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Renormalization of Neutral Mesons in Three-Field Problems

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It is shown that a finite S-matrix can be obtained to low orders by renormalization if neutral spin zero mesons with (pseudo) scalar coupling are added to the mixture of nucleons, photons, and charged spin zero mesons previously considered, provided that a limited number of contact interactions are introduced. A general proof for any order is not given; but it is shown that at no stage can divergences of a new form appear, requiring the introduction of additional contact terms.

Renormalization certainly fails for the pseudovector interaction and the possibility of defining a finite S-matrix, formally, with the use of regulators, is considered. It is shown that it is not possible to do this in a manner which is both self-consistent and consistent with the renormalization of the pseudoscalar coupling. This result is related to the "ambiguities" found in the calculation of three-field problems with the use of regulators only, and is illustrated in the appendix by a detailed discussion of the two-photon decay of a neutral spin zero meson. It is concluded that cross sections for interactions, for which renormalization is inadequate, can certainly not be calculated in this manner, and that if, in fact, such theories have any meaning, it must be possible to remove the divergences in a more fundamental way.

I. INTRODUCTION

IT has already been shown[1,2] that, at least to fourth order in the coupling constants, a finite S-matrix can be obtained by renormalization for the combined interactions of nucleons, photons, and charged spin zero mesons with (pseudo) scalar coupling. For any interaction involving mesons it is necessary to introduce a direct interaction between four mesons in order to cancel the divergences arising from scattering of mesons by mesons.[3] No additional contact term is required for the above three-field mixture where the mesons are charged. The problem is complicated in higher orders by secondary infinities from the contact interaction[1] and the "overlapping" of divergences.[4] However, these problems have now been solved in detail by Salam[5] for the interactions of the separate pairs of fields, and there is reason to believe that the S-matrix for the above charged meson mixture is also finite to any order after renormalization.

In view of the very definite experimental evidence[6] for neutral spin zero mesons, it is of interest to consider what modifications are required if neutral mesons are included in the above mixture with (pseudo) scalar coupling.[7] It is found that the number of possible types of primitive divergent is again strictly limited. For pseudoscalar mesons it is necessary to introduce two extra contact terms coupling four mesons, and for scalar mesons three more contact terms coupling three mesons. No divergences of any new form can ever appear in the calculations requiring the introduction of still further contact terms.

The difficulties of secondary infinities from the four-meson contact terms are the same as in the charged meson mixture. The three-meson terms are particularly innocuous in this respect, giving rise to only one divergent self-energy graph.[8] Thus, the introduction of neutral spin zero mesons with (pseudo) scalar coupling introduces no essentially new difficulty into the three-field renormalization problem, and it is again probable that a finite S-matrix can be defined up to any order.

The method of renormalization can certainly not be applied if pseudovector coupling is assumed.[3] It is here shown, further, that it is not even possible to define a finite S-matrix for this coupling by the formal use of regulators[9,10] in a manner which is both self-consistent and consistent with the renormalization of the pseudoscalar coupling. This is discussed in relation to the three-field calculations of Ozaki, Onenda, and Sasaki[11] and of Fukuda, Miyamoto, and Hayakawa[12] (to be referred to as OOS and FMH, respectively). These authors applied regulators, formally, to both pseudoscalar and pseudovector coupling and found "ambiguities" in their results for meson decay life times. These calculations can now be separated into those which are equivalent to renormalization (and thus have a reasonable theoretical basis) and those which are definitely inconsistent with it and quite without foundation. This point is illustrated in the appendix, where the two-photon decay of a neutral pseudoscalar meson is considered in detail. For this process the two possible couplings are equivalent to lowest order, and regularization must not disturb this equivalence. With this

[1] P. T. Matthews, Phys. Rev. **80**, 292 (1950). It is hoped to publish elsewhere a more detailed account of this work.
[2] A. Salam, Phys. Rev. **79**, 910 (1950).
[3] P. T. Matthews, Phil. Mag. **41**, 185 (1950); Phys. Rev. **80**, 292 (1950). F. Rohrlich, Phys. Rev. **80**, 666 (1950).
[4] F. J. Dyson, Phys. Rev. **76**, 1736 (1949), §VII.
[5] A. Salam (to be published). Salam has treated the case when the contact interaction just cancels the divergences and no "real" contact interaction is allowed for.
[6] Steinberger, Panofsky, and Steller, Phys. Rev. **78**, 802 (1950). References to further experimental evidence are given in this paper.
[7] Particular calculations have been made in this frame work by K. M. Case, Phys. Rev. **76**, 1 (1949), and K. A. Brueckner, Phys. Rev. **79**, 641 (1950).
[8] J. C. Ward, Phys. Rev. **79**, 406 (1950).
[9] W. Pauli and F. Villars, Revs. Modern Phys. **21**, 434 (1949).
[10] J. Steinberger, Phys. Rev. **76**, 1180 (1949).
[11] Ozaki, Onenda, and Sasaki, Prog. Theor. Phys. **4**, 524 (1949); **5**, 25, 165, 373 (1950). Also Y. Katayama, Prog. Theor. Phys. **5**, 272 (1950).
[12] H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4**, 347, 392 (1949). Fukuda, Miyamoto, and Hayakawa, Prog. Theor. Phys. **5**, 283, 352 (1950). Also a joint paper by Tomonaga, *et al.*, Prog. Theor. Phys. **4**, 477 (1949).

