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Covariant Transformation Law for the Field Equations*

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Ordinarily, the existence of Bianchi identities is proven on the strength of the transformation properties of the lagrangian of the theory. In this paper, nothing is assumed concerning the lagrangian, except that the field equations themselves are covariant with respect to general coordinate transformations. It is then shown that at least the coefficients of the second-order derivatives in the field equations satisfy the usual relationships. Furthermore, a very weak restriction on the transformation law of the field equations is sufficient to derive conservation laws that hold even in the presence of matter.

I. INTRODUCTION

In his first paper on covariant field theories, Bergmann[1] introduced the field equations as the Euler-Lagrange equations of a variational principle. If the lagrangian is designated by L and if L is a function of the field variables y_A and their first derivatives with respect to the coordinates, $y_{A,\rho}$, the field equations will take the form

$$L^A \equiv \partial^A L - (\partial^{A\rho} L)_{,\rho} = 0, \qquad \partial^A L = \partial L/\partial y_A, \quad \partial^{A\rho} L = \partial L/\partial y_{A,\rho}. \tag{1.1}$$

In a physical theory, the field equations must be covariant; i.e., if they are satisfied in one permissible coordinate system, they must automatically be satisfied in every permissible system. For a completely covariant theory, the permissible coordinate systems include all those which can be transformed into each other with a nonvanishing jacobian. To assure this covariance, it was assumed in **I** that the lagrangian would, in the face of an infinitesimal coordinate transformation, transform as

$$\delta L = Q^{\rho}{}_{,\rho}. \tag{1.2}$$

The field variables themselves were to obey a homogeneous linear transformation law, whose infinitesimal form is denoted by

$$\bar{\delta} y_A = F^{B\mu}{}_{A\nu} \xi^{\nu}{}_{,\mu} y_B - y_{A,\rho} \xi^{\rho}. \tag{1.3}$$

The condition (1.2) is sufficient to assure not only covariance of the field equations, but also the formulation of the algebraic relationships between the canonical momenta (**I**-5.6) and the construction of the "strong" conservation laws (**I**-3.11). In this paper we shall investigate whether the conditions necessary for the formulation of the algebraic relationships between the canonical momenta and the construction of the "strong" conservation laws are not independent of the assumption (1.2). It turns out that covariance of the field equations (1.1) alone suffices for the algebraic constraints. For the "strong" conservation laws we must assume a particular transformation law for the field equations.

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[1] P. Bergmann, Phys. Rev. 75, 680–685 (1949). Referred to hereafter as I. Formulas in I are referred to by symbols such as (I-5.6).

II. TRANSFORMATION LAW OF THE FIELD EQUATIONS

If condition (1.2) is satisfied, the expressions L^A transform according to the equation

$$\bar{\delta} L^B = -F^{B\mu}{}_{A\nu} \xi^{\nu}{}_{,\mu} L^A - (L^B \xi^{\mu})_{,\mu}. \tag{2.1}$$

If we merely wish to assume that the expressions L^A transform in accordance with some linear, homogeneous transformation law, then it appears reasonable to start with an infinitesimal transformation law of the form

$$\bar{\delta} L^B = -G^{B\mu}{}_{A\nu} \xi^{\nu}{}_{,\mu} L^A - (L^B \xi_{\mu})_{,\mu} + H^{B\mu}{}_{A\nu} L^A{}_{,\mu} \xi^{\nu}, \tag{2.2}$$

where the $G^{B\mu}{}_{A\nu}$ and $H^{B\mu}{}_{A\nu}$ are constant coefficients as yet undetermined. The law (2.2) was chosen as the most general linear homogeneous transformation law with the correct order of differentiation.

The requirement of group character immediately leads to restrictions on the transformation coefficients. By forming the commutator of two arbitrary, infinitesimal transformations ξ'^{μ} and ξ''^{μ}, we find the two conditions

$$G^{C\nu}{}_{A\mu} G^{B\sigma}{}_{C\rho} - G^{C\sigma}{}_{A\rho} G^{B\nu}{}_{C\mu} = \delta^{\nu}{}_{\rho} G^{B\sigma}{}_{A\mu} - \delta^{\sigma}{}_{\mu} G^{B\nu}{}_{A\rho} \tag{2.3}$$

