The Production of Mesons by Protons on Deuterons*

H. P. NOYES
Radiation Laboratory, Department of Physics, University of California, Berkeley, California
(Received September 25, 1950)

An attempt is made to predict the energy and angular distribution of charged mesons to be expected from the bombardment of deuterium by 345-Mev protons assuming that the cross section for production in two-nucleon collisions is known. The problem is simplified by assuming that the third particle enters the reaction only by giving to the struck particle a momentum distribution at some time prior to the meson production, by carrying off the complementary momentum, and by limiting the states available through the exclusion principle. The possibility of the reformation of a deuteron more than doubles the cross section for the production of positive if it occurs. Hence the positive-negative ratio offers an experimental means of determining the spin dependence of positive meson production in the final state, assuming that the ratio of p–n to p–p production is known. Alternatively, this approach offers a method for a rough determination of the p–n meson production cross section using high energy proton beams.

I. INTRODUCTION

THE purpose of this paper is to attempt to predict the energy and angular distribution of mesons resulting from the bombardment of deuterium by 345-Mev protons, under the assumption that the cross section for the production of mesons in two-nucleon collisions is known. Alternatively, since high energy neutron sources of sufficient intensity and energy resolution for an accurate, direct study of meson production in n–p collisions have not yet been achieved, this analysis can offer a method for learning something of this n–p cross section. The production of mesons in proton-proton collisions in being investigated experimentally by Richman, Wilcox, Whitehead, Cartwright, and Peterson;[1] and Brueckner has conducted a theoretical investigation of the problem in the light of these experiments.[2]

The general method of attacking the problem was suggested to the author by Chew, in analogy with an approach he is using in the study of n–d inelastic scattering.[3] It rests upon two assumptions, of which the first is probably justified, while the second is open to considerable question. The first assumption is that the production takes place in a time so short compared with the period of the deuteron that the impulse approximation may be used. Since for the incident proton $v/c = 0.682$, the time for it to cross a meson Compton wavelength is only two percent of the deuteron period. Hence it seems to be reasonable to assume that the problem can be treated in terms of the production of mesons by two nucleons, one of which has the momentum distribution of a particle in the deuteron, while the third particle simply carries off the complementary momentum without otherwise entering the reaction. The general examination of the errors made in impulse approximations of this type is to be discussed in a forthcoming paper by Wick and Chew.

The more dubious assumption is that the particles which produce the meson do not then interact with the third particle except insofar as the exclusion principle limits the states available to them. This is essentially the same approximation as the neglect of double scattering from alternate particles in the inelastic scattering of nucleons by deuterons. In the latter problem one of the three particles must have a large momentum relative to a pair of particles that are interacting strongly. As Chew has shown, this allows a straightforward treatment in terms of two-body interactions. When a meson is formed, however, it is quite possible for all three nucleons to have comparable momenta, so that exact treatment would require a solution of the three-body problem.

Still, it can be argued that the situation is not so desperate as to invalidate the method used below. Firstly, when the three momenta are comparable, the exclusion principle causes a compensating reduction in the cross section. A further limitation occurs in that the phase space available to the three particles in this energy region (i.e., the region of high meson energy) is small. Outside of this region the effect is small. In fact, if the third particle has a relative energy of 40 Mev or greater, as is true on the average for most of the distribution except the high energy tail already discussed, its cross section at the average distance from the two interacting particles (4 or 5 wavelengths at this energy) covers less than six percent of the solid angle into which these particles may go. Moreover, the interaction between the two particles which produce the meson can be taken into account to the same extent as was done by Brueckner. This interaction completely alters the two nucleon distribution both in magnitude and shape; it increases the magnitude of the p–d cross section by a corresponding amount but has much less influence on the shape. Thus, the results given below should be at least qualitatively correct, except possibly

* The work described in this paper was performed under the auspices of the AEC.
[1] Wilcox, Cartwright, Richman, and Whitehead, Phys. Rev. **79**, 198 (1950). C. Richman and H. A. Wilcox, private communication. V. Peterson, Phys. