum_{a,b} |(\chi_2 b|V_l|\chi_1 a)|^2, \tag{5}$$

where V_l is the interaction between the projectile and the electric 2^l-pole moment of the nucleus, χ_1 and χ_2

[8] G. C. Wick, Ricerca Scient. **XI**, 49 (1940).

are the initial and final states of the projectile (normalized, respectively, to unit flux and unit energy), a is one of the $2j_A+1$ initial nuclear states belonging to the initial level A, and b is one of the $2j_B+1$ final nuclear states belonging to the final level B. The integrals over the projectile's coordinates extend from r_0 to infinity.

In the usual case, in which the nuclear moments are randomly oriented, the angular distribution of the inelastically scattered particles is symmetric about the direction of incidence of the projectiles. In all cases which we shall consider it is possible to choose a direction in terms of which the product $\chi_2^*\chi_1$ has axial symmetry. Taking this axis of symmetry in, say, the $\mathbf{n}$ direction, and using the addition theorem for the Legendre polyomials, the matrix element occuring in Eq. (5) can be factored, and we obtain

$$d\sigma/d\Omega=[2\pi/\hbar(2j_A+1)]M_{AB}^2M_{12}^2, \qquad (6)$$

where

$$M_{AB}^2=\sum_{a,b}|(b|\sum_{\mu=1}^{Z}R_\mu{}^lP_l[\cos(\mathbf{n},\mathbf{R}_\mu)]|a)|^2$$

and

$$M_{12}^2=|(\chi_2|ze^2P_l[\cos(\mathbf{n},\mathbf{r})]/r^{l+1}|\chi_1)|^2.$$

The electric 2^l-pole photo-excitation cross section $\sigma_l^{(\gamma)}$ is proportional to M_{AB}^2. Writing $\sigma_l^{(\gamma)}=\beta M_{AB}^2/(2j_A+1)$

$$\sigma_l=(2\pi\sigma_l^{(\gamma)}/\hbar\beta)\int M_{12}^2d\Omega.$$

The number of virtual electric 2^l-pole quanta in the electric field of the projectile is thus given by

$$N_l=(2\pi/\hbar\beta)\int M_{12}^2d\Omega. \qquad (7)$$

(A) Cross Sections in the Born Approximation

In the Born approximation the incident and scattered particles are represented by plane waves with wave numbers $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. The cross section for an electric 2^l-pole transition is given by

$$\frac{d\sigma}{d\Omega}=\left(\frac{n_1}{2\pi Z}\right)^2\frac{k_1k_2}{2j_A+1}M_{AB}^2M_{12}^2 \qquad (8)$$

with

$$M_{12}^2=\left|\int\frac{\exp(i\mathbf{K}\cdot\mathbf{r})\cdot P_l[\cos(\mathbf{K},\mathbf{r})]}{r^{l+1}}d\mathbf{r}\right|^2,$$

$$M_{AB}^2=\sum_{a,b}\left|\sum_{\mu=1}^{Z}(b|R_\mu{}^nP_l[\cos(\mathbf{K},\mathbf{R}_\mu)]|a)\right|^2,$$

$$n_1=zZe^2/\hbar v_1,$$

$$\mathbf{K}=\mathbf{k}_1-\mathbf{k}_2,\quad K=|\mathbf{K}|=\{k_1^2+k_2^2-2k_1k_2\cos\theta\}^{\frac{1}{2}},$$

where θ is the scattering angle in the center-of-mass system. The integral over the particles coordinates can be carried out readily and yields

$$M_{12}^2=16\pi^2K^{2(l-2)}[j_{l-1}(Kr_0)/(Kr_0)^{l-1}]^2.$$

$j_q(x)$ is the spherical bessel function of order q.

The cross section for electric 2^l-pole excitation is then

$$\frac{d\sigma}{d\Omega}=4\left(\frac{n_1}{Z}\right)^2\frac{k_1k_2}{2j_A+1}K^{2(l-2)}\left[\frac{j_{l-1}(Kr_0)}{(Kr_0)^{l-1}}\right]^2M_{AB}^2. \qquad (9)$$

The angular distribution is given by the factor $K^{2(l-2)}[j_{l-1}(Kr_0)/(Kr_0)^{l-1}]^2$. The effect of the finite size of the nucleus on the angular distribution shows up in the factor in brackets. If one allows r_0 to go to zero, this factor becomes a constant, and the angular distribution is given simply by the factor $K^{2(l-2)}$.

On integrating over all angles of scattering, one has

$$\sigma=8\pi(n_1/Z)^2M_{AB}^2B_l/(2j_A+1), \qquad (10)$$

where

$$B_l=\frac{1}{r_0{}^{2l-2}}\int_{(k_1-k_2)r_0}^{(k_1+k_2)r_0}\frac{j_{l-1}{}^2(y)}{y}dy.$$

The integration for B_l can be carried out, but the general result is rather complicated and will not be given here. For $l=1$, 2, and 3 we shall obtain relatively simple expressions for B_l.

From the transformation properties of $P_l[\cos(\mathbf{K},\mathbf{R})]$ it is evident that the square of the nuclear matrix element (M_{AB}^2) is proportional to the sum of the squares of the elements of the irreducible (traceless) multipole moment tensor for the 2^l-pole transition $A\rightarrow B$. In the following paragraphs we shall evaluate M_{AB}^2 in terms of the multipole moments for dipole, quadrupole, and octupole interactions.

(1) *Electric Dipole Transitions*

We introduce the electric dipole moment of the nucleus, $\mathbf{D}$, which has the cartesian components

$$D_i=x_i=\sum_{\mu=1}^{Z}x_{i\mu}.$$

The cross section for electric dipole transitions is obtained by setting $l=1$ in Eq. (9). If we average over all directions of $\mathbf{R}_\mu$ we obtain

$$\left\{\left[\sum_{\mu=1}^{Z}R_\mu P_1[\cos(\mathbf{K},\mathbf{R}_\mu)]\right]^2\right\}_{\text{Av}}=\tfrac{1}{3}|\mathbf{D}|^2.$$

Hence, for randomly oriented nuclear moments

$$M_{AB}^2=D_{AB}^2/3,$$

where

$$D_{AB}^2=\sum_{a,b}\sum_i|(b|D_i|a)|^2$$

is the square of the electric dipole moment of the nucleus for the transition $A\rightarrow B$. Thus, for electric dipole transitions

$$\frac{d\sigma}{d\Omega}=\frac{4}{3}\left(\frac{n_1}{Z}\right)^2\frac{D_{AB}^2}{2j_A+1}\frac{k_1k_2}{K^2}j_0^2(Kr_0), \tag{11}$$

$$\sigma=\frac{8\pi}{3}\left(\frac{n_1}{Z}\right)^2\frac{D_{AB}^2}{2j_A+1}B_1, \tag{12}$$

where

$$B_1=f[2(k_1+k_2)r_0]-f[2(k_1-k_2)r_0],$$

$$f(x)=Ci(x)+(\cos x-x\sin x-1)x^2.$$

$Ci(x)$ is the cosine integral:

$$Ci(x)=-\int_x^\infty(\cos t/t)dt.$$

The relatively complicated function B_1 can be approximated in a very simple way in certain cases of physical interest. Thus in the case that $2(k_1+k_2)r_0$ is small compared to unity,

$$B_1\cong\ln[(k_1+k_2)/(k_1-k_2)].$$

In the physically important case that $2(k_1-k_2)r_0\ll1$ and $2(k_1+k_2)r_0\gg1$,

$$B_1\cong\ln[\epsilon/2\gamma(k_1-k_2)r_0],$$

where ϵ = base for natural logarithms, $\ln\gamma$ = Euler's constant: $\gamma=1.781$. It is interesting to note that the last formula yields a total cross section which differs very little from that which is obtained by using a point nucleus in performing the integrations over the projectile's configuration space and taking account of the nonzero size of the nucleus by limiting the recoil momentum imparted to the nucleus to values $\leqslant\hbar/r_0$. This latter procedure, which has been used by Dancoff[9] in describing the electric break-up of the deuteron, yields the result

$$B_1=\ln[1/(k_1-k_2)r_0],$$

which is just slightly larger than our value of B_1.

(2) *Electric Quadrupole Transitions*

The differential and total cross sections for electric quadrupole transitions are obtained by setting $l=2$ in Eqs. (9) and (10). We introduce the quadrupole moment tensor Q which has elements $Q_{ij}=x_ix_j$ and the irreducible quadrupole moment tensor Q' which has elements $Q_{ij}'=Q_{ij}-\frac{1}{3}I\delta_{ij}$, where $I=x^2+y^2+z^2$ is the trace of the quadrupole moment tensor. From the transformation properties of $P_2[\cos(\mathbf{K},\mathbf{R})]$ it is evident that the square of the nuclear matrix element, M_{AB}^2, is proportional to $Q_{AB}'^2$, that is, proportional to the square of the traceless quadrupole moment for the transition $A\rightarrow B$.

[9] S. M. Dancoff, Phys. Rev. 72, 1017 (1947).

The elements of the quadrupole moment tensor satisfy the equation:

$$\sum_{ij}|Q_{ij}'|^2=\tfrac{2}{3}\sum_{ij}|Q_{ij}|^2.$$

Since averaging over all directions of the vector $\mathbf{R}$ yields

$$[\{R^2P_2[\cos(\mathbf{K},\mathbf{R})]\}^2]_{\text{Av}}=\tfrac{1}{5}\sum_{ij}|Q_{ij}|^2=\tfrac{3}{10}\sum_{ij}|Q_{ij}'|^2,$$

it follows that for randomly oriented nuclear moments

$$M_{AB}^2=3Q_{AB}'^2/10,$$

where

$$Q_{AB}'^2=\sum_{a,b}\sum_{ij}|(b|Q_{ij}'|a)|^2$$

is the square of the traceless quadrupole moment for the transition $A\rightarrow B$. Consequently, from Eq. (9) we obtain

$$\frac{d\sigma}{d\Omega}=\frac{6}{5}\left(\frac{n_1}{Z}\right)^2k_1k_2\frac{Q_{AB}'^2}{2j_A+1}\left[\frac{j_1(Kr_0)}{Kr_0}\right]^2, \tag{13}$$

and

$$\sigma=\frac{12\pi}{5}\left(\frac{n_1}{Z}\right)^2\frac{Q_{AB}'^2}{2j_A+1}B_2 \tag{14}$$

with

$$B_2=F[2(k_1+k_2)r_0]-F[2(k_1-k_2)r_0],$$

$$F(x)=(2/r_0^2x^4)[\cos x+x\sin x-1-\tfrac{1}{2}x^2].$$

In certain cases of physical interest, simple approximations for B_2 can be obtained. Thus, if $2(k_1+k_2)r_0<1$, $B_2\cong2k_1k_2/9$. If $2(k_1-k_2)r_0<1$ but $2(k_1+k_2)r_0\gg1$, $B_2\cong1/4r_0^2$. Use of a cutoff in momentum space rather than in configuration space leads to the value $B_1\cong1/18r_0^2$ in this latter case.

