would not be expected according to the statistical theory, and these protons show an asymmetrical angular distribution. This indicates that the (γ,p) reaction is not necessarily produced by photons grouped in a narrow interval around W_e. This should be noticeable from the shape of the transition curve corresponding to the (γ,p) reaction. It is unfortunate that the accuracy obtained in this case does not warrant any conclusions along this line.

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Effective Photon Energies of High Energy Photo-Nuclear Reactions*

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An attempt has been made to use shower theory to evaluate the effective energies of the photo-nuclear reactions measured by Strauch. It seems that these energies can be determined most accurately from the area under the transition curve, the so-called "track length." A theoretical formula for the track length is discussed. The shape of the transition curve at small thicknesses can also be calculated quite accurately and serves as a rough check on the effective energies as derived from the track length. A comparison with experiment of the theoretical shape of the whole transition curve is given; and, as one would expect, the agreement is not very good.

I. INTRODUCTION

IN this paper we try to use shower theory to evaluate some of Strauch's[1] results on high energy photo-nuclear reactions. As Strauch has described, the cross sections for these reactions have more or less sharp maxima for some photon energy. For most of our calculations it will be adequate to assume that the width at this maximum is very small; i.e., that the reactions take place for only one photon energy, which we shall call W_e, the "effective energy." The effect of this approximation is discussed later. If it were easy to make accurate calculations with present shower theory, there would be no problem; one would simply calculate shower curves for various energies W_e, and for some value of W_e would obtain a fit with the experimental curve. For the energies in which we are interested, however, around 20 Mev, it is well known that shower theory cannot be relied upon to predict an accurate cascade curve, mainly because the cross sections for pair production and bremsstrahlung vary considerably over the range of energies of interest, which is from about 20 to 300 Mev. We must look, therefore, to some quantity that can be calculated more accurately than can the shape of the entire transition curve and yet one that gives us the information we desire.

It is clear that one does not really need to know the whole transition curve in order to find the energy to which it corresponds. If we consider transition curves corresponding to different energies, but to the same initial conditions, then at any thickness there is a unique correlation between the energy and the height of the curve. Thus, any one point on the transition curve determines the energy, in principle. Of course, this is no real help, for if we could calculate an arbitrary point accurately, we could calculate the detailed shape. There is a particular point on the transition curve however, which can be calculated rather more accurately than can any other point; namely, the height of the maximum. The reason is, as Rossi and Greisen[2] have pointed out, that at the maximum of the shower one can take into account approximately the variation of the pair production cross section with energy. This enhances the accuracy considerably. Thus, if the shower curve corresponding to an energy W_e shows a maximum, one might hope to determine W_e by the position and height of the maximum. Therein lies the difficulty. Although some of Strauch's curves have a maximum, those corresponding to higher energies do not. We must find a different method if we wish it to be universally applicable.

For very large thicknesses multiplication becomes

* This work was performed under the auspices of the AEC.
[1] K. Strauch, Phys. Rev. 81, 973 (1950).

[2] B. Rossi and K. Greisen, Revs. Modern Phys. 13, 274 (1941).

unimportant, and the "shower" curves simply become the exponential absorption curve of photons[3] of energy W_e. One might hope to determine the absorption coefficient from the slope of the experimental curve and from this to get the energy W_e. There is no difficulty in principle with this idea; in practice one must go to such large thicknesses before pure absorption sets in that the intensity becomes impractically small.

The beginning of Strauch's shower curves have a characteristic shape. There is a drop at very small thicknesses owing to the absorption of photons of energy W_e; multiplication soon sets in, however, and the curve becomes less steep and may even rise again. The initial slope is entirely due to self-absorption; and this slope, in principle, determines the energy. Unfortunately, this slope is very difficult to measure with any accuracy. On the other hand, one might hope that since the first part of the shower curve, up to perhaps a half-radiation length, is mainly an absorption curve, multiplication processes being secondary, one might be able to calculate this multiplication with sufficient accuracy to predict the behavior of the beginning of the curve with reasonable accuracy. This expectation is fulfilled. One can calculate the shower curve to almost a radiation length with considerable accuracy. Unfortunately, there is an experimental limitation. It is difficult to get good statistics on the beginning part of the curve. Therefore, one cannot obtain a very accurate value for W_e in this manner.