condition, there are two ways in which regulators can be applied. One of these is rejected because it would not lead to a finite S-matrix for the pseudovector coupling and the other because it removes finite gauge invariant terms, in contradiction to the renormalization of the pseudoscalar coupling.

II. RENORMALIZATION

The hamiltonian for the three-field mixture with the addition of neutral pseudoscalar mesons with pseudoscalar coupling is the same as that considered in reference 1 with the addition of a term H_4, where

$$H_4 = i\bar{\psi}\gamma_5(g_1\tau_P + g_2\tau_N)\psi\phi', \quad (1)$$

and ϕ' is the neutral meson potential ($\phi'(x)$ real). (Scalar mesons are obtained by replacing $(\gamma_5)_{\alpha\beta}$ by $-i\delta_{\alpha\beta}$.)

This introduces two new types of vertex with one neutral meson line and either two neutron or two proton lines (to be referred to as neutron and proton vertices, respectively).

If the number of external neutral meson lines is E_m', the condition for primitive divergents is found by Dyson's[4] method to be

$$\tfrac{3}{2}E_n + E_p + E_m + E_m' < 5. \quad (2)$$

Any graph with $E_n = E_m = 0$, and E_p odd, is excluded by Furry's theorem.[13] The only remaining new primitive divergents (apart from self-energy and vertex parts to be considered below) are (i) $E_p=1$, $E_m'=1$, $E_m=2$; (ii) $E_p=2$, $E_m'=1$; (iii) $E_p=2$, $E_m'=2$; (iv) $E_m'=3$; (v) $E_m'=1$, $E_m=2$; (vi) $E_m'=4$; (vii) $E_m'=2$, $E_m=2$.

Graphs of type (i) are at most logarithmically divergent; but from considerations of invariance the whole expression is a vector. Since the leading term cannot be a vector, it must vanish identically and the contribution from such graphs is convergent.

Graphs of type (ii) give the decay of a neutral meson into two photons. This process has been observed[6] and has been considered in low orders by Steinberger,[10] by Fukuda and Miyamoto,[12] and by OOS.[11] From considerations of invariance the integral must be expressible in the form (k_1 and k_2 are momentum vectors of the photons)

$$D\delta_{\mu\nu} + (k_{1\lambda}k_{2\lambda} - k_{1\mu}k_{2\nu})E + k_{1\mu}k_{2\nu}F + \epsilon_{\mu\nu\sigma\rho}k_{1\sigma}k_{2\rho}G + \text{conv. integral},$$

where D may be infinite, but E, F, and G are finite constants; but for gauge invariance to be satisfied D must be zero,[14] (The terms in E, F, and G have been chosen to satisfy gauge invariance), and the integral converges.

Type (iii) is another possible real process which is potentially logarithmically divergent. The leading term $A\delta_{\mu\nu}$ can be separated off by putting the photon momenta zero and the meson momenta equal and satisfying the free field equation. For gauge invariance A is identically zero,[15] and again all integrals from such graphs converge.

If the neutral mesons are pseudoscalar, graphs of types (iv) or (v) are excluded by a "Lorentz invariant" type of selection rule as introduced by FMH.[16] No such argument is applicable for scalar mesons and in this case genuine divergences occur. These must be cancelled by the introduction of two more direct interaction terms,

$$\chi_1\phi'^3 + \chi_2\phi^*\phi\phi',$$

where χ_1 and χ_2 are infinite constants, so chosen as to cancel the infinite contributions from the graphs, and leave a finite matrix element in agreement with the experimental value, given by the scattering of mesons in an external meson field. The only secondary divergence introduced by the inclusion of the χ-terms is a single self-energy part.[8] The infinite terms may be absorbed in the mass and "charge" renormalizations *after* the cancellation of the infinities from the three meson parts of the χ-terms.