It is of interest to note that our quadrupole cross section is proportional to the square of the traceless quadrupole moment, $Q_{AB}'^2$. Wick,[8] on the other hand, obtains a cross section which contains a term proportional to $Q_{AB}'^2$ and a second term proportional to the square of the trace of the quadrupole moment tensor (I_{AB}^2). This latter term, which corresponds to a "monopole" interaction, and which gives rise to $j=0\rightarrow j=0$ transitions, occurs nowhere in our cross sections because we have deleted the region occupied by the nucleus from the projectile's configuration space. The manner in which the monopole term enters into the cross sections can be seen easily by extending our integrations over the projectile's configuration space to include the region occupied by the nucleus. In this case the cross section is

$$\frac{d\sigma}{d\Omega}=\frac{4}{2j_A+1}\left(\frac{n_1}{Z}\right)^2\frac{k_1k_2}{K^4}\sum_{a,b}\left|\left(b\left|\sum_{\mu=1}^{Z}\exp(i\mathbf{K}\cdot\mathbf{R}_\mu)\right|a\right)\right|^2.$$

The exponential can be expanded in a Taylor series or in a series of spherical harmonics. The Taylor series classifies terms according to powers of $\mathbf{K}\cdot\mathbf{R}$. The

$(\mathbf{K}\cdot\mathbf{R})^l$ term mixes together terms corresponding to electric 2^l-pole, 2^{l-2}-pole, 2^{l-4}-pole, etc., interactions. Thus $(\mathbf{K}\cdot\mathbf{R})^2=\frac{1}{3}K^2R^2\{2P_2[\cos(\mathbf{K},\mathbf{R})]+P_0[\cos(\mathbf{K},\mathbf{R})]\}$

yields a term which corresponds to quadrupole interaction (P_2) and a term which corresponds to a monopole interaction (P_0). The pure quadrupole interaction yields a cross section proportional to $Q_{AB}'^2$; the monopole interaction gives a cross section proportional to I_{AB}^2. Similarly,

$$(\mathbf{K}\cdot\mathbf{R})^3=\tfrac{1}{5}K^3R^3\{2P_3[\cos(\mathbf{K},\mathbf{R})]+3P_1[\cos(\mathbf{K},\mathbf{R})]\}$$

mixes octupole and dipole interactions. The pure electric 2^l-pole interaction always yields a cross section expressible in terms of the elements of the irreducible 2^l-pole moment tensor. If the exponential $\exp(i\mathbf{K}\cdot\mathbf{R})$ is expanded in a series of spherical harmonics:

$$\exp(i\mathbf{K}\cdot\mathbf{R})=\sum_{l=0}^{\infty} i^l(2l+1)P_l[\cos(\mathbf{K},\mathbf{R})]j_l(KR),$$

the lth-order term of the series corresponds to the pure electric 2^l-pole interaction. In this series the monopole transitions arise from the $l=0$ term, the dipole transitions from the $l=1$ term, etc.

(3) *Electric Octupole Transitions*

The differential and total cross sections for electric octupole transitions are obtained by setting $l=3$ in Eqs. (9) and (10). We introduce the octupole moment tensor S with elements $S_{ijk}=x_ix_jx_k$, and the irreducible octupole moment tensor S' with elements $S'_{jik}=x_ix_jx_k-\frac{1}{5}(x_i\delta_{jk}+x_j\delta_{ki}+x_k\delta_{ij})$. From the transformation properties of $P_3[\cos(\mathbf{K},\mathbf{R})]$ it is evident that the square of the nuclear matrix element, M_{AB}^2, is proportional to $S_{AB}'^2$, where

$$S_{AB}'^2=\sum_{a,b}\sum_{i,j,k}|(b|S'_{ijk}|a)|^2$$

is the square of the irreducible octupole moment tensor for the transition $A\rightarrow B$. The elements of the octupole moment tensor satisfy the equation

$$\sum_{ijk}|S'_{ijk}|^2=\tfrac{2}{5}\sum_{ijk}|S_{ijk}|^2.$$

Since averaging over all directions of $\mathbf{R}$ yields

$$[\{R^3P_3[\cos(\mathbf{K},\mathbf{R})]\}^2]_{\text{Av}} = (1/7)\textstyle\sum_{ijk}|S_{ijk}|^2=(5/14)\sum_{ijk}|S'_{ijk}|^2,$$

we obtain, for randomly oriented nuclear moments,

$$M_{AB}^2=(5/14)S_{AB}'^2.$$

Thus from Eq. (9)

$$\frac{d\sigma}{d\Omega}=\frac{10}{7}\left(\frac{n_1}{Z}\right)^2 k_1k_2\frac{S_{AB}'^2}{2j_A+1}K^2\left[\frac{j_2(Kr_0)}{(Kr_0)^2}\right]^2 \tag{15}$$

and

$$\sigma=\frac{20\pi}{7}\left(\frac{n_1}{Z}\right)^2\frac{S_{AB}'^2}{2j_A+1}B_3 \tag{16}$$

with

$$B_3=\phi[2(k_1+k_2)r_0]-\phi[2(k_1-k_2)r_0],$$

$$\phi(x)=\frac{48}{r_0^4x^6}\left[\left(1-\frac{3}{8}x^2\right)\cos x+\left(1-\frac{x^2}{24}\right)x\sin x-\left(1+\frac{x^2}{8}+\frac{x^4}{48}\right)\right].$$

In certain cases of physical interest simple approximations for B_3 can be obtained. Thus if $2(k_1+k_2)r_0<1$, $B_3\cong(2/225)k_1k_2(k_1^2+k_2^2)$. If $2(k_1-k_2)r_0<1$, but

$$2(k_1+k_2)r_0\gg1,\quad B_3\cong1/12r_0^4.$$

(B) Electric Dipole and Quadrupole Excitation Using Coulomb Field Wave Functions

(1) *Electric Dipole*

In this section we shall give a more accurate treatment of electric dipole excitation, using the exact Coulomb wave functions to describe the motion of the projectile.[10] The motion of the incident projectile is described by a coulomb wave asymptotic to a plane wave moving in the direction $\mathbf{k}_1$ plus an outgoing spherical wave; this wave function is normalized to unit flux. The scattered projectile's wave function is taken as a Coulomb wave asymptotic to a plane wave moving in the $\mathbf{k}_2$ direction plus an incoming spherical wave; this wave function is energy normalized. The wave functions are

$$\begin{aligned}\chi_1&=N_{fl}\exp(i\mathbf{k}_1\cdot\mathbf{r})L_{in_1}(\rho_1)=N_{fl}U_1,\\ N_{fl}&=[2\pi mn_1/hk_1(e^{2\pi n_1}-1)]^{\frac{1}{2}},\\ \chi_2&=N_E\exp(i\mathbf{k}_2\cdot\mathbf{r})L_{-in_2}(-\rho_2)=N_EU_2,\\ N_E&=[mk_2n_2/(2\pi)^2h^2(e^{2\pi n_2}-1)]^{\frac{1}{2}},\end{aligned} \tag{17}$$

with $\rho_1=i(kr-\mathbf{k}\cdot\mathbf{r})$, and $\rho_2=i(kr+\mathbf{k}\cdot\mathbf{r})$. $L_q(x)$ is the Laguerre function (S-119) and is a special case of a confluent hypergeometric function: $L_q(x)=F(-q,1,x)$.

Using these wave functions, the cross section for an electric dipole transition may be obtained from Eq. (5):

$$\frac{d\sigma}{d\Omega}=\frac{2\pi}{h}\frac{D_{AB}^2}{3(2j_A+1)}\sum_i\left|\left(\chi_2\left|\frac{ze^2x_i}{r^3}\right|\chi_1\right)\right|^2. \tag{18}$$

To simplify the calculations we shall neglect the extension of the nucleus in the integration over the projectile's

[10] A. Sommerfeld, *Atombau und Spektrallinien* (II Band, F. Vieweg and Sohn, Braunschweig, 1939). Hereafter, references to this book will be given as S followed by the appropriate page number.

configuration space. However, we shall take this extension into account after integration.

The matrix elements occurring in (18) can be simplified by use of the equation of motion (Ehrenfest's theorem) for a particle moving in a Coulomb field. The force acting on such a particle is given by

$$F_{x_i}=m\ddot{x}_i=-\partial(zZe^2/r)/\partial x_i=zZe^2x_i/r^3.$$

Using this relation and the relation between the operator x_i and its time derivative, we have

$$\left(\chi_2\left|\frac{ze^2x_i}{r^3}\right|\chi_1\right)=-\frac{1}{Z}\left(\chi_2\left|\frac{\partial}{\partial x_i}\frac{zZe^2}{r}\right|\chi_1\right)=\frac{m}{Z}(\chi_2|\ddot{x}_i|\chi_1)$$
$$=-\frac{m}{Z}\left(\frac{E_2-E_1}{\hbar}\right)^2(\chi_2|x_i|\chi_1).$$

Thus the matrix elements in (18) can be reduced to the well-known matrix elements for the production of bremsstrahlung by a charged particle scattered in a coulomb field.