and

$$H^{B\mu}{}_{A\nu} = 0. \tag{2.4}$$

III. IDENTITIES

We can obtain the complete conditions satisfied by the transformation law of the field equations by forming $\bar{\delta} L^A$ (the infinitesimal change in L^A) as specified functions of the field variables and their first and second derivatives, and by equating the resulting expression to the right-hand side of (2.2), taking into account the restrictions already obtained in (2.3) and (2.4). On both sides of the equation we shall have terms containing as factors ξ^{μ} and their first, second, and third derivatives. These functions and their derivatives which generate the infinitesimal coordinate transformation are completely arbitrary at one point of space-time, except that the higher derivatives must satisfy the usual symmetry relations

$$\xi^{\alpha}{}_{,\mu\nu} = \xi^{\alpha}{}_{,\nu\mu}, \tag{3.1}$$

etc. Since, however, the transformation law for L^A must come out the same, whether we obtain it through the use of Eq. (2.2) or by the lengthy, straightforward calculation, the coefficients of the arbitrary functions and their various derivatives on the left must equal the corresponding coefficients on the right. By carrying out this computation, the following three sets of necessary identities were obtained:

$$F^{C\beta}{}_{B\alpha}\partial^{B\mu\nu}L^A\cdot y_C+F^{C\mu}{}_{B\alpha}\partial^{B\nu\beta}L^A\cdot y_C +F^{C\nu}{}_{B\alpha}\partial^{B\beta\mu}L^A\cdot y_C=0, \tag{3.2}$$

$$F^{C\beta}{}_{B\alpha}(\partial^{B\mu}L^A\cdot y_C+2\partial^{B\mu\nu}L^A\cdot y_{C,\nu})-\partial^{B\mu\beta}L^A\cdot y_{B,\alpha} +F^{C\mu}{}_{B\alpha}(\partial^{B\beta}L^A\cdot y_C+2\partial^{B\beta\nu}L^A\cdot y_{C,\nu}) -\partial^{B\beta\mu}L^A\cdot y_{B,\alpha}=0, \tag{3.3}$$

$$F^{C\beta}{}_{B\alpha}(\partial^{B}L^A\cdot y_C+\partial^{B\mu}L^A\cdot y_{C,\mu}+\partial^{B\mu\nu}L^A\cdot y_{C,\mu\nu}) +\delta^\beta{}_\alpha L^A-\partial^{B\beta}L^A\cdot y_{B,\alpha}-2\partial^{B\beta\nu}L^A\cdot y_{B,\alpha\nu} +G^{A\beta}{}_{B\alpha}L^B=0. \tag{3.4}$$

These three sets contain all the restrictions on the form of covariant Euler-Lagrange equations and their transformation laws. While it appeared to be very difficult to get the information contained in Eqs. (3.3) and (3.4) into a readily usable form, Eqs. (3.2) are identical with Eq. (I-3.6). Since it is these conditions which are necessary for the formulation of the algebraic relationships between the canonical momenta, (I-5.6), the further development of a theory with the more general transformation law (2.2) could be carried out along the same lines as one with the special law (2.1).

IV. FORMATION OF FURTHER IDENTITIES BY THE USE OF THE COMMUTATOR

Identities can be obtained in a different form [though contained in Eqs. (3.2), (3.3), (3.4)] if we form $(\bar\delta_1\bar\delta_2-\bar\delta_2\bar\delta_1)L^A$ in two different ways and equate the results. First, we can easily verify that

$$\bar\delta_1 L=L^A\bar\delta_1 y_A+(\partial^{A\rho}L\cdot\bar\delta_1 y_A)_{,\rho}. \tag{4.1}$$

From this relation we form

$$\bar\delta_3 L=(\bar\delta_1\bar\delta_2-\bar\delta_2\bar\delta_1)L =L^A(\bar\delta_1\bar\delta_2-\bar\delta_2\bar\delta_1)y_A+[\partial^{A\rho}L\cdot(\bar\delta_1\bar\delta_2-\bar\delta_2\bar\delta_1)y_A]_{,\rho}. \tag{4.2}$$

We can also form the transformation $\bar\delta_1$ of $\bar\delta_2$ directly:

$$\bar\delta_1(\bar\delta_2 L)=\bar\delta_1[L^A\bar\delta_2 y_A+(\partial^{A\rho}L\cdot\bar\delta_2 y_A)_{,\rho}] =\bar\delta_1 L^A\cdot\bar\delta_2 y_A+L^A\bar\delta_1\bar\delta_2 y_A+[\bar\delta_1(\partial^{A\rho}L\cdot\bar\delta_2 y_A)]_{,\rho}. \tag{4.3}$$