Finally, it is possible to determine the effective energy W_e from the area under the transition curve, the so-called "track length." This quantity has the advantage that one can take the variation of the pair production cross section with energy into account, just as in the calculation of the height at the maximum. Moreover, it has an advantage over the latter quantity: although not all of Strauch's transition curves have a maximum, all have an area. It is the track length which we have mainly used in calculating W_e, although we have also used the initial behavior of the shower curve, up to almost a radiation length, as a rough check.

The remainder of this paper is in four parts: in Sec. II we have calculated the photon spectrum to be expected from the synchrotron target; in Sec. III we have calculated the track length as a function of energy, and applied our formula to Strauch's results; in Sec. IV we have calculated the detailed shape of the curves at their beginning; finally, in Sec. V, we have calculated, as best we could, the detailed shape of the transition curves, neither expecting nor getting very good agreement, with experiment.

For convenient reference, we give here the usual shower equations, using the notation of Rossi and Greisen unless otherwise indicated.

$$\frac{\partial\pi(E,t)}{\partial t}=2\int_0^1\gamma\left(\frac{E}{u},t\right)\psi\left(\frac{E}{u},u\right)\frac{du}{u}-\int_0^1\pi(E,t)\phi(E,v)dv+\int_0^1\frac{1}{1-v}\pi\left(\frac{E}{1-v},t\right)\phi\left(\frac{E}{1-v},v\right)dv+\epsilon\frac{\partial\pi(E,t)}{\partial E},\quad(1a)$$

$$\frac{\partial\gamma(W,t)}{\partial t}=\int_0^1\pi\left(\frac{W}{v},t\right)\phi\left(\frac{W}{v},v\right)\frac{dv}{v}-\sigma(W)\gamma(W,t).\quad(1b)$$

Here $\pi(E,t)$ is the number of electrons of energy E at thickness t and $\gamma(W,t)$ is the same for photons. $\psi(W,u)$ is the probability per radiation length that a photon of energy W produce a pair, one particle of which has fractional energy u. $\phi(E,v)$ is the probability per radiation length that an electron of energy E emit a photon with fractional energy v. The usual shower theory deals with high energies where ψ and ϕ are functions of u and v only; but for our purposes, keeping the dependence on W and E explicit facilitates discussion. ϵ in the above equations is the critical energy, and thicknesses are measured in radiation units. We also depart slightly from Rossi and Greisen by letting $\sigma(W)$ be the *total* absorption coefficient for photons of energy W. This will be discussed later. If we call $\sigma_c(W)$ the absorption coefficient for the Compton effect and $\sigma_p(W)$ that due to pair production, then

$$\sigma(W)=\sigma_p(W)+\sigma_c(W).$$

II. PHOTON SPECTRUM FROM THE TARGET

The 322-Mev electrons from the beam of the synchrotron are allowed to fall on a target of Pt, 0.020 in. thick, producing the beam of photons used in the experiment. If the target were infinitely thin, the distribution of photon energies W whould be given by $\phi(E_0,v)$, where $E_0=322$ Mev and $v=W/E_0$. Actually, the finite thickness of the target introduces a correction, which we shall calculate in this section. First, we make explicit an assumption inherent in our use of the function $\phi(E_0,v)$. This function gives the energy distribution of the photons produced by an electron of energy E_0, integrated over the angles between the electron and the photon directions. At first sight, one might think that the appropriate function for our purposes should be the energy distribution of photons produced in essentially the same direction as the electron. In passing through the target, however, the electrons are multiply scattered; and in the present geometry the effect of these

[3] If we take the finite width of the reaction cross sections into account, the shower curve at large thicknesses really becomes the absorption curve of the photons of lowest energy that can produce the reaction.

deviations arising from scattering will be taken care of to a good approximation[4] by using the integrated function $\phi(E_0, v)$.

We can find the photon spectrum by simply putting into the shower equations a power series expansion corresponding to the correct initial conditions, i.e.,

$$\pi(E_0, E, t)=\delta(E_0-E)+P(E_0, E)t+\cdots,$$
$$\gamma(E_0, W, t)=Q(E_0, W)t+R(E_0, W)t^2+\cdots.$$