In terms of the wave functions

$$U_1=\exp(i\mathbf{k}_1\cdot\mathbf{r})L_{in_1}(\rho_1),\quad U_2=\exp(i\mathbf{k}_2\cdot\mathbf{r})L_{-in_2}(-\rho_2)$$

and the matrix elements

$$M_{x_i}=(U_2|x_i|U_1),$$

the cross section can be written as

$$\frac{d\sigma}{d\Omega}=\frac{A_1A_2}{3(2\pi)^2}\left(\frac{m}{\hbar^2}\right)^4\frac{k_2}{k_1}\frac{(E_2-E_1)^4}{Z^2}\exp[-2\pi(n_1+n_2)]$$
$$\times\frac{D_{AB}^2}{2j_A+1}\sum_i|M_{x_i}|^2,$$

where

$$A_{(1,2)}=2\pi n_{(1,2)}/1-\exp(-2\pi n_{(1,2)}).$$

The matrix elements, M_{x_i}, have been evaluated by Sommerfeld. Taking the x-axis in the $\mathbf{k}_1$ direction and θ as the scattering angle, the results are (S-502 and 509)

$$M_x=C\{i(n_2-n_1\cos\theta)F+(1-\cos\theta)(1-x)F'\}(1-x)^{-in_1-in_2-1},$$
$$M_{(x,y)}=-C\begin{Bmatrix}\cos\phi\\ \sin\phi\end{Bmatrix}\sin\theta[in_1F+(1-x)F']\times(1-x)^{-in_1-in_2-1},\qquad(19)$$

where $F=F(-in_1,-in_2,1,x)$ is the hypergeometric function,

$$F'=dF/dx,\quad x=-[4k_1k_2/(k_1-k_2)^2]\sin^2\tfrac{1}{2}\theta$$

and

$$C=-16\pi e^{\pi n_1}\frac{k_1k_2}{(k_1^2-k_2^2)^2(k_1-k_2)^2}\left(\frac{k_1+k_2}{k_1-k_2}\right)^{i(n_1+n_2)}.$$

Hence the cross section is given by

$$\frac{d\sigma}{d\Omega}=\frac{4}{3}\frac{A_1A_2e^{-2\pi n_2}}{Z^2}\frac{k_2}{k_1}\frac{k_1^2k_2^2}{(k_1-k_2)^4}\frac{D_{AB}^2}{2j_A+1}$$
$$\cdot\{|i(n_2-n_1\cos\theta)F+(1-\cos\theta)(1-x)F'|^2+\sin^2\theta|in_1F+(1-x)F'|^2\}.\qquad(20)$$

Still disregarding the finite size of the nucleus, this expression can be integrated over all angles of scattering (S-526), yielding the total cross section:

$$\sigma=\frac{2\pi}{3}\frac{A_1A_2e^{-2\pi n_2}}{Z^2}\frac{k_2}{k_1}\frac{D_{AB}^2}{2j_A+1}x_0\frac{d}{dx_0}|F(x_0)|^2,\qquad(21)$$

where $x_0=-4k_1k_2/(k_1-k_2)^2$ and

$$F(x_0)=F(-in_1,-in_2,1,x_0).$$

The general expressions for the cross sections given by (20) and (21) are rather complicated functions of energy and scattering angle. In certain limiting cases they reduce to much simpler expressions. We shall consider three limiting cases which are of some physical interest.

(*a*) *Cross Sections When n_1 Is Small.*—We shall consider the case $n_1\to0$, with n_2 arbitrary. For small n_1, $F\cong g$, $F'\cong0$, where $g=1$ if n_2/n_1 is not approximately equal to unity and $g=\sinh\pi n_1/\pi n_1$ if n_2/n_1 is approximately equal to unity.[11] Thus for small n_1,

$$\frac{d\sigma}{d\Omega}=\frac{4}{3}\left(\frac{n_1}{Z}\right)^2A_1A_2e^{-2\pi n_2}g^2\frac{D_{AB}^2}{2j_A+1}\frac{k_1k_2}{K^2}f^2,\qquad(22)$$

$$K^2=k_1^2+k_2^2-2k_1k_2\cos\theta,$$

$$g=\begin{Bmatrix}\sinh\pi n_1/\pi n_1 & \text{if } n_2\cong n_1\\ 1 & \text{otherwise}\end{Bmatrix}.$$

The factor f^2 has been included to take account of the finite size of the nucleus. The value of f may be obtained readily in the limiting cases of small and large n_2. Thus if n_2 is small, Eq. (22) gives the first order coulomb corrections to the Born formula. Consequently for small n_2 we may take for f the value obtained in the Born approximation: $f=j_0(Kr_0)$. If, on the other hand, n_2 is large (i.e., v_2 is small), the angular distribution should be isotropic; for this case we may therefore take $f=1$. Thus we have the results

$$f=j_0(Kr_0)\quad\text{if } n_2 \text{ is small},$$
$$f=1\quad\text{if } n_2 \text{ is large}.$$

[11] As $n_1\to0$ and $n_2/n_1\neq1$, then x is small and one can expand F and retain only the first term, namely, unity. As $n_1\to0$ and $n_2/n_1\cong1$, x is large (except for $\theta=0$) and one can express $F(x)$ in terms of $F(1/1-x)$; expanding $F(1/1-x)$ and retaining only the first term leads to the value $g=(\sinh\pi n_1)/\pi n_1$ in this case.

Integration over all angles then leads to the results:

$$\sigma=\frac{8\pi}{3}\left(\frac{n_1}{Z}\right)^2 g^2 A_1 A_2 e^{-2\pi n_2}\frac{D_{AB}{}^2}{2j_A+1}C_1, \qquad (23)$$

where

$C_1=\ln[(k_1+k_2)/(k_1-k_2)]$ if n_2 is large (k_2 small),
$C_1=B_1$ of Eq. (12) if n_2 is small.

(*b*) *Cross Sections for Strong Coulomb Field.*—We consider next the case in which the influence of the coulomb field is very large. In particular, we consider the case $n_2\rightarrow\infty$ and $n_1\gg 1$. Let

$$|P|^2=|i(n_2-n_1\cos\theta)F+1(1-\cos\theta)(1-x)F'|^2,$$
$$|Q|^2=|in_1F+(1-x)F'|^2,$$
$$\rho=n_1/n_2 \text{ so that } 0\leqslant\rho\leqslant 1.$$

Again we take the x-axis along the $\mathbf{k}_1$ direction. Then as $n_2\rightarrow\infty$ and $n_1\gg 1$, the majority of the particles are scattered backwards, and we need consider values of $\theta\cong\pi$ only. We then obtain (S-806)

$$\begin{Bmatrix}|P|^2\\|Q|^2\end{Bmatrix}=\frac{1}{64}\frac{n_2{}^2}{3}e^{2\pi n_1}\begin{Bmatrix}\alpha^2 e^{5\pi i/6}H_{2/3}{}^{(1)}(is)\\ \alpha e^{2\pi i/3}H_{1/3}{}^{(1)}(is)\end{Bmatrix}^2,$$

where

$$s=\frac{n_2\alpha^3\rho(1-\rho)}{6(1+\rho)^2}, \quad \alpha=\pi-\theta,$$

and $H_q{}^{(1)}$ is the (cylindrical) hankel function. Hence

$$\begin{Bmatrix}|M_x|^2\\|M_y|^2\\|M_z|^2\end{Bmatrix}=\frac{4\pi^2}{3}\left(\frac{a}{Z}\right)^8\frac{n_1{}^{18}e^{4\pi n_1}}{(1-\rho^2)(1+\rho)^8}\alpha^4 \times\begin{Bmatrix}e^{5\pi i/6}H_{2/3}{}^{(1)}(is)\\ \cos\phi\ e^{2\pi i/3}H_{1/3}{}^{(1)}(is)\\ \sin\phi\ e^{2\pi i/3}H_{1/3}{}^{(1)}(is)\end{Bmatrix}^2, \qquad (24a)$$

where $a=\hbar^2/me^2$. Thus

$$\frac{d\sigma}{d\Omega}=\frac{A_1A_2}{144}\frac{k_2k_1{}^3}{(k_1-k_2)^4}\left(\frac{n_1}{Z}\right)^2\frac{D_{AB}{}^2}{2j_A+1}\exp[-2\pi(n_2-n_1)]\alpha^4 \cdot\{|e^{5\pi i/6}H_{2/3}{}^{(1)}(is)|^2+|e^{2\pi i/3}H_{1/3}{}^{(1)}(is)|^2\}. \qquad (24)$$

It is readily seen that the cross section is very small unless n_1 is quite large. For large n_1, the projectiles are not likely to penetrate into the nucleus, and it seems unnecessary, therefore, to make any correction for the finite size of the nucleus.

Integrating over all angles of scatter, we obtain (S-560)

$$\sigma=\frac{8\pi^2}{3\sqrt{3}}\left(\frac{n_1}{Z}\right)^2\frac{D_{AB}{}^2}{2j_A+1}\exp[-2\pi(n_2-n_1)]. \qquad (25)$$

(*c*) *Cross Sections for $n_2\cong n_1$, with n_1 Arbitrary.*—A case of considerable physical interest is that in which the energy transferred to the nucleus is small. For this case we write

$$k_1-k_2=\epsilon$$

so that,

$$n_2-n_1=n_1\epsilon/k_2.$$

The hypergeometric functions occurring in Eq. (20) can then be evaluated by use of the theorems:

$$\frac{d}{dx}F(a,b,c,x)=\frac{ab}{c}F(a+1,b+1,c+1,x),$$

$$(1-x)^{a+b-c}F(a,b,c,x)=F(c-a,c-b,c,x),$$

$$\lim_{\epsilon\rightarrow 0}\epsilon^{-1}F(1+in_1,1+in_2,1,x)=0,$$

$$\lim_{\epsilon\rightarrow 0}\epsilon^{-2}|F(1+in_1,1+in_2,1,x)+in_1F(1+in_1,1+in_2,2,x)|^2=\frac{1}{K^2|\Gamma(1+in_1)|^2}.$$

Consequently, for this case

$$|M_x|^2\cong\frac{|c|^2n_1{}^2(k_1-k_2)^4}{|\Gamma(1+in_1)|^4}\cdot\frac{(1-\cos\theta)^2}{K^4},$$
$$|M_y|^2=|M_z|^2\cong\frac{|c|^2n_1{}^2(k_1-k_2)^4}{|\Gamma(1+in_1)|^4}\cdot\frac{\sin^2\theta}{K^4}. \qquad (26a)$$

On substitution of these results into Eq. (20) one obtains

$$\frac{d\sigma}{d\Omega}\cong\frac{4}{3}\exp[-2\pi(n_2-n_1)]n_1{}^2\frac{k_1k_2}{K^2}\frac{D_{AB}{}^2}{2j_A+1}j_0{}^2(Kr_0). \qquad (26)$$

The factor $j_0{}^2(Kr_0)$ has been included in (26) to take account of the finite size of the nucleus. Since the case $n_2\cong n_1$ corresponds to small effective interaction between projectile and nucleus we may expect this factor, which was obtained by the Born approximation [compare Eq. (11)] to be approximately correct here.

Integration over angles yields for the total cross section:

$$\sigma=\frac{8\pi}{3}\left(\frac{n_1}{Z}\right)^2\exp[-2\pi(n_2-n_1)]\frac{D_{AB}{}^2}{2j_A+1}B_1. \qquad (27)$$

(2) *Electric Quadrupole*

If the motion of the projectile is described by the coulomb wave functions U_1 and U_2 given in the preceding section, the matrix element for electric quadrupole transitions may be evaluated rather easily if the finite size of the nucleus is neglected in the integration over the projectile's coordinates. The effects of the finite size of the nucleus may be included, in limiting cases, by the procedures used in the preceding Section (II-B-1).