The last two types, (vi) and (vii), are new forms for the scattering of mesons by mesons, which give rise to genuine divergences. They must be cancelled by the introduction of direct interactions $\lambda_1\phi^*\phi\phi^2$ and $\lambda_2\phi^4$, where the infinite constants λ can be chosen to give agreement with experiment.[17] They introduce difficulties of secondary infinities similar to those for separate pairs of fields.[5]

It remains to be seen what renormalizations of the coupling constants are necessary to remove infinities from self-energy and vertex parts. The proton and neutron self-energy parts are each modified by virtual neutral mesons. The charged meson vertex parts are also modified, but from any such part with the proton entering (τ_- vertex) a corresponding part with the neutron entering (τ_+ vertex) can be obtained by reversing all arrows in the graph. Thus, the same infinities are associated with both τ_+ and τ_- vertices. Since with both such vertices are associated half a proton and half a neutron line, the same renormalization is applicable to both, as required by the single coupling constant f.

For renormalization, the neutron vertex of a neutral meson is associated with a neutron line, the proton vertex with a proton line. Further, there is no one-to-one correspondence between neutron vertex parts and proton vertex parts. The possibility that these two discrepancies cancel, leading to the same infinite con-

[13] W. Furry, Phys. Rev. **51**, 125 (1937).

[14] For scalar mesons D does not vanish on account of γ-factors, but from symmetry considerations employed by Schwinger for the photon self-energy. J. Schwinger, Phys. Rev. **6**, 790 (1949).

[15] This can also be proved by the method due to Ward, since $A\delta_{\mu\nu}$ can be expressed as the integral of a double derivative of a neutral meson self-energy. J. C. Ward, Phys. Rev. **77**, 293 (1950).

[16] An explicit proof due to F. J. Dyson is to be published in a paper by A. Salam.

[17] The constant is taken to be $\lambda = \lambda' + \delta\lambda(\lambda', e, f, g)$ where λ' is independent of e, f, and g and chosen to fit experiment, and $\delta\lambda$ is chosen to cancel divergences.

stants for both vertices, can be ruled out by considering the terms dependent on e^2. Thus different renormalizations are required for the two constants g_1 and g_2.[18]

Summarizing, to obtain a finite S-matrix by renormalization for the interaction of nucleons, photons, and charged spinless mesons with (pseudo) scalar coupling the interaction must involve the three constants e, f, and λ. If the mesons are pseudoscalar and neutral pseudoscalar mesons are also included, it is necessary to introduce the constants e, f, g_1, g_2, λ, λ_1, and λ_2. If all the mesons are scalar, then the additional constants χ_1 and χ_2 are also certainly necessary for a finite theory. The renormalized value of all these constants can (theoretically) be determined by experiment.

With this scheme a finite S-matrix can certainly be obtained by renormalization to fourth order in the "charges" e, f, and g. In higher orders the problem is complicated by the overlapping of divergences[4] and the secondary infinities arising from the contact terms.[1] A general proof has not yet been given that renormalization can be carried out consistently to any order. However, it has been shown here that at no stage can divergences of a new form appear in the theory, involving the necessity of introducing any additional constants.

Finally we remark that it would be possible to include the electron-positron field coupled to the electromagnetic field with the same renormalization of the constant e.[19] It would also be possible to add neutral vector mesons with vector coupling and to remove all additional divergences by a renormalization of the single new coupling constant.[20] If any other type of particle or coupling were included, Dyson's method sets no limit on the possible types of primitive divergent[21] and renormalization certainly fails to define a finite S-matrix. (We have not considered here the possibility of mixing meson fields with relations between the coupling constants.)

III. REGULARIZATION AND "AMBIGUITIES"

The clear distinction between meson theories for which renormalization is effective and those for which it is not provides a criterion for settling the "ambiguities" which have been met by OOS[11] and by FMH[12] in the calculation of meson decay lifetimes.

In applying renormalization, the theory is always set up in gauge invariant form; and the condition that the resulting S-matrix elements must be gauge invariant is used to *define* the values of certain integrals, which would otherwise be ambiguous owing to the singularities of the Δ-functions at the origin. A detailed evaluation of these integrals is then correct only if it yields a result consistent with this definition. In practice, it is often found that the required result is obtained because the γ-factors of potentially divergent terms vanish identically (as for the scattering of light-by-light); but in other cases the correct result is only derived by making full use of the symmetry properties of the Δ-functions inherent in their general definitions.[22] This has been illustrated by the calculations of the photon self-energy by Schwinger[14] and Wentzel.[23] "Making full use of symmetry properties" is not a very well-defined mathematical process in momentum space, and in order to make the procedure more precise the ambiguous integrals can be defined by the regulators of Pauli and Villars.[9] This is a purely formal procedure in agreement with renormalization for any theory to which renormalization is applicable.