Equating to zero various powers of t in the shower equation, the unknown functions P, Q, and R are found in succession as easily evaluable integrals. $Q(E_0, W)$ comes out to be just $\phi(E_0, v)$, as it must. If we use the the approximate expression $\phi(E_0, v)\approx 1/v$ to calculate the small correction term $R(E_0, W)$, we obtain

$$R(E_0, W)=-\tfrac{1}{2}[\sigma(W)-\ln(1-W/E_0)].$$

Thus our corrected spectrum from the target is

$$\phi(E_0, v)-\tfrac{1}{2}t\{\sigma(W)-\ln(1-v)\}. \quad (2)$$

This corrected spectrum is plotted in Fig. 7 of Strauch's paper.[1] The physical interpretation of this spectrum is clear. For v small, $\ln(1-v)\approx 0$ and the important term in the correction is just the absorption of photons in the target. The correction term $-\tfrac{1}{2}\sigma(W)t$ represents this self-absorption. For large v the term in $\ln(1-v)$ is important. This term *diminishes* the number of high energy photons. This represents a double radiation process: there is an overwhelmingly probability for emitting a low energy quantum in which the energy of the electron is diminished below 322 Mev; therefore, it can no longer emit a quantum with this upper limit. Thus the effect of the finite target thickness is to diminish appreciably the number of very high energy photons.

After leaving the target, the beam must pass through a quartz donut about $\frac{5}{8}$ in. thick. The main effect of this on energies above 18 Mev is to reduce the intensity uniformly, since the absorption coefficient is small and varies slowly with energy in this region. We therefore neglect this correction.

III. THE TRACK LENGTH

The most accurate calculation of the track length of photons to date is the numerical work by Richards and Nordheim[5] in which collision loss of the electrons and the Compton effect are taken into account, as well as the variation with energy of the radiation and pair production cross sections. These are not very convenient for our purpose, however, since they are made for a single incident photon. To apply them to the present problem, one would need to integrate their results numerically over the photon spectrum emerging from the synchrotron. Moreover, aside from the labor involved, there is the difficulty that Richards and Nordheim's results hold only when the single incident photon has an energy much larger than the energy W_e of the photons which one is considering; in these circumstances it is not clear how to carry out the integration over the photon beam from the synchrotron, since, of course, it contains photons with energies arbitrarily close to W_e.

Fortunately, the photon energies with which we have to deal are always greater than about 17 Mev; i.e., about two and a half times the critical energy in lead. In this case, it is possible to make a slight adaptation of the formulas for the track length in Rossi and Greisen so that they apply with considerable accuracy. Before we do this, there is a somewhat peculiar feature of the usual track length formulas, which we should consider. For the sake of discussion, suppose we are interested in the track length of photons of energy W, due to an initial spectrum which goes as $1/W$ up to some maximum energy W_0. Now, following Nordheim and Hebb[6] the track length of photons of energy W, due to an arbitrary initial spectrum $\gamma(W_0, W, 0)$ is

$$Z(W_0, W)=\frac{1}{\sigma\cdot 2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{A(s)M(s, 0)W^{-(s+1)}}{[A(s)-B(s)C(s)/\sigma]}ds, \quad (3)$$

where

$$M(s, 0)=\int_0^\infty W^s\gamma(W_0, W, 0)dW.$$

For a $1/W$ spectrum up to W_0, $M(s, 0)=W_0{}^s/s$; and we therefore have

$$Z(W_0, W) =\frac{1}{\sigma W\cdot 2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{A(s)}{[A(s)-B(s)C(s)/\sigma]}\frac{e^{ys}}{s}ds, \quad (4)$$

where $y=\ln(W_0/W)$ and the integration path is to the right of all the singularities of the integrand. Here $A(s)$, $B(s)$, $C(s)$ are defined as by Rossi and Greisen.[2,7] Now the integrand of (4) has simple poles at $s=1, -2.6, -3.6\cdots$. If we evaluate the residues at these poles and divide the result by the initial spectrum dW/W to get the track length relative to the number of photons at $t=0$, we find

$$Z_{\text{rel}}(W_0, W)=\frac{1}{\sigma}\left[0.437\frac{W_0}{W}-0.02\left(\frac{W}{W_0}\right)^{2.6} -0.005\left(\frac{W}{W_0}\right)^{3.6}+\cdots\right]. \quad (5)$$

Now it is clear that (5) becomes invalid[8] when W approaches W_0. The relative number of photons of energy W, when W is very close to W_0, will be given by $e^{-\sigma t}$, since there will be essentially no multiplication.

[4] L. I. Schiff, Phys. Rev. **70**, 87 (1946).

[5] J. Richards and L. Nordheim, Phys. Rev. **74**, 1106 (1948).