If the coulomb wave functions are expanded in terms of momentum functions[12]

$$U_1=\int \psi_k \exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k},\quad U_2^*=\int \psi_{k'}\exp(-i\mathbf{k}'\cdot\mathbf{r})d\mathbf{k}', \tag{28}$$

the electric matrix element, M, for 2^l-pole transitions can be written as

$$M_l=\langle U_2|V_l|U_1\rangle=\int\int \psi_{k'}\psi_k M_{l(\mathrm{Born})}(\mathbf{k},\mathbf{k}')d\mathbf{k}d\mathbf{k}', \tag{29}$$

where $M_{l(\mathrm{Born})}(\mathbf{k},\mathbf{k}')$ is the matrix element in the Born approximation. In the quadrupole ($l=2$) case, $M_{l(\mathrm{Born})}$ is independent of $\mathbf{k}$ and $\mathbf{k}'$. Thus,

$$M_2=M_{2(\mathrm{Born})}\cdot\int\int \psi_{k'}\psi_k d\mathbf{k}d\mathbf{k}'$$

$$=M_{2(\mathrm{Born})}\cdot U_2^*(0)U_1(0),$$

where $U_1(0)$ and $U_2(0)$ are the coulomb wave functions evaluated at the origin. Consequently, the use of coulomb field wave functions modifies the differential and total cross sections only by the factor

$$[2\pi n_1/(e^{2\pi n_1}-1)]\cdot[2\pi n_2/(e^{2\pi n_2}-1)]. \tag{30}$$

(C) Elastic Scattering

If both elastic and inelastic scattering cross sections are measured at an angle for which the elastic scattering is relatively independent of the properties of the target nucleus, the elastic scattering cross section can be used as a convenient normalization. The well-known formula for the elastic scattering cross section[13] can be simplified by making use of the semiclassical approximation that those projectiles for which the angular momentum l is greater than $l_c=kr_0(1-Ze^2/Er_0)^{\frac{1}{2}}$ miss the nucleus; these projectiles suffer no phase shift. The coulomb phase shift, η_l, can be evaluated by use of the relation

$$\eta_l=\arg\Gamma(l+1+in)=\arg\prod_{s=0}^{l-1}(l-s+in)\Gamma(1+in)$$

$$=\eta_0+\beta_l,$$

where

$$\beta_l=\sum_{s=1}^{l}\tan^{-1}n/s.$$

A case of particular simplicity is that in which projectiles which penetrate into the nucleus ($l\leqslant l_c$) are not likely to be elastically scattered. These projectiles may be omitted from the elastic scattering by setting their phase shifts equal to $i\infty$. In this case the elastic scattering cross section is given by

$$\frac{d\sigma}{d\Omega}=\left|\frac{n}{2k}\csc^2\tfrac{1}{2}\theta\exp[-2in\ln(\sin\tfrac{1}{2}\theta)]+\frac{1}{2ik}\left\{1+\sum_{l=1}^{l_c}(2l+1)e^{2i\beta_l}P_l(\cos\theta)\right\}\right|^2. \tag{31}$$

This expression is independent of all properties of the nucleus except its radius and charge.

III. COMPARISON BETWEEN PARTICLE CROSS SECTIONS AND PHOTO CROSS SECTIONS: NUMBER OF ELECTRIC 2^l-POLE QUANTA

In Sec. II [Eq. (6)] we have seen that if the nuclear moments are randomly oriented, the cross section for the electric 2^l-pole excitation of a nucleus by the field of a nonrelativistic charged projectile may be factored into a product of the number of virtual electric 2^l-pole quanta in the field of the projectile and the photoexcitation cross section for the transition. We shall now give an explicit formulation of this "method of virtual quanta" for the coulomb field excitation of nuclei. We shall give explicitly the numbers of virtual electric dipole, quadrupole, and octupole quanta in the field of the charged projectile; and in the electric dipole case we establish the relation between the number of virtual and the number of real quanta (bremsstrahlung) associated with the deflection of the projectile. In the present discussion we shall give the results for a "point" target nucleus; the effects of the finite size of the nucleus may be taken into account, at least approximately, by the method discussed in Sec. II.

In order to obtain the number of virtual quanta, we must obtain an expression for the electric 2^l-pole photoexcitation cross section. The interaction between the radiation electric field $\boldsymbol{\varepsilon}$ and the electric 2^l-pole moment of the nucleus is taken as

$$U_l=\frac{1}{l!}e(\mathbf{E}\cdot\mathbf{R})(\mathbf{k}_\nu\cdot\mathbf{R})^{l-1}, \tag{32}$$

where $\mathbf{R}$ stands for the coordinates of the protons in the nucleus, $\mathbf{k}_\nu$ is the wave number of the incident radiation ($k_\nu=|\mathbf{k}_\nu|=2\pi/\lambda=E/\hbar c$; $E=$ transition energy); and $\boldsymbol{\pi}$ is a unit vector in the direction of polarization of the incident radiation. Thus

$$\sigma_l^{(\gamma)}=4\pi^2\frac{e^2}{\hbar c}\frac{E}{2j_A+1}\sum_{a,b}\left|\left(b\left|\frac{(\boldsymbol{\pi}\cdot\mathbf{R})(\mathbf{k}_\nu\cdot\mathbf{R})^{l-1}}{l!}\right|a\right)\right|^2. \tag{33}$$

(1) *Electric Dipole Excitation*

Setting $l=1$ in Eq. (33) yields for the electric dipole photo-excitation cross section:

$$\sigma_1^{(\gamma)}=\frac{4\pi^2e^2k_\nu}{2j_A+1}\sum_{a,b}|(b|\boldsymbol{\pi}\cdot\mathbf{R}|a)|^2=\frac{4\pi^2}{3}e^2k_\nu\frac{D_{AB}^2}{2j_A+1}. \tag{34}$$

[12] This procedure was suggested to one of us by R. Serber in connection with another problem.
[13] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), pp. 116 ff.

Thus from Eqs. (21) and (1) the cross section for electric dipole excitation of the nucleus by the field of a positively charged projectile is given by

$$\sigma_1 = N_1 \sigma_1^{(\gamma)},$$

where N_1, the number of virtual electric dipole quanta in the field of the charged projectile, is given by

$$N_1 dE = \frac{dE}{E} \frac{A_1 A_2}{2\pi Z^2} \frac{1}{e^2/\hbar c} \frac{k_2}{k_1} e^{-2\pi n_2} x_0 \frac{d}{dx_0} |F(x_0)|^2, \quad (35)$$

where E is the transition energy.

It is interesting to note that the cross section for the production of real quanta (bremsstrahlung), when the charged particle makes the transition due to deflection in a pure coulomb field, is expressible in terms of N_1 in a simple way. In the electric dipole approximation, the cross section for the production of bremsstrahlung in dE is (S-527 and 564)

$$\sigma_x dE = \frac{dE}{E} z^2 \frac{e^2}{\hbar c} \left(\frac{\hbar}{mc}\right)^2 \frac{n_1}{n_2} A_1 A_2 e^{-2\pi n_2} x_0 \frac{d}{dx_0} |F(x_0)|^2. \quad (36)$$

Comparison with (35) shows that the bremsstrahlung cross section may be written:

$$\sigma_x dE = 2\pi z^2 Z^2 (e^2/mc^2)^2 N_1 dE. \quad (37)$$

This simple result is of the form to be expected on the basis of semi-classical arguments.

In certain limiting cases the rather formidable expression (35) for N_1 may be replaced with relatively simple approximate expressions. Thus, from Eqs. (23), (27), and (25) we have

$$N_1 dE = \frac{dE}{E} \frac{2z^2}{\pi} \frac{e^2}{\hbar c} \left(\frac{c}{v_1}\right)^2 A_1 A_2 \times e^{-2\pi n_2} \ln\left(\frac{k_1 + k_2}{k_1 - k_2}\right), \quad \begin{matrix} n_1 \to 0 \\ n_2 \text{ arbitrary} \end{matrix} \quad (38)$$

$$N_1 dE = \frac{dE}{E} \frac{2z^2}{\pi} \frac{e^2}{\hbar c} \left(\frac{c}{v_1}\right)^2 \frac{A_2}{A_1} \times e^{-2\pi(n_2 - n_1)} \ln\left(\frac{k_1 + k_2}{k_1 - k_2}\right), \quad n_1 \cong n_2 \quad (39)$$

$$N_1 dE = \frac{dE}{E} \frac{2z^2}{\sqrt{3}} \frac{e^2}{\hbar c} \left(\frac{c}{v_1}\right)^2 e^{-2\pi(n_2 - n_1)}, \quad \begin{matrix} n_1 \gg 1 \\ n_2 \neq n_1 \end{matrix}. \quad (40)$$

The Born approximation result can be obtained from the first or second of these limiting cases by setting $A_1 = A_2 = 1 = e^{-2\pi n_2}$. The Born result is essentially the same as that obtained by the semi-classical method.

(2) *Electric Quadrupole Transitions*

Setting $l = 2$ in Eq. (33) gives for the electric quadrupole photo-excitation cross section:

$$\sigma_2^{(\gamma)} = \frac{\pi^2 e^2 k_\nu}{(2j_A + 1)} \sum_{a,b} |(b|(\boldsymbol{\pi} \cdot \mathbf{R})(\mathbf{k}_\nu \cdot \mathbf{R})|a)|^2$$

$$= \frac{\pi^2 e^2 k_\nu}{(2j_A + 1)} \sum_{a,b} |\sum_{i,j} (b|Q_{ij}|a) \pi_i k_{\nu j}|^2.$$

From a theorem which is given in Part 1 of the appendix to this paper it follows that

$$\sum_{a,b} |\sum_{i,j} (b|Q_{ij}|a) \pi_i k_{\nu j}|^2 = k_\nu^2 Q_{AB}'^2/10 \quad (41)$$

and, therefore,

$$\sigma_2^{(\gamma)} = (\pi^2 e^2/10) Q_{AB}'^2/(2j_A + 1) k_\nu^3. \quad (42)$$

Writing $\sigma_2 = N_2 \sigma_2^{(\gamma)}$ and using Eq. (14), one has for the number of electric quadrupole quanta

$$N_2 dE = (dE/E)(16z^2 e^2/3\pi\hbar c)(c/v_1)^2 k_1 k_2/k_\nu^2. \quad (43)$$

(3) *Electric Octupole Transitions*

Setting $l = 3$ in Eq. (33) yields for the electric octupole photo-excitation cross section:

$$\sigma_3^{(\gamma)} = \frac{\pi^2}{9} \frac{e^2}{\hbar c} \frac{E}{2j_A + 1} \sum_{a,b} |(b|(\boldsymbol{\pi} \cdot \mathbf{R})(\mathbf{k}_\nu \cdot \mathbf{R})^2|a)|^2$$

$$= \frac{\pi^2}{9} \frac{e^2}{\hbar c} \frac{E}{2j_A + 1} \sum_{a,b} |\sum_{ijk} (b|S_{ijk}|a) \pi_i k_{\nu i} k_{\nu k}|^2.$$