Now the procedure of regularization has been extended by Steinberger[10] to determine a finite S-matrix for *any* of the standard meson interactions by imposing just six conditions on the subsidiary fictitious masses which are introduced. However, these extended theories are purely formal, as demonstrated by Pais and Uhlenbeck;[24] and there is really no reason to believe them to be correct. Renormalization, on the other hand, has a sound (though not rigorous) theoretical basis and for electrodynamics is in very good agreement with experiment. Thus, it would seem that if a conflict should arise between a renormalized and a regularized theory, it is certainly the latter which should either be modified or possibly discarded

This is precisely the situation with the pseudoscalar and pseudovector interactions of pseudoscalar mesons. The former can be renormalized; the latter is finite only when regularized; but the two theories can be shown to be equivalent for certain effects in low orders.[25] If the coupling constants are taken to be the same for both couplings, the equivalence theorem, when it is valid, has the form

$$[pv] = -2\kappa_0[ps], \tag{3}$$

where $[ps]$ and $[pv]$ are the S-matrix elements and κ_0 is the nucleon reciprocal Compton wavelength. The difficulty arises because of the term κ_0 in Eq. (3). If the equivalence theorem is to be maintained, one must either

(A) apply Steinberger's conditions to $\kappa_0[ps]$ and $[pv]$

or

(B) apply Steinberger's conditions to $[ps]$ and $[pv]/\kappa_0$.

These alternatives were suggested by OOS,[11] who have calculated according to (A); but this is not equivalent

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[18] If electromagnetic effects are excluded, the interaction of nucleons with both charged and neutral spin zero mesons can be expressed in Kemmer's symmetric form with a single coupling constant g and one λ- and one χ-term; but this breaks down when the photon field is included. N. Kemmer, Proc. Cambridge Phil. Soc. **34**, 354 (1938).

[19] A particular calculation with this combination of fields has been made by M. N. Rosenbluth, Phys. Rev. **79**, 615 (1950).

[20] P. T. Matthews, Phys. Rev. **76**, 1254 (1949).

[21] P. T. Matthews, Phil. Mag. **41**, 185 (1950).

[22] J. Schwinger, Phys. Rev. **75**, 1439 (1948).

[23] G. Wentzel, Phys. Rev. **74**, 1070 (1948).

[24] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

[25] F. J. Dyson, Phys. Rev. **73**, 929 (1948). K. M. Case, Phys. Rev. **76**, 14 (1949).

to the renormalization of the pseudoscalar coupling and is ruled out by the criterion developed above. (It has the effect of dropping finite and gauge invariant terms through the new condition $\Sigma C_i/\kappa_i=0$ on $[ps]$.)

Alternative (B) is equally unsatisfactory because, when applied in general to the pseudovector coupling, it will not give a finite S matrix (since effectively the condition $\Sigma C_i=0$ has now been removed).

There remains the third possibility of abandoning the equivalence theorem and applying Steinberger's conditions to both $[ps]$ and $[pv]$. This is the procedure which has been followed by FMH,[12] but it is equally objectionable as (A) or (B) because the equivalence theorem is based on the evaluation of integrals by using just those symmetry properties of the Δ-functions which have given the correct values (that is, in accordance with gauge invariance) for renormalized theories. It is quite unreasonable to accept this method of calculation in one case and reject it in another.

Thus, we can say that when renormalization is applicable, it leads to unique results. There is no way consistent with renormalization of extending the formal procedure of regularization to theories to which renormalization is not applicable. The only self-consistent application of regularization to all theories, (A), is incompatible with renormalization and has the effect of removing finite and gauge invariant terms.

We conclude that a formal extension of regularization definitely fails for interactions which cannot be renormalized If, in fact, such theories have any physical significance, it must be possible to remove the divergences from the S-matrix in a more fundamental way.