[6] L. W. Nordheim and M. H. Hebb, Phys. Rev. **56**, 494 (1939).

[7] Note that $B(s)$ as defined in reference 2 has a factor σ in it so that the integrand in (4) is really independent of σ.

[8] This was called to my attention by Dr. Strauch.

Hence, Z_{rel} will be just $1/\sigma$; and for smaller W where there is multiplication Z_{rel} must be greater than $1/\sigma$. This condition fails to hold for (5) when $W \approx 0.41 W_0$. It is not clear why (5) is incorrect for $W \sim W_0$, when (4) is almost certainly correct. It may be that the integrand in (4) has singularities off the real axis, although Nordheim and Hebb have made a search near $s=1$, and we also have made a rather perfunctory search, without finding any. We are concerned with this point, not because we want to use a formula like (5) for W close to W_0—in Strauch's experiments W/W_0 is always fairly small—but we really would like to know whether (5) can be considered to be correct for W/W_0 small, where the higher order terms are negligible. This is not obvious, since the fact that (5) breaks down for $W \sim W_0$ throws suspicion on it.

Some light can be shed on this question in the following way. If one uses the Carlson-Oppenheimer[9] approximation to the shower equations, it is easy to show that this is equivalent to using

$$A(s)=2s/(s+1), \quad B(s)=2\sigma/(s+1), \quad C(s)=1/s.$$

Then

$$Z_{\text{rel}}(W_0, W)=\frac{1}{\sigma\cdot 2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{s e^{ys}}{s^2-1}ds = \frac{0.500}{\sigma}\left(\frac{W_0}{W}+\frac{W}{W_0}\right). \tag{6}$$

This is a most reasonable result, since Z_{rel} as given by (6) is always greater than $1/\sigma$ and approaches $1/\sigma$ as W approaches W_0. Unfortunately, Eq. (6), however reasonable in appearance, cannot really be trusted for W close to W_0, since the Carlson-Oppenheimer approximation is not very good in this region. For smaller values of W, Eq. (6) shows that Z_{rel} is proportional to W_0/W with a correction term of order $(W/W_0)^2$, which is small. This is probably a trustworthy qualitative conclusion in general, since the Carlson-Oppenheimer approximation is not bad for $W \ll W_0$.

We have also tried to check (5) in the following manner. We have calculated the shower curve as a function of thickness for various values of $y=\ln(W_0/W)$ and integrated these numerically to find the track length; we have used an expansion in powers of t for small t and the usual saddle point method for larger t. This method gives reasonable results; e.g., as W approaches W_0 the relative track length approaches $1/\sigma$. The major difficulty is that it is not very accurate, since the saddle point method can be off by 10 or 15 percent for the smaller values of y and t. We have corrected for the inaccuracies of the saddle point method as best we could by comparison of the answers which it gives with the quite accurate results given by the power series in t. for those values of t for which one can get an answer by both methods. Comparison of our answers for Z_{rel} by this method with (5) makes it appear that (5) is correct for $y=3$, is a few percent low for $y=2$, about 30 percent low for $y=1$, and, of course, off by a factor 1/0.41 for $y=0$. We have done the same sort of calculation for an initial spectrum

$$1/W[(4/3)(1-W/W_0)+0.38(W/W_0)^2],$$

which is a rough approximation to the spectrum from the synchrotron, and find that the formula corresponding to (5) is more accurate than for the $1/W$ spectrum, being off, for example, only by a few percent for $y=1$. All in all, then, it seems clear that although formulas like (5) are not correct for W close to W_0, they are probably valid for $W \ll W_0$.

Now we turn to the real problem of interest, that of calculating as accurately as possible the track length of photons using the initial spectrum given by (2). We are interested in energies from about 17 Mev up. Energy loss of electrons by ionization is not negligible in this range; but, as Rossi and Greisen[2] have shown, one can correct for this by using an asymptotic expansion in powers of ϵ/W. The variation with energy of the pair production cross section can also be included in the manner indicated by Rossi and Greisen,[10] i.e., by writing

$$\psi(W, u)du=\sigma_p(W)du \tag{7}$$

and considering that the unknown function in the shower equations for the track length is not $Z(W_0, W)$ but $\sigma_p(W)\cdot Z(W_0, W)$. The above approximation for $\psi(W, u)$ means that the pair spectrum is taken to be flat; i.e., that the probability for the production of an electron of any energy is independent of energy. This is a quite good approximation in the range 18 to 322 Mev. The variation of the radiation cross section with energy can be included by taking some average expression appropriate to the region 18 to 322 Mev. The expression we have chosen is

$$\phi(v)=\frac{1}{v}\left[\frac{4}{3}(1-v)+\frac{3}{4}v^2\right]. \tag{8}$$

Reference to Rossi and Greisen will show that this seems to be a reasonable approximation except for v close to unity, where $\phi(v)$ is relatively small anyway.