From a theorem given in Part 2 of the appendix it follows that

$$\sum_{a,b} |\sum_{ijk} (b|S_{ijk}|a) \pi_i k_{\nu j} k_{\nu k}|^2 = \frac{4k_\nu^4}{105} [S_{AB}'^2 + (7/20)(R^2 D)_{AB}^2], \quad (44)$$

where

$$(R^2 D)_{AB}^2 = \sum_i |(b|R^2 x_i|a)|^2.$$

Consequently

$$\sigma_3^{(\gamma)} = \frac{4\pi^2}{945} e^2 k_\nu^5 \frac{S_{AB}'^2 + (7/20)(R^2 D)_{AB}^2}{2j_A + 1}. \quad (45)$$

The $(R^2 D)_{AB}^2$ term results from an interaction between the radiation and the electric dipole moment of the nucleus, and should be added as a second order term to the electric dipole cross section. Since this term does not correspond to an octupole transition, we shall drop it from our expression for $\sigma_3^{(\gamma)}$. Thus we take

$$\sigma_3^{(\gamma)} = (4\pi^2 e^2/945) S_{AB}'^2 k_\nu^5/(2j_A + 1). \quad (46)$$

Writing $\sigma = N_3\sigma_3^{(\gamma)}$ and using Eq. (16) for σ_3 we obtain, for the distribution of octupole quanta in the electric field of the charged projectile,

$$N_3dE = \frac{dE}{E}\frac{6z^2}{\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{c}{v_1}\right)^2\frac{k_1k_2(k_1^2+k_2^2)}{k_\nu^4}. \quad (47)$$

IV. ELECTRIC BREAK-UP OF THE DEUTERON

When a beam of deuterons strikes a target the deuterons may be disintegrated and emergent beams of neutrons and protons thus obtained. In a number of experiments the angular and energy distributions of the neutrons and protons have been studied. Helmholz, McMillan, and Sewell[14] have studied the angular and energy distributions of the neutrons obtained by bombarding targets with 190-Mev deuterons. Angular and energy distributions of the neutrons obtained when various targets are bombarded with 14- to 18-Mev deuterons have been studied by Falk, Creutz, and Seitz,[15] Roberts and Abelson,[16] and Ammiraju.[17] Angular and energy distributions of the protons obtained by bombarding thin targets with 14-kev deuterons have been studied by the M.I.T. group.[18] The results of all these experiments seem to indicate that the deuteron break-up which yields the protons or neutrons is due to stripping or to electric break-up. In the stripping process one of the nucleons of the deuteron strikes the target nucleus and is stripped off the deuteron; the other nucleon misses the nucleus and continues its flight. This process has been discussed for high energy deuterons by Serber,[19] and for low energy deuterons by Falk and Wolfenstein[20] and by French.[21] In the electric break-up process, the deuteron misses the nucleus but is disintegrated by its electric field. The possibility of electric break-up of the deuteron was first discussed by Oppenheimer.[22] A detailed theory of the process for high energy deuterons has been given by Dancoff,[9] using the Born approximation. The results of Dancoff and Serber show that for high energy deuterons nuclear stripping is considerably more probable than is electric break-up. However, at somewhat lower deuteron energies the electric process may play a significant role. Consequently, in this section we shall apply the results of Sec. II to the electric break-up of the deuteron.

In our discussion we include the effects of the coulomb field on the motion of the center of gravity of the deuteron, but we neglect the effects of the coulomb field on the outgoing proton. This procedure is probably adequate for a description of the behavior of the outgoing neutrons, but does not give an adequate description of the outgoing protons unless the deuteron energy is very high. We confine our attention to the outgoing neutrons.[23]

We describe the motion of the c.g. of the deuteron by means of the wave functions χ_1 and χ_2 of Eq. (17). The initial and final states of the deuteron's relative motion we shall describe by the "zero-range" wave functions:[24]

$$\begin{aligned} D_1 &= (\alpha/2\pi)^{\frac{1}{2}}e^{-\alpha\rho}/\rho, \quad \alpha=(M\epsilon_0/\hbar^2)^{\frac{1}{2}},\\ D_2 &= N_2\exp(i\mathbf{k}_\rho\cdot\boldsymbol{\rho}), \quad N_f=[Mk_\rho/(2\pi)^3 2\hbar^2]^{\frac{1}{2}}, \end{aligned} \quad (48)$$

where $\epsilon_0 \cong 2.2$ Mev is the deuteron's binding energy, M is the nucleon mass, $\boldsymbol{\rho}$ is the relative coordinate, and $\mathbf{k}_\rho$ is the wave number of the relative motion. In the electric dipole approximation, the cross section for deuteron break-up with disintegration energy in $d\epsilon$, the center of mass being scattered into the solid angle $d\Omega_2$, and the neutron ejected into $d\Omega_1$, is

$$\sigma d\epsilon d\Omega_1 d\Omega_2 = \frac{2\pi}{\hbar}\left|\left(D_f\chi_f\left|\frac{Ze^2(\mathbf{r}\cdot\boldsymbol{\rho})}{2r^3}\right|D_i\chi_i\right)\right|^2 d\epsilon d\Omega_1 d\Omega_2$$

$$= \frac{\pi N_f^2|D|^2}{2\hbar k\rho^2}\left|\left(\chi_f\left|\frac{Ze^2(\mathbf{k}_\rho\cdot\mathbf{r})}{r^3}\right|\chi_i\right)\right|^2 d\epsilon d\Omega_1 d\Omega_2$$

with

$$|D|^2 = \frac{1}{k_\rho^2}\left|\int \exp(-i\mathbf{k}_\rho\cdot\boldsymbol{\rho})(\mathbf{k}_\rho\cdot\boldsymbol{\rho})D_1 d\rho\right|^2 = \frac{32\pi\alpha k_\rho^2}{(k_\rho^2+\alpha^2)^4}.$$

Applying the equation of motion discussed in Sec. II-B we find

$$\sigma d\epsilon d\Omega_1 d\Omega_2$$

$$= \frac{2\pi}{\hbar}\frac{N_f^2|D|^2M^2}{k_\rho^2}\left(\frac{E_2-E_1}{\hbar}\right)^4|(\chi_2|\mathbf{k}_\rho\cdot\mathbf{r}|\chi_1)|^2 d\epsilon d\Omega_1 d\Omega_2$$

$$= (2\pi/\hbar)N_f^2N_E^2N_{fl}^2|D|^2M^2(E_2-E_1/\hbar)^4$$

$$\times\{\cos^2\theta_1\cdot|M_x|^2+\sin^2\theta_1\cdot\tfrac{1}{2}(M_y^2+M_z^2)\}d\epsilon d\Omega_1 d\Omega_2,$$

where θ_1 is the angle between the wave number vectors $\mathbf{k}_\rho$ and $\mathbf{k}_1$, with E_2 and E_1 as the final and initial energies of the motion of the center of mass, and with N_{fl}, N_E, $M_{x,y,z}$ as given by Eqs. (17) and (19). The x-direction has been chosen as the direction of the incident deuteron beam. When the intergrations over the

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[20] C. E. Falk, thesis, reference 15.

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[23] In an independent treatment of the electric break-up of the deuteron M. L. Goldberger (private communication) has taken into account in the influence of the coulomb field on the outgoing protons. M. L. Goldberger, Phys. Rev. **79**, 221 (1950). In all discussions the effects of polarization of the deuteron on the wave function of the incident deuteron are neglected.

[24] The finite range of the neutron-proton interaction can be included in an approximate way by using the factor $(1+\alpha b_t)$, where b_t is the range of the n–p interaction in the triplet ground state.

angles of scattering of the center of mass of the deuteron have been carried out we find that

$$\sigma d\epsilon d\Omega_1=(2\pi/\hbar)N_f^2N_E^2N_{fl}^2|D|^2M^2(E_2-E_1/\hbar)^2 \times\left\{\cos^2\theta_1\cdot\int|M_x|^2d\Omega_2+\sin^2\theta_1 \cdot\tfrac{1}{2}\int(M_y^2+M_z^2)d\Omega_2\right\}d\epsilon d\Omega_1. \quad (49)$$

This relatively complicated result can be simplified in certain limiting cases. Thus, if the deuteron moves in a strong coulomb field ($n_1\gg 1$, $n_2\neq n_1$) we can use the approximate expressions given by Eq. (24a) for $M_{x,y,z}$. Then

$$\sigma d\epsilon d\Omega_1=\frac{8n_1^2\hbar^2(\epsilon_0\epsilon^3)^{\frac{1}{2}}}{3\sqrt{3}M(\epsilon+\epsilon_0)^4}\cdot\exp[-2\pi(n_2-n_1)] \cdot\{\cos^2\theta_1+\tfrac{1}{4}\sin^2\theta_1\}d\epsilon d\Omega_1. \quad (50)$$

A limiting case of importance from the experimental viewpoint is that in which the deuteron's initial energy exceeds the coulomb barrier energy by considerably more than the binding energy of the deuteron. We shall now investigate this case in detail. Inspection of the form of the factor $|D|^2$ shows that the cross section becomes small if the disintegration energy, ϵ, is large. Consequently the center of mass of the deuteron is not likely to lose more than a rather small fraction of its energy in the disintegration. For this reason we can substitute for M_x, M_y, and M_z the approximate values suitable for small energy transfer. From Eqs. (19) and (26a) we see that if n_2-n_1 is small, *or* if n_1 and n_2 are both small,

$$\begin{aligned}|M_x|^2&\cong|C|^2(k_1-k_2)^4(\sinh n_1\pi/n_1\pi)^2\times(n_2-n_1\cos\theta_2)^2/K^4,\\ |M_y|^2+|M_z|^2&\cong|C|^2(k_1-k_2)^4\times(\sinh n_1\pi/n_1\pi)^2\sin^2\theta_2/K^4.\end{aligned} \quad (51)$$

Substitution of these expressions into Eq. (49) yields:

$$\sigma d\epsilon d\Omega_1=\frac{2}{\pi}n_1^2\exp[-2\pi(n_2-n_1)]\frac{\hbar^2}{M}\frac{\epsilon_0^{\frac{1}{2}}\epsilon^{\frac{3}{2}}}{(\epsilon+\epsilon_0)^4} \times\left\{\frac{(k_1+k_2)^2+K_m^2}{4k_1^2}\left(1-\frac{1}{\Gamma^2}\right)P_2(\cos\theta_1) +\sin^2\theta_1\ln\Gamma\right\}d\epsilon d\Omega, \quad (52)$$

where $\Gamma=K_m/(k_1-k_2)$ and $\hbar K_m$ is the maximum recoil suffered by the center of gravity of the deuteron. If the finite size of the nucleus is neglected, $K_m=k_1+k_2$; the effects of the finite size of the nucleus can be taken into account by limiting the maximum recoil momentum to $\hbar K_m=1/r_0$, where r_0 is the sum of the "radii" of the deuteron and target nucleus. Apart from the "coulomb factor," $\exp[-2\pi(n_2-n_1)]$, the cross section given by Eq. (52) is equivalent to that obtained by Dancoff[9] by using the Born approximation.