Forming the commutator from (4.3) and equating the result to the expression (4.2), we obtain

$$\bar\delta_1 L^A\cdot\bar\delta_2 y_A-\bar\delta_2 L^A\cdot\bar\delta_1 y_A +[\bar\delta_1(\partial^{A\alpha}L)\cdot\bar\delta_2 y_A-\bar\delta_2(\partial^{A\alpha}L)\cdot\bar\delta_1 y_A]_{,\alpha}=0. \tag{4.4}$$

The last term in this expression is a complete divergence, and hence the first two terms must also be a complete divergence. Writing out these first two terms and using (2.2) and (1.3), we can arrange them into an expression of the form $A_\mu\xi^\mu{}_2+B^\nu{}_\mu\xi^\mu{}_{2,\nu}$, where the coefficients A_μ and $B^\nu{}_\mu$ are themselves functions of the field variables, the coordinate variations $\xi^\alpha{}_1$, and their derivatives. This expression will be a complete divergence only if A_μ equals $B^\nu{}_{\mu,\nu}$. Since the $\xi^\alpha{}_1$ and their derivatives are arbitrary, except for symmetry of higher derivatives, at any one point of space-time, the coefficients of $\xi^\alpha{}_1$ and its derivatives must vanish identically. Following this straightforward but lengthy calculation, we find the following identities (the first being the coefficient of $\xi^\alpha{}_1$ and the others of the successive derivatives):

$$(F^{B\beta}{}_{A\kappa}y_B L^A{}_{,\iota}+G^{B\beta}{}_{A\kappa}y_{B,\iota}L^A+\delta^\beta{}_\iota y_{A,\kappa}L^A)_{,\beta}=0, \tag{4.5}$$

$$(G^{C\alpha}{}_{A\iota}F^{B\beta}{}_{C\kappa}-G^{C\beta}{}_{A\kappa}F^{B\alpha}{}_{C\iota}+\delta^\alpha{}_\iota F^{B\beta}{}_{A\kappa} -\delta^\beta{}_\kappa F^{B\alpha}{}_{A\iota})(y_B L^A)_{,\beta}+2\{F^{B\alpha}{}_{A\iota}y_B L^A{}_{,\kappa} +G^{B\alpha}{}_{A\iota}y_{B,\kappa}L^A+\delta^\alpha{}_\iota y_{C,\kappa}L^C\}_{(\iota\kappa)}\equiv 0, \tag{4.6}$$

and

$$\{y_C L^B F^{C\beta}{}_{A\kappa}(G^{A\alpha}{}_{B\iota}+\delta^A{}_B\delta^\alpha{}_\iota)\}^{(\alpha\beta)}{}_{[\iota\kappa]}=0. \tag{4.7}$$

In these three equations, the symmetrization or antisymmetrization with respect to certain indices is indicated by the symbols $\{\ \}^{(\alpha\beta)}$ and $\{\ \}^{[\alpha\beta]}$, respectively.

In these three sets of identities, if we let $G^{A\mu}{}_{C\nu}=F^{A\mu}{}_{C\nu}$, we find that (4.7) is satisfied identically, while (4.5) and (4.6) yield the contracted Bianchi identities. When $F^{A\mu}{}_{C\nu}\neq G^{A\mu}{}_{C\nu}$, (4.5), (4.6), and (4.7) are all together 210 identities, which may, however, possess some mutual algebraic dependence.

V. CONSERVATION LAWS

In order to form "strong" conservation laws, we must find sixteen functions whose divergence is identically zero whether the field equations are satisfied or not. The sixteen functions

$$t^\rho{}_\iota=\delta^\rho{}_\iota L-y_{A,\iota}\partial^{A\rho}L \tag{5.1}$$

have zero divergences when the field equations are satisfied, for

$$t^\rho{}_{\iota,\rho}\equiv y_{A,\iota}L^A. \tag{5.2}$$

In the presence of matter the field equations are not equal to zero; but

$$L^A=P^A, \tag{5.3}$$

where P^A is representative of the distribution of matter in the field. When $P^A\neq 0$, we must subtract from (5.1) a function whose divergence will cancel the expression $y_{A,\iota}L^A$ or $y_{A,\iota}P^A$. We shall construct such a function by using the identities that can be established in the formalism. With no restrictions on $F^{A\alpha}{}_{B\beta}$ and $G^{A\alpha}{}_{B\beta}$, it appears impossible to form such expressions. However, with a relation between $F^{A\alpha}{}_{B\beta}$ and $G^{A\alpha}{}_{B\beta}$, it is possible to form sixteen functions whose divergence is always zero.