If one carries out the calculations according to the above sketch, using the boundary condition that the incident spectrum is that given by (2) he obtains

$$Z(W_0, W)=0.346W_0/\sigma_p(W)\cdot W^2\cdot \nu(\epsilon/W), \tag{9}$$

where

$$\nu(\epsilon/W)=1+0.71(\epsilon/W)-0.32(\epsilon/W)^2+0.41(\epsilon/W)^3+\cdots.$$

We have not included the dubious negative powers of W_0/W which one gets from the evaluation of (4) at the poles on the negative real axis, since the discussion at the beginning of this section implies that they are

[9] J. F. Carlson and J. R. Oppenheimer, Phys. Rev. 51, 220 (1937).

[10] See reference 2, p. 293.

negligible for the energies which interest us. The factor 0.346 appears in (9) instead of the usual 0.437 for two reasons. First, $M(1,0)$, as evaluated by numerical integration of the spectrum given by (2), turns out to be 0.816 W_0/W instead of W_0/W as before. Second, we have used the $A(s)$, $B(s)$, and $C(s)$ corresponding to the $\phi(v)$ and $\psi(u)$ given above instead of those calculated with the usual asymptotic $\phi(v)$ and $\psi(u)$, as by Rossi and Greisen. This alters the residue at $s=1$. This change is rather insensitive to the choice of $\phi(v)$. For example, if we use the $A(s)$ corresponding to $v\phi(v)=(4/3)(1-v)+v^2$ and to $(4/3)(1-v)+\frac{1}{2}v^2$, the factor 0.346 changes to 0.355 and 0.330, respectively. Thus, using the average $\phi(v)$ given by (8) introduces only a small error.

At the lowest energies for which we wish to use (9) the cross section for Compton effect is about 15 percent of that for pair production, and is therefore not negligible. To take this into account accurately one would need to supplement Eqs. (1) by a term describing the production of electrons with energy greater than W in the Compton effect, and a term describing the photons with energy greater than W that get an energy W in a Compton scattering. This is difficult and we shall not attempt it, since the effect of these terms is probably small anyway. In addition to these effects the Compton effect acts to absorb the photons of energy W in which we are interested. One takes this into account roughly in the following manner. One replaces $\sigma_p(W)$ in (7) by $\sigma(W)$ and as mentioned before uses the total absorption coefficient $\sigma(W)$ in (1b). A glance at Eq. (1b) shows that this means that we take the absorption of photons of energy W into account correctly, but that we falsify the spectrum of electrons with energy greater than W, since replacing $\sigma_p(W)$ by $\sigma(W)$ makes the pair cross section too large. But a photon of 18 Mev is produced on the average by an electron of, say, twice that energy, where the Compton effect is very small anyway, so that it doesn't matter that we have allowed $\psi(W,u)$ to include the Compton effect. Also, and this is probably a stronger argument, we have checked (9) by comparison of the analogous formula for a single incident photon with the numerical results of Richards and Nordheim and found agreement within five percent from 14 Mev up, if we use $\sigma(W)$ and not $\sigma_p(W)$.

In evaluating the experiments, one wants not the track length given by (9), but $Z_{\mathrm{rel}}(W_0,W)$, the track length relative to the number of photons initially present in dW.

If we use the notation

$$\gamma(W_0,W,0)=f(W_0,W)/W$$

and make the change from $\sigma_p(W)$ to $\sigma(W)$ just mentioned, the relative track length is given by

$$Z_{\mathrm{rel}}(W_0,W)=0.346W_0/\sigma(W)W\nu(\epsilon/W)f(W_0,W). \quad (10)$$

In applying (10), as Strauch has explained, we have increased the radiation length by 10 percent over the value given by Rossi and Greisen and have decreased the pair production cross section per cm by 10 percent. It is obviously difficult to estimate the error in (10); but if forced to guess we would say that it is probably good to 15 percent at 17 Mev, and perhaps 10 percent at twice this energy. As was discussed earlier in this section, (10) must break down for $W\sim W_0$; but if we can extrapolate from our previous results, it should still be reasonably accurate for $W/W_0<\frac{1}{3}$.