To facilitate comparison with experiment, we shall carry out a transformation of variables from (ϵ, θ_1) to (E_n, θ), where E_n is the neutron's energy in the system of the center of mass of the target nucleus and deuteron, and θ is the angle between the direction of the outgoing neutron and the direction of the incident deuteron beam. We neglect the finite size of the target nucleus, taking $K_m=k_1+k_2$. This omission will tend to overemphasize the role of the neutrons at the upper and of the energy spectrum and will yield an upper limit for the total cross section. Since with $n_2\cong n_1$ the probability of large lateral deflection of the deuteron's center of gravity is small (see Eq. (26)), we shall assume that the deuteron's center of gravity moves in a straight line. With these approximations one can take

$$\epsilon/E_D'\cong f^2+\theta^2, \quad (\epsilon/E_D')\sin^2\theta_1\cong\theta^2$$

so that

$$(\epsilon/E_D')P_2(\cos\theta_1)\cong f^2-\tfrac{1}{2}\theta^2.$$

The jacobian of the transformation is

$$J\cong 2\cos\theta/[f^2+\theta^2]^{\frac{1}{2}},$$

where $E_D'=E_D-E_B$; E_D = initial deuteron energy; E_B = coulomb barrier energy of the deuteron at the time of break-up and

$$f=[E_N-\tfrac{1}{2}(E_D'-\epsilon_0)]/E_D'. \quad (53)$$

Consequently the cross section for electric break-up, with the neutron being ejected into the solid angle $d\Omega$ with energy in dE_N

$$\sigma d\Omega dE_N\cong\frac{8Z^2e^4C_1}{\pi E_DE_D'^2}\left(\frac{\epsilon_0}{E_D'}\right)^{\frac{1}{2}}\frac{[f^2+\frac{1}{2}\theta^2(\ln\lambda-1)]}{[f^2+\theta^2+(\epsilon_0/E_D')]^4}d\Omega dE_N \quad (54)$$

with

$$\begin{aligned}C_1&=\exp\{-(\pi n_1E_D'/E_D)[f^2+\theta^2+(\epsilon_0/E_D')]\},\\ &=4/[f^2+\theta^2+(\epsilon_0/E_D')].\end{aligned}$$

The angular and energy distributions can be obtained from Eq. (55) by integration. In general, these integrations cannot be carried out analytically, and we shall perform them explicitly only for the case that n_1 is sufficiently small that $\pi n_1\epsilon_0/E_D\ll 1$, so that $C_1\cong 1$. Since the logarithm is a slowly varying function of the angle, the integrals of the type

$$I=\int f(x)\ln\phi(x)dx,$$

in which $f(x)$ is a rather rapidly varying function of x which is appreciably different from zero only over a limited range of x, can be evaluated by the following approximation

$$I\cong\ln\phi_{\mathrm{Av}}\cdot\int f(x)dx,$$

where

$$\phi_{Av}=\int \chi(x)f(x)dx \Big/ \int f(x)dx.$$

Furthermore, since the cross section diminishes rapidly to zero as θ becomes large, or as $[E_N-\frac{1}{2}(E_D'-\epsilon_0)]$ becomes large, the limits of the integral can be taken to be zero and infinity. Carrying out the integrations for $C_1\cong 1$ we find for the angular and energy distributions of the outgoing neutrons

$$\sigma d\Omega=\frac{Z^2e^4}{2E_DE_D'}\left(\frac{\epsilon_0}{E_D'}\right)^{\frac{1}{2}} \times\frac{[(\epsilon_0/E_D')+\frac{1}{2}\theta^2(5\ln\xi_1-3)]}{[(\epsilon_0/E_D')+\theta^2]^{7/2}}\cdot\cos\theta\cdot d\Omega, \quad (55)$$

$$\sigma dE_N=\frac{2Z^2e^4}{3E_DE_D'^2}\left(\frac{\epsilon_0}{E_D'}\right)^{\frac{1}{2}} \times\frac{[(3+\ln\xi_2)f^2+(\ln\xi_2-1)\epsilon_0/E_D']}{[f^2+(\epsilon_0/E_D')]^3}dE_N \quad (56)$$

with

$$\xi_1=10/3(\theta^2+\epsilon_0/E_D'),\quad \xi_2=4/3(f^2+\epsilon_0/E_D').$$

The total cross section can be obtained by an integration of (56) over all neutron energies. We thus find

$$\sigma=(\pi Z^2e^4/3\epsilon_0E_D)\ln(E_D'/3^{\frac{1}{2}}\epsilon_0). \quad (57)$$

The total cross section can also be obtained by a direct integration of Eq. (52). In case the exponential member is taken to be unity this yields

$$\sigma=\frac{16Z^2e^4\epsilon_0^{\frac{1}{2}}}{3E_D}\cdot\int_0^{\epsilon_{\max}}\frac{\epsilon^{\frac{3}{2}}}{(\epsilon+\epsilon_0)^4}\ln\Gamma\cdot d\epsilon \cong\frac{16Z^2e^4\epsilon_0^{\frac{1}{2}}}{3E_D}\ln\Gamma_{Av}\cdot\int_0^{\infty}\epsilon^{\frac{3}{2}}/(\epsilon+\epsilon_0)^4d\epsilon$$

or

$$\sigma=(\pi Z^2e^4/3\epsilon_0E_D)\ln\Gamma_{Av}, \quad (58)$$

where

$$\Gamma=(k_1+k_2)/(k_1-k_2)\cong 4E_D/(\epsilon+\epsilon_0).$$

We then have from our definition of average values

$$\Gamma_{Av}=\frac{\int_0^\infty \Gamma(\epsilon)\epsilon^{\frac{3}{2}}/(\epsilon_0+\epsilon)^4d\epsilon}{\int_0^\infty \epsilon^{\frac{3}{2}}/(\epsilon_0+\epsilon)^4d\epsilon}=2E_D/3\epsilon_0.$$

The very slight difference between the cross section formulas (57) and (58) arises from the approximations used in the transformation to the laboratory system of coordinates.

V. DISCUSSION OF THE RESULTS

We shall now apply the theory developed in the preceding sections to a discussion of two processes in which electric transitions may play a significant role.

(A) Inelastic Scattering of Deuterons

We have pointed out that the inelastic scattering of deuterons should take place primarily through electric transitions. Experimental data on the deuterons scattered inelastically on aluminum and magnesium targets have been obtained by Greenlees, Kempton, and Rhoderick,[2] and by Holt and Young.[3] The angular distribution of the inelastically scattered deuterons offers the best criterion for the determination of the type of interaction responsible for the transition. Transitions occuring as a result of compound nucleus formation should yield an almost isotropic distribution of the inelastically scattered deuterons. Electric dipole interaction, on the other hand, leads to the differential cross section [see Eq. (26)]:

$$\frac{d\sigma}{d\Omega}=\frac{4}{3}\left(\frac{n_1}{Z}\right)^2\exp[-2\pi(n_2-n_1)]\frac{D_{AB}^2}{2j_A+1}\frac{k_1k_2}{K^2}j_0^2(Kr_0). \quad (59)$$

In making quantitative use of Eq. (59) for deuterons it must be remembered that to the approximations used in the derivation of this equation we must add the following: (1) the finite extension of the projectile's charge distribution is neglected; the deuteron's charge is assumed to be concentrated at the deuteron's center of mass, (2) the polarization (stretch) of the deuteron in the field of the target nucleus is neglected.

Because of the approximations used, we can expect Eq. (59) to have only semi-quantitative significance at large angles. However, the angular distributions predicted by Eq. (59) on the one hand, and by the theory of compound nucleus formation on the other, differ so greatly that despite the omission of the second-order effects involved in our approximations we should be able to distinguish readily between the two types of angular distribution.

The angular distribution obtained by Holt and Young for 7.5-Mev deuterons cattered inelastically on magnesium is reproduced for convenience in Fig. 1(a). The excitation energy is 1.36 Mev. The angular distribution for elastically scattered deuterons (7.5-Mev deuterons on the same magnesium target) is also given in this figure.

The theoretical angular distribution given by Eq. (59) depends somewhat upon the choice of the "cut-off" radius, r_0. We may write

$$r_0=1.5(24)^{\frac{1}{3}}\times 10^{-13}+R_D \text{ cm}$$

where R_D is the mean radius of the deuteron. The choice of R_D is certainly not unambiguous; and we have chosen it so that the theoretical angular distribution has its second minimum at $\theta=60°$ as required by the experimental results. This requires that $Kr_0=2\pi$ at $\theta=60°$; thus, $r_0=7.58\times 10^{-13}$ cm, and $R_D=3.25\times 10^{-13}$ cm.

The value of the nuclear dipole moment,

$$D_{AB}[=(D_{AB}^2)^{\frac{1}{2}}],$$

for the transition which causes the inelastic scattering may be obtained by comparison of the experimental and theoretical ratios of inelastic and elastic scattering. Since compound nucleus formation is not likely to play a significant role in the elastic scattering of deuterons, the appropriate expression for the elastic scattering cross section is given by Eq. (31). Since the elastic scattering at a given angle depends rather critically on the choice of the ambiguous quantity l_c, we have integrated the differential cross sections over all angles from 75° to 135° and then equated the theoretical and experimental ratios of inelastic to elastic cross sections. One obtains $D_{AB}=7.5\times10^{-13}$ cm; this is about 1.74 R_N, where R_N is the nuclear radius ($R_N=1.5\times(24)^{\frac{1}{3}}\times10^{-13}=4.33\times10^{-13}$ cm) of ${}_{12}Mg^{24}$. In view of the approximations involved, this result is not unreasonable. The theoretical angular distribution ($d\sigma/d\Omega$) for the inelastic scattering of 7.5-Mev deuterons on Mg^{24} is given in Fig. 1(b). The total cross section for inelastic scattering is given by

$$\sigma=0.041D_{AB}^2=2.1\times10^{-26}\ \text{cm}^2.$$

Comparison of the theoretical angular distribution with the experimental one seems to indicate rather clearly that the inelastic scattering is due to electric interaction.†

The experimental data obtained by Rhoderick[1] on the total cross sections for protons scattered inelastically on aluminum and magnesium targets seem to indicate that the energy transfer takes place primarily by compound nucleus formation. Unfortunately, no angular distributions are given in the published results of these experiments.