From (4.6), contracting on κ and α and solving for $y_{A,\iota}L^A$, we obtain

$$5y_{C,\iota}L^C+(F^{B\rho}{}_{A\rho}y_B L^A{}_{,\iota}+G^{B\rho}{}_{A\rho}y_{B,\iota}L^A) +(F^{B\rho}{}_{A\iota}y_B L^A{}_{,\rho}+G^{B\rho}{}_{A\iota}y_{B,\rho}L^A) +(G^{C\sigma}{}_{A\iota}F^{B\rho}{}_{C\sigma}-G^{C\rho}{}_{A\sigma}F^{B\sigma}{}_{C\iota})(y_B L^A)_{,\rho}\equiv 0. \tag{5.4}$$

We note that the last term on the right is a divergence, but the others are not unless

$$G^{B\alpha}{}_{A\iota} = F^{B\alpha}{}_{A\iota}. \tag{5.5}$$

In that case, we can form the sixteen functions

$$T^{\rho}{}_{\iota} = t^{\rho}{}_{\iota} + F^{B\rho}{}_{A\iota} L^{A} y_{B}, \tag{5.6}$$

whose divergence is identically zero whether the L^A vanish or are equal to P_A. This expression is identical with (I-3.11).

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The Mass of S^{35} from Microwave Spectroscopy*

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The $J=1\rightarrow2$ rotational absorption transition in OCS has been observed for the molecules containing S^{35} and S^{34}. From the frequencies, the frequency differences, and the previously known frequencies of S^{34}, S^{33}, and S^{32} we have evaluated the mass difference ratios $(S^{35}-S^{32})/(S^{34}-S^{32})$ and $(S^{35}-S^{32})/(S^{33}-S^{32})$. From these values and values of the stable S masses two independent values of the $(S^{35}-S^{32})$ mass differences are calculated to be 2.99844 ± 0.00042 and 2.99770 ± 0.00048, respectively.

I. INTRODUCTION

We have remeasured[1] the frequencies of the $J=1\rightarrow2$ molecular rotational absorption transitions of OCS containing S^{32}, S^{33}, S^{34}, and S^{35} for the purpose of determining the mass of S^{35} and of evaluating the nuclear quadrupole interaction to a higher accuracy.

II. METHOD

Our apparatus, somewhat similar to various other K-band spectroscopes described in the literature,[2,3] is illustrated schematically in Fig. 1. It utilizes 100-kc/sec Stark effect modulation and for maximum sensitivity a phase-sensitive detector at the output. Our frequencies were measured by means of variable microwave frequency markers obtained from a frequency standard somewhat similar to those used in other laboratories.[4] Figure 2 shows a schematic arrangement of the system. The basis of our measurements is a General Radio 100-kc/sec crystal-controlled secondary frequency standard calibrated against Radio Station WWV.

As shown below in Eq. (3) the mass of S^{35} can be expressed in terms of frequency differences and ratios. In determining such differences small systematic errors, such as are caused by delays in the spectroscope amplifier, tend to cancel out of differences and ratios of nearly equal frequencies.

Because of the electric quadrupole moment of the odd S isotopes, the $J=1\rightarrow2$ transition is split into several components which are only partially resolved.[1] From the shape of the pattern one can infer the nuclear spin, while for the magnitude of the separations one may evaluate the quadrupole coupling constant.[5]

The significant spectral frequency referred to in the Eqs. (2) and (3) are those of the center of gravity of the $J=1\rightarrow2$ group of lines for one isotopic molecule. The displacement of the strong central line can be evaluated from the quadrupole coupling constant.[5]

Table I contains a summary of our data on S^{35} along with comparable results for the stable isotopes as measured by Geschwind and Gunther-Mohr.[6]

By taking the ratio of the intervals between the upper and lower minor components and the central one for S^{35} one gets a value of 1.48, which is in excellent agreement with the theoretical ratio of 1.46 for a nuclear spin of $\frac{3}{2}$. Clearly, the sign of the quadrupole moment of S^{35} is opposite that of S^{33}, since the patterns are inverted with respect to each other.[7] The value of the quadrupole constant is 20.5 ± 0.2 Mc/sec. Townes and Dailey[8] have made a rough calculation of the molecular electric field gradient in OCS to evaluate the electric quadrupole moment of S^{33}. Using their figures, we get a value of 0.06×10^{-24} cm^2 for the electric quadrupole moment of S^{35}, which is considered to be good to within a factor of 2.

As a result of the quadrupole interaction, the strong central line is shifted by 0.440 ± 0.003 Mc/sec down in frequency. For the most reliable value of the OCS^{35} frequency we took the value of the OCS^{34} frequency

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