As Strauch has noted, a kind of internal check on the experiments and theory can be had by carrying out an experiment on a given element with two different maximum energies W_{01} and W_{02} of the photon beam. The ratio of the relative track lengths is then

$$\frac{Z_{\mathrm{rel}}(W_{01},W)}{Z_{\mathrm{rel}}(W_{02},W)}=\frac{W_{01}f(W_{02},W)}{W_{02}f(W_{01},W)}. \quad (11)$$

This theoretical expression for the ratio should be quite accurate, since most of the approximations involved in the derivation of (10) effectively cancel in forming it. Equation (11) has been checked for the two reactions $Cu^{63}(\gamma,n)Cu^{62}$ and $C^{12}(\gamma,n)C^{11}$, which were carried out at maximum energies of 322 and about 200 Mev. The results are given in Table I of Strauch's paper.[1]

IV. SMALL THICKNESSES

If we wish to find the shape of the transition curve for small thickness, an expansion in powers of t suggests itself. As we have seen, the transition curve must drop at the very beginning, since absorption of the photons in the incident beam is a first-order effect proportional to t, and the production of photons is at least of second order. This suggests the use of an expansion of the form

$$\gamma(W_0,W,t)=e^{-\sigma(W)t}[\gamma(W_0,W,0)+K(W_0,W)t+L(W_0,W)t^2+\cdots], \quad (12a)$$

$$\pi(W_0,E,t)=[M(W_0,E)t+N(W_0,E)t^2+\cdots]. \quad (12b)$$

This expansion obviously satisfies the boundary conditions. The factor $e^{-\sigma(W)t}$ in (12a), of course, represents the absorption of photons of energy W initially in the beam and the coefficients $K(W_0,W)$, $L(W_0,W)$, etc. describe their subsequent multiplication.

These functions can be found by putting (12a) and (12b) into the shower equations (1) and equating to zero successive powers of t. The functions then come out to be simple, but sometimes tedious integrals. Alternately, one can find, in much the same manner, the Mellin transforms of the shower equations as a series in t and can then invert this transform, evaluating the complex integrals that result by the method of residues. Both methods, lead to the same result, of course, which we first write down and then discuss. As before, the quantity of interest is not $\gamma(W_0,W,t)$ but

$$\gamma_{\mathrm{rel}}(W_0,W,t)=\gamma(W_0,W,t)/\gamma(W_0,W,0).$$

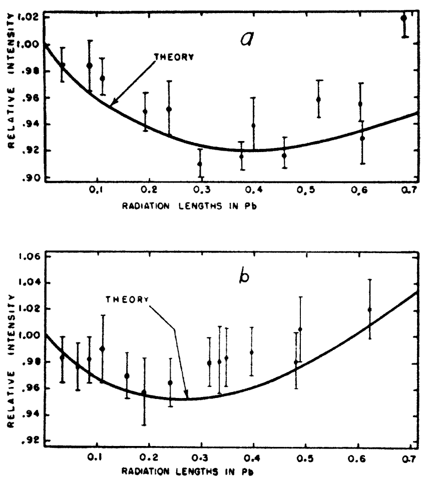

FIG. 1. (a) Relative intensity in Pb of photons causing $C^{12}(\gamma,n)C^{11}$. (b) Relative intensity of Pb of photons causing $Cu^{63}(\gamma,n)Cu^{62}$.

Our results are

$$\gamma_{\mathrm{rel}}(W_0, W, t) = e^{-\sigma(W)t}\left[1+\frac{\bar{\sigma}_p F(y)t^2}{f(W_0, W)}+\frac{G(y)t^3}{3f(W_0, W)}+\cdots\right], \quad (13)$$

where

$$y=\ln(W_0/W), \quad \bar{\sigma}_p=\sigma_p[(W_0W)^{\frac{1}{2}}],$$

$$F(y)=(16/9)(3e^{-y}-3+y+\tfrac{1}{2}y^2e^{-y}+2ye^{-y}) +1.8(\tfrac{1}{2}+\tfrac{1}{2}e^{-2y}-ye^{-y}-e^{-2y}) +0.45(e^{-y}-e^{-2y}-ye^{-2y}),$$