(B) Electric Break-Up of the Deuteron in Flight

First of all we shall give a comparison of the cross sections for the competitive processes of stripping and electric break-up. In both processes, we shall restrict ourselves to the case in which the escaping *neutron* is observed. In the computation of the electric break-up cross section, we shall neglect the finite size of the target nucleus;[25] this procedure yields an upper limit for the cross section. In the point nucleus approximation, the cross section for electric break-up may be obtained by setting $K_m=k_1+k_2$ in Eq. (52). We shall assume that in the majority of the disintegrations, the energy lost by the center of gravity of the deuteron is small compared

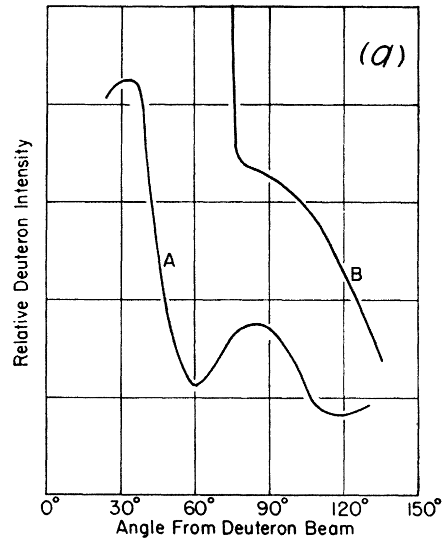

FIG. 1(a). Empirical results obtained by Holt and Young for the angular variation of the intensity of deuterons scattered inelastically (curve A) and elastically (curve B) on magnesium.

to the initial energy of the deuteron; consequently, we may make the approximations: $k_1-k_2\cong2E_1/(\epsilon+\epsilon_0)$, $k_1+k_2\cong2k_1$. Setting $x=\epsilon/\epsilon_0$, the total cross section for electric break-up is given by

$$\sigma=\frac{16Z^2e^4}{3\epsilon_0E_1}e^{-\pi n_1\epsilon_0/E_1}\int_0^{x_m}\frac{x^{\frac{3}{2}}}{(1+x)^4}\times e^{-\pi n_1\epsilon_0x/E_1}\ln\left[\frac{4E_1}{E_0(1+x)}\right]dx.$$

The upper limit, x_m, is determined by energy conservation; for deuterons having energy of 15 Mev or more, no appreciable error is incurred by taking $x_m=\infty$. The electric break-up cross section has been evaluated by numerical integration for 15-Mev deuterons for four values of Z; the results are given in Table I. Inclusion

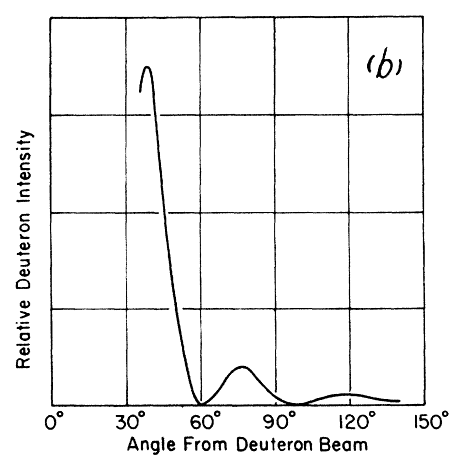

FIG. 1(b). Theoretical angular variation of intensity of deuterons scattered inelastically from magnesium.

† If the electric dipole moment, D_{AB}, is very small, the scattering may be due to electric quadrupole interaction.

[25] For 200-Mev deuterons, Dancoff has taken the finite size of the nucleus into account by limiting the recoil momentum of the deuteron's center of gravity to values less than or equal to $\hbar/r_0$, where r_0 is the sum of the radii of the deuteron and target nucleus. For high energies this cutoff in momentum space is approximately equivalent to the more exact procedure of modifying (in configuration space) the interaction between the target nucleus and the deuteron to take account of the nuclear forces. For the much lower energies (~15 Mev) in which we shall be interested, the momentum space cutoff is a very poor approximation, and the more exact procedure must be used. Modification of the interaction to include the effects of nuclear forces leads to very complicated integrations and cumbersome formulas; we believe that that the labor involved in these computations is not warranted at the present time.

TABLE I. Comparison of stripping cross section with cross section for electric break-up on a point nucleus. (Deuteron energy =15 Mev.)

Target nucleus	Cross section (barns) Stripping	Electric
Be	0.09	0.016
Mg	0.12	0.094
Cu	0.14	0.22
Au	0.08	0.17

of the effects of the finite size of the target nucleus would decrease these cross sections, the amount of decrease being greater for the heavy nuclei than for the light nuclei.

The stripping cross section for high energy ($E_1 \cong 200$ Mev) deuterons is given by Serber[19] as

$$\sigma_s \cong 5A^{\frac{1}{3}} \times 10^{-26} \text{ cm}^2, \tag{60}$$

where A is the mass number of the target nucleus. For lower deuteron energies an estimate of the (d,n) stripping cross section can be obtained by modifying Eq. (60) to include the effect of the coulomb repulsion on the incident deuteron.[26] This modification may be achieved in an approximate way by multiplying the right hand side of Eq. (60) by a factor $\rho(R)$ which takes account of the decrease in the density of protons at the nuclear surface due to the coulomb repulsion.[27] This density factor has been evaluated by Konopinski and Bethe.[28] For the target nuclei $_4Be^9$, $_{12}Mg^{24}$, $_{29}Cu^{63}$, and $_{79}Au^{197}$ one obtains the values $\rho = 0.9$, 0.8, 0.72, and 0.28, respectively. Values of the stripping cross sections for the four nuclei are given in Table I. Since we have assumed tacitly the proton's sticking probability to be unity in each case, the values listed in the table really give upper limits for the stripping cross sections.

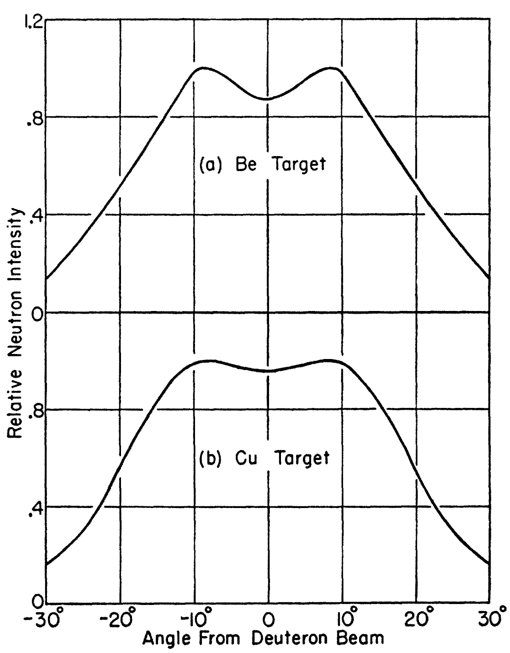

FIG. 2. Angular distribution of neutrons obtained from electric break-up of the deuteron.

[26] In the derivation of Eq. (60) Serber neglected the Coulomb repulsion on the deuteron; this neglect is justified, of course, for the large deuteron energies considered by Serber. A second assumption involved in the derivation of Eq. (60) is that the period of the internal motion of the deuteron is much greater than the collision time. The validity of this assumption is questionable for deuteron energies as low as 15 Mev, but we shall assume that no large errror is introduced in the total cross section by the use of this assumption.

The results given in Table I indicate that cross sections for the electric break-up process may be large enough, in some cases, to enable this process to compete favorably with the stripping process.

The theoretical angular and energy distributions for the neutrons obtained by electric break-up of the deuteron in flight are given by Eqs. (54), (55), and (56). Plots of the angular distributions obtained with 15-Mev deuterons incident on Be and Cu targets are given in Fig. 2. In the consideration of the theoretical distributions it should be remembered that a "point nucleus" model has been used. Furthermore, spreading of the deuteron beam because of coulomb deflection and multiple scattering has been neglected; this spreading of the beam will widen the angular distribution somewhat. It should be noted that in the angular distributions given in Fig. 2, neutrons of all permitted energies are counted. In most of the experiments, neutron detectors having rather high threshold energies have been employed. At the high end of the neutron spectrum our expressions are not quantitatively valid, for we have assumed the loss of energy of the deuteron's center of gravity to be small compared to the deuteron's initial energy.

The predictions of the electric break-up theory seem to be consistent with the experimental results. Thus, neglecting the spreading of the deuteron beam, the theoretical angular distribution (for neutrons of all permitted energies) has a half width of about 40° for both Be and Cu targets. Thus, at least up to $Z=30$, the half-width is roughly independent of the atomic number of the target nucleus. And, again in agreement with experiment, the theory predicts that the yield of neutrons from a heavy target such as gold is less than the yield from a light target such as magnesium.

The electric break-up of the deuteron seems to lead characteristically to a "double-peaked" angular distribution for the escaping neutrons if neutrons of all energies are counted. If, however, one counts only the neutrons at the upper end of the energy spectrum, the theory yields a "single-peaked" distribution; it must be remembered, however, that our expressions do not have quantitative validity for the extreme energies of the neutron spectrum.

Lack of sufficient experimental data in the range of

[27] D. C. Peaslee, Phys. Rev. 74, 1001 (1948).

[28] E. J. Konopinski and H. A. Bethe, Phys. Rev. 54, 130 (1938). In the notation of these authors $\rho = l_c^2/l_0^2$.

neutron energies for which our theory has quantitative significance, and lack of an adequate theory of the stripping process for "low" energy deuterons make it impossible to assess, at present, the relative contributions of stripping and electric break-up to the observed neutron intensities. Of course, any neutrons observed to have energy greater than $E_1-\epsilon_0$ ($\cong$12.8 Mev for 15 Mev deuterons) cannot result from electric break-up and must be attributed to stripping or to compound nucleus formation in which the target nucleus absorbs the incident deuteron and subsequently emits a neutron.

APPENDIX

(A). The result given in Eq. (41) of the text follows from the more general theorem[29]

$$\sum_{a,b}|\sum_{ij}(b|Q_{ij}|a)u_iv_j|^2=(Q_{AB}'^2/10)[u^2v^2+\tfrac{1}{3}(\mathbf{u}\cdot\mathbf{v})^2]+I_{AB}^2(\mathbf{u}\cdot\mathbf{v})^2/9, \quad (61)$$

where $\mathbf{u}$ and $\mathbf{v}$ are arbitrary vectors. We shall now prove this theorem.