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APPENDIX

The decay of a neutral pseudoscalar meson into two photons (k_1, μ) and (k_2, ν) is here discussed in detail. For pseudovector coupling the matrix element is

$$[pv]=(-i/2\hbar c)^3\int dx_1dx_2dx_3Tr[\gamma_5\gamma_\rho S_F(x_1-x_2)\times\gamma_\mu S_F(x_2-x_3)\gamma_\nu S_F(x_3-x_1)]A_\mu(x_2)A_\nu(x_3)d\phi(x_1)/dx_{1\rho}. \quad (4)$$

Integration by parts with respect to x_1 and use of the relation

$$\gamma_\mu\partial S_F(x)/\partial x_\mu=\kappa_0S_F(x)-2i\delta(x), \quad (5)$$

yields

$$[pv]=-2\kappa_0[ps]+2i(-i/2\hbar c)^3\int dx_2dx_3\times\{Tr[\gamma_5\gamma_\mu S_F(x_1-x_3)\gamma_\nu S_F(x_3-x_1)]+Tr[\gamma_5S_F(x_1-x_2)\gamma_\mu S_F(x_2-x_1)\gamma_\nu]\}. \quad (6)$$

For the equivalence theorem to be valid for this problem, it must be shown that the terms arising from the δ-function in Eq. (5) vanish. (In Case's[25] proof of the equivalence theorem it is assumed tacitly that the nucleons are free and these terms are not considered.) Using the symmetry property, $\Delta_F(x)=\Delta_F(-x)$ and noticing that only the γ-dependent terms of the S_F factors give a non-zero trace,

$$Tr[\gamma_5\gamma_\mu S_F(x_1-x_3)\gamma_\nu S_F(x_3-x_1)]=-Tr[\gamma_5\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta]\partial\Delta_F(x_1-x_3)\partial\Delta_F(x_1-x_3)/\partial x_{1\alpha}\partial x_{1\beta}. \quad (7)$$

This can be shown to vanish identically by interchanging the suffixes α and β. The final term vanishes similarly, thus proving the validity of the equivalence theorem for this effect. (Similar proofs have been given OOS[10] and FMH.[11])

The physically significant term $[ps]$ has been calculated by Steinberger.[10] It is potentially logarithmically divergent, but is in fact, absolutely convergent in accordance with gauge invariance.

It is of interest to consider the two vanishing terms in momentum space, because it is through them that the so called "ambiguities" arise. The first is easily shown to be proportional to

$$\int d^4p\frac{Tr[\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu]p_\beta k_{2\alpha}}{[(p+k_2/2)^2+\kappa_0^2][(p-k_2/2)^2+\kappa_0^2]}, \quad (8)$$

which is the form given by Steinberger.[10] Using Feynman's identity[26]

$$1/ab=\int_0^1 dx/[(a-b)x+b], \quad (9)$$

and making the transformation

$$p=q+k(\tfrac{1}{2}-x),$$

the integral reduces to

$$-\tfrac{1}{2}k_{2\alpha}k_{2\beta}\int d^4q\int_0^1 dx(2x-1)/[k_2^2(x^2-x)-q^2-\kappa_0^2]^2, \quad (10)$$

which vanishes identically when integrated over x, in accordance with calculation in configuration space. A similar calculation shows that the third term also vanishes. Thus, $[ps]$ is the unique result given by renormalization, consistent with gauge invariance and the equivalence theorem.

The same result would have been obtained for pseudoscalar coupling by regularization, but it is not possible to introduce regulators for pseudoscalar and pseudovector coupling without inconsistences. Alternative (A), used by OOS,[11] would remove finite and gauge invariant terms from $[ps]$. Alternative (B) would give the above result for this effect but would not, in general, give a finite S-matrix for pseudovector coupling.[27] The other possibility is to apply Steinberger's conditions separately to pseudoscalar and pseudovector coupling so that the same integrals are evaluated differently for the two cases. This was done by Steinberger[10] and FMH,[12] both stressing the inconsistency of their method. (These calculations are in agreement, however, with renormalization for pseudoscalar coupling.)[28]

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[26] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

[27] Note that (B) is really not even sufficient for the present problem, because the condition $\Sigma C_i=0$ on $[pv]$ (excluded by (B)), is required to make the integral in Eq. (6) definitely zero, if no symmetry arguments are used and one relies on regulators alone to evaluate ambiguous integrals.

[28] After the completion of this work, a method of calculation equivalent to a much more fundamental introduction of regulators was given by Schwinger. In its application to the meson—two-photon decay problem Schwinger, in effect, regularized according to (B). As stated in the text, this does not, in general, lead to a finite S-matrix for pseudovector coupling, because one of the required conditions is missing. This shows itself in Schwinger's calculation in that the "regulators" operate effectively on the partial derivative of $[pv]$ and the integrated terms [corresponding exactly to the integral in Eq. (6)], must be removed by what is essentially a symmetry argument as described in footnote 27 (J. Schwinger, Physics Seminar, Institute for Advanced Study, Princeton, October 17, 1950).