$$G(y)=\bar{\sigma}^2(y-1+e^{-y})-\bar{\sigma}H(y)-\bar{\sigma}(\epsilon/W_0)(e^y-1),$$

$$H(y)=1.645-\sum_{n=1}^{\infty}(e^{-ny}/n^2)+(1-e^{-y})\ln(1-e^{-y})-ye^{-y}.$$

The following approximations have been used in calculating the above result. As an analytic approximation to the initial spectrum, we have used

$$W\gamma(W_0, W, 0)=\left[\frac{4}{3}\left(1-\frac{W}{W_0}\right)+0.6\frac{W^2}{W_0{}^2}\right].$$

In calculating the term in t^2, we have used, as for the track length, $v\phi(v)=(4/3)(1-v)+\frac{3}{4}v^2$, and $\psi(W, u)du = \mathrm{const}\ du$. In calculating the small term in t^3, we have, for simplicity, used $\phi(v)=1/v$. In terms in t^2 and t^3 describing the multiplication, there is the factor $\sigma_p[(W_0W)^{\frac{1}{2}}]$. This enters in the following way. Multiplication takes place because electrons of some average energy between W_0 and W are created and then radiate photons of energy W; for this average energy we have chosen the geometrical mean of W_0 and W; hence, the pair production cross section at this energy is $\sigma_p[(W_0W)^{\frac{1}{2}}]$. We have checked this approximation by writing $\sigma_p(W)$ as a linear function of $\ln(W)$, which is a fair representation in the energy range of interest here, and then calculating the coefficient of t^2 using the initial spectrum $\gamma(W_0, W, 0)=1/W$, and the approximation $\phi(v)=1/v$. One can then carry out the integrations and it turns out that to a very close approximation one gets the same results for the coefficient of t^2 as if he had started from the beginning with the average value $\sigma_p[(W_0W)^{\frac{1}{2}}]$.

For $W=W_0$ it is clear that (13) must become $e^{-\sigma(W)t}$, since there can no longer be any multiplication. Thus, for this case $(y=0)$, $F(y)$, and $G(y)$ must vanish. This provides a useful check on the calculations. In the terms in t^3 there enters a correction due to collision loss. This correction diverges as W goes to zero, but for $W>\epsilon$ it is probably correct.

We have used the above formula to calculate the beginning shapes of the transition curves for the reactions $Cu^{63}(\gamma,n)Cu^{62}$ and $C^{12}(\gamma,n)C^{11}$, using for the respective "resonance" energies the values 18 Mev and 27 Mev derived from the track length. The results are shown in Fig. 1. The experiments, of course, do not give very reliable results at these thicknesses, since one is trying to measure changes in intensity of the order of a few percent and very long counting periods are needed to get good statistical accuracy. Within the experimental errors, however, the theory seems to give fair agreement. If anything, the theoretical curve seems to be too low. The theoretical curves would be raised if one assumed that the effective energies were somewhat lower than 18 and 27 Mev, but the poor accuracy of the experiments does not justify this.

One should note that the expansions given above are quite accurate where they apply. This is because the main phenomenon at small thicknesses is simply the absorption of original photons and one knows accurately this absorption coefficient. The shower theory enters, of course, in giving the coefficients $F(y)$, etc. for the higher order terms, but it turns out that these are not at all sensitive to the approximate expression for the cross sections one chooses for radiation and pair production.

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We have also calculated as best we could, the detailed shape of the transition curves for the $Cu^{63}(\gamma,n)Cu^{62}$ and $C^{12}(\gamma,n)C^{11}$ reactions using the usual saddle point method. As in the calculation of the track lengths, one can take some reasonable average value for the radiation cross section and can take into account the ionization loss by using the asymptotic expansions as given by Rossi and Greisen. One cannot take into account the variation of the pair production cross section, however, in even the approximate way in which it was done in calculating the track length. Since the pair production varies by almost a factor of two between maximum and minimum energies with which we deal, i.e., between 322 and 18 Mev, considerable uncertainty is introduced into the results. Nonetheless, we thought it might be of some interest to present them.

Since one cannot take into account the variation of the absorption cross section $\sigma(W)$ with energy, he must choose some average value in carrying out the calculations. The question arises as to what is the most reasonable value. We have chosen to use $\sigma(W_e)$ for the following reasons. This value is roughly correct for the track length and, for the same reasons that apply there, for the maximum of the shower curve; also, for very large thicknesses, the cascade curve approaches a pure absorption curve with absorption coefficient $\sigma(W_e)$.