We introduce the tensor T which has elements $T_{ij}=u_iv_j+u_jv_i$. And we introduce the spherical harmonics Y_2^m defined by

$$Y_2^0(Q)=(\tfrac{2}{3})^{\frac{1}{2}}(3z^2-r^2),\qquad Y_2^0(T)=(\tfrac{2}{3})^{\frac{1}{2}}(3u_zv_z-\mathbf{u}\cdot\mathbf{v}),$$
$$Y_2^{\pm1}(Q)=\mp 2z(x\pm iy),\qquad Y_2^{\pm1}(T)=\mp\{u_z(v_x\pm iv_y)+v_z(u_x\pm iu_y)\},$$
$$Y_2^{\pm2}(Q)=(x\pm iy)^2,\qquad Y_2^{\pm2}(T)=(u_x\pm iu_y)(v_x\pm iv_y).$$

We then have the following relations

$$\sum_m|Y_2^m(Q)|^2=4\sum_{ij}|Q_{ij}'|^2,$$
$$\sum_{a,b}|(b|Y_2^m(Q)|a)|^2=\tfrac{4}{5}Q_{AB}'^2 \text{ for all } m,$$
$$\sum_m|Y_2^m(T)|^2=2[u^2v^2+\tfrac{1}{3}(\mathbf{u}\cdot\mathbf{v})^2],$$
$$\sum_m Y^{2m}(Q)Y_2^{m*}(T)=2\sum_{ij}Q_{ij}'(u_iv_j+u_jv_i-\tfrac{2}{3}\mathbf{u}\cdot\mathbf{v}\delta_{ij}).$$

Consequently,

$$\begin{aligned}\sum_{a,b}|\sum_{ij}(b|Q_{ij}|a)u_iv_j|^2&=\sum_{a,b}|\sum_{ij}\{(b|Q_{ij}'|a)+\tfrac{1}{3}(b|I|a)\delta_{ij}\}\tfrac{1}{2}(u_iv_j+u_jv_i)|^2\\&=\sum_{a,b}|\tfrac{1}{2}\sum_{ij}(b|Q_{ij}'|a)(u_iv_j+u_jv_i-\tfrac{2}{3}\mathbf{u}\cdot\mathbf{v}\delta_{ij})+\tfrac{1}{3}\mathbf{u}\cdot\mathbf{v}(b|I|a)\delta_{ij}|^2\\&=\sum_{a,b}|\tfrac{1}{4}\sum_m(b|Y_2^m(Q)|a)Y_2^{m*}(T)+\tfrac{1}{3}\mathbf{u}\cdot\mathbf{v}(b|I|a)|^2\\&=\sum_{a,b}\{\tfrac{1}{16}\sum_m|(b|Y_2^m(Q)|a)|^2|Y_2^m(T)|^2+[(\mathbf{u}\cdot\mathbf{v})^2/9]|(b|I|a)|^2\}\\&=(Q_{AB}'^2/10)[u^2v^2+\tfrac{1}{3}(\mathbf{u}\cdot\mathbf{v})^2]+(I_{AB}^2/9)(\mathbf{u}\cdot\mathbf{v})^2.\end{aligned}$$

The special choice $\mathbf{u}=k_\nu$, $\mathbf{v}=\boldsymbol{\pi}$ yields the result given in Eq. (41) of the text.

(B). The result given in Eq. (44) of the text follows from the more general theorem

$$\begin{aligned}\sum_{a,b}|\sum_{ijk}(b|S_{ijk}|a)u_iv_jw_k|^2=&\frac{S_{AB}'^2}{210}\{5u^2v^2w^2-2(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{u})\\&+3[(\mathbf{u}\cdot\mathbf{v})^2w^2+(\mathbf{v}\cdot\mathbf{w})^2u^2+(\mathbf{w}\cdot\mathbf{u})^2v^2]\}\\&+\frac{(r^2D)_{AB}^2}{75}\{6(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{u})\\&+[(\mathbf{u}\cdot\mathbf{v})^2w^2+(\mathbf{v}\cdot\mathbf{w})^2u^2+(\mathbf{w}\cdot\mathbf{u})^2v^2]\},\quad (62)\end{aligned}$$

where

$$(r^2D)_{AB}=\sum_{a,b}\sum_i|(b|r^2x_i|a)|^2$$

and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are arbitrary vectors. We shall now prove this theorem.

We introduce the tensor T having elements $T_{ijk}=\sum_P u_iv_jw_k$ and the tensor T' having elements

$$T_{ijk}'=\sum_P\{u_iv_jw_k-\tfrac{1}{5}\delta_{ij}[u_k(\mathbf{v}\cdot\mathbf{w})+v_k(\mathbf{w}\cdot\mathbf{u})+w_k(\mathbf{u}\cdot\mathbf{v})]\},$$

where the sums are extended over all permutations of ijk. And we introduce the spherical harmonics Y_3^m defined by

$$Y_3^0(S)=(\tfrac{4}{5})^{\frac{1}{2}}z(5z^2-3r^2),$$
$$Y_3^{\pm1}(S)=\mp(\tfrac{3}{5})^{\frac{1}{2}}(5z^2-r^2)(x\pm iy),$$
$$Y_3^{\pm2}(S)=(6)^{\frac{1}{2}}z(x\pm iy)^2,$$
$$Y_3^{\pm3}(S)=\mp(x\pm iy)^3,$$
$$Y_3^0(T)=(4/45)^{\frac{1}{2}}\{u_z(5v_zw_z-3\mathbf{v}\cdot\mathbf{w})+v_z(5w_zu_z-3\mathbf{w}\cdot\mathbf{u})+w_z(5u_zv_z-3\mathbf{u}\cdot\mathbf{v})\},$$
$$Y_3^{\pm1}(T)=\mp(1/15)^{\frac{1}{2}}\{(5u_zv_z-\mathbf{u}\cdot\mathbf{v})(w_x\pm iw_y)+(5v_zw_z-\mathbf{v}\cdot\mathbf{w})(u_x\pm iu_y)+(5w_zu_z-\mathbf{w}\cdot\mathbf{u})(v_x\pm iv_y)\},$$
$$Y_3^{\pm2}(T)=(\tfrac{2}{3})^{\frac{1}{2}}\{u_z(v_x\pm iv_y)(w_x\pm iw_y)+v_z(w_x\pm iw_y)(u_x\pm iu_y)+w_z(u_x\pm iu_y)(v_x\pm iv_y)\},$$
$$Y_3^{\pm3}(T)=\mp(u_x\pm iu_y)(v_x\pm iv_y)(w_x\pm iw_y).$$

We then have the following relations:

$$\sum_{ijk}S_{ijk}'T_{ijk}'=\tfrac{3}{4}\sum_m Y_3^m(S)Y_3^{m*}(T),$$
$$\sum_m|Y_3^m(S)|^2=8\sum_{ijk}|S_{ijk}'|^2,$$
$$\sum_{a,b}|(b|Y_3^m(S)|a)|^2=(8/7)S_{AB}'^2 \text{ for all } m,$$
$$\sum_m|Y_3^m(T)|^2=(4/15)\{5u^2v^2w^2-2(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{u})+3[(\mathbf{u}\cdot\mathbf{v})^2w^2+(\mathbf{v}\cdot\mathbf{w})^2u^2+(\mathbf{w}\cdot\mathbf{u})^2v^2]\},$$
$$\sum_i[(\mathbf{u}\cdot\mathbf{v})w_i+(\mathbf{v}\cdot\mathbf{w})u_i+(\mathbf{w}\cdot\mathbf{u})v_i]^2=6(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{u})+[(\mathbf{u}\cdot\mathbf{v})^2w^2+(\mathbf{v}\cdot\mathbf{w})^2u^2+(\mathbf{w}\cdot\mathbf{u})^2v^2].$$

Consequently,

$$\begin{aligned}\sum_{a,b}|\sum_{ijk}(b|S_{ijk}|a)u_iv_jw_k|^2&=\frac{1}{36}\sum_{a,b}|\sum_{ijk}(b|S_{ijk}|a)\sum_P u_iv_jw_k|^2\\&=\frac{1}{36}\sum_{a,b}|\sum_{ijk}\{(b|S_{ijk}'|a)+\frac{1}{10}(b|r^2\sum_P x_i\delta_{jk}|a)\}\sum_P u_iv_jw_k|^2\\&=\frac{1}{36}\sum_{a,b}|\sum_{ijk}(b|S_{ijk}'|a)T_{ijk}'+\frac{1}{10}(b|r^2\sum_P x_i\delta_{jk}|a)\sum_P u_iv_jw_k|^2\\&=\frac{1}{36}\sum_{a,b}|\tfrac{3}{4}\sum_m(b|Y_3^m(S)|a)Y_3^{m*}(T)+\frac{1}{10}\sum_{ijk}(b|r^2\sum_P x_i\delta_{ik}|a)\sum_P u_iv_jw_k|^2\\&=\frac{1}{36}\sum_{a,b}|\tfrac{3}{4}\sum_m(b|Y_3^m(S)|a)Y_3^{m*}(T)+(6/5)\sum_i(b|r^2x_i|a)[(\mathbf{u}\cdot\mathbf{v})w_i+(\mathbf{v}\cdot\mathbf{w})u_i+(\mathbf{w}\cdot\mathbf{u})v_i]|^2\\&=\frac{1}{64}\sum_{a,b}\sum_m|(b|Y_3^m(S)|a)|^2|Y_3^m(T)|^2+\frac{(r^2D)_{AB}^2}{75}\sum_i[(\mathbf{u}\cdot\mathbf{v})w_i+(\mathbf{u}\cdot\mathbf{w})u_i+(\mathbf{w}\cdot\mathbf{u})v_i]^2\\&=\frac{S_{AB}'^2}{210}\{5u^2v^2w^2-2(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{u})+3[(\mathbf{u}\cdot\mathbf{v})^2w^2+(\mathbf{v}\cdot\mathbf{w})^2u^2+(\mathbf{w}\cdot\mathbf{u})^2v^2]\}\\&\quad+\frac{(r^2D)_{AB}^2}{75}\{6(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{u})+[(\mathbf{u}\cdot\mathbf{v})^2w^2+(\mathbf{v}\cdot\mathbf{w})^2u^2+(\mathbf{w}\cdot\mathbf{u})^2v^2]\}.\end{aligned}$$

The special choice $\mathbf{u}=\mathbf{v}=\mathbf{k}_\nu$, $\mathbf{w}=\boldsymbol{\pi}$ leads to the result given in Eq. (44).

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[29] This theorem is used by G. C. Wick, reference 8. A similar theorem has been proved by D. L. Falkoff (thesis, University of Michigan, 1948). Our proof is based on the method used by Falkoff.