The saddle point method leads to the following expression

$$\gamma_{\mathrm{rel}}(W_0, W, t)$$

$$=\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{H_2(s)\exp[\lambda_1(s)t]M(s,0)}{f(W_0,W)[\nu_1(s,\epsilon/W)\cdot W]^s[\lambda_1''(s)t+1/s^2]^{\frac{1}{2}}}, \quad (14)$$

where

$$t=(1/s-y)/\lambda_1'(s).$$

Here, the functions $\lambda_1(s)$, $H_2(s)$, and $\nu_1(s, \epsilon/W)$ are defined in terms of $A(s)$, $B(s)$, and $C(s)$ as by Rossi and Greisen; but in the actual calculation of the latter functions we have used $\psi(u)du=\sigma(W_e)du$ and the $\phi(v)$ given by (8). $M(s, 0)$ is calculated numerically from the curve in Fig. 7 of Strauch's paper. Using the above expression, we have calculated the transition curves for the reactions $Cu^{63}(\gamma,n)Cu^{62}$ and $C^{12}(\gamma,n)C^{11}$. The curves are shown in Fig. 2. Whether the agreement is better or worse than one should expect is a moot equation. The agreement for Cu with a resonance energy of 18 Mev is not as good as that for C for which the resonance energy is 27 Mev. This is not implausible qualitatively, since the various approximations involved in taking average cross sections and neglecting Compton effect are somewhat more serious at 18 than at 27 Mev. One might perhaps have expected better agreement at the maximum, for the reasons given in the introduction.

It may be, however, that the errors in the saddle point method are not negligible. It is altogether possible that the saddle point method gives too low a value by perhaps 10 percent near the maximum; if this is true, the shower theory proper is in better agreement with

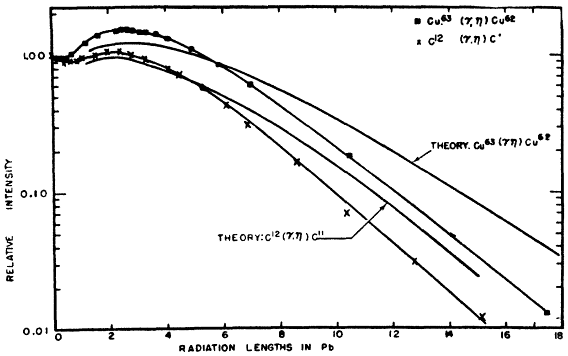

FIG. 2. Relative intensity in Pb of photons causing $Cu^{63}(\gamma,n)Cu^{62}$ and $C^{12}(\gamma,n)C^{11}$.

the experiment than evaluation by the saddle point method would seem to imply.

In all the work thus far, we have assumed that the (γ,n) cross sections are infinitely sharp, i.e., if we call $\Sigma(W)$ the cross section as a function of energy, that $\Sigma(W)=\delta(W-W_e)$, where W_e ist he "resonance" energy. What then is the effect of the finite width? Suppose for illustration that $\Sigma(W)$ is constant, and has a square shape centered about a value W_e, and with width Δ, i.e.,

$$\Sigma(W)=\text{constant},\quad W_e-\tfrac{1}{2}\Delta<W<W_e+\tfrac{1}{2}\Delta,$$
$$\Sigma(W)=0 \text{ otherwise.}$$

It is then easy to see, e.g., that if $\frac{1}{2}\Delta\ll W_e$, Eq. (9) is replaced approximately by the following relation:

$$Z_{\mathrm{rel}}(W_0, W_e)=\frac{0.346W_0(1+\Delta^2/6W_e^2)}{\sigma(W_e)W_e\cdot f(W_0,W_e)\nu(\epsilon/W_e)}. \quad (15)$$

The effect of the finite width is quite small. For example, if $W_e=20$, $\Delta=10$, this effect increases Z_{rel} by about four percent. One can see also that the effect of the finite width varies with depth in the shower. The spectrum of photons goes as $1/W^{s+1}$, where s is 0 at the beginning of the shower, is unity at the maximum, and increases slowly with thickness thereafter. Thus, the effect of the finite width in raising the shower curve increases slowly with thickness. Taking it into account would therefore slightly increase the discrepancy between experiment and theory shown in Fig. 2.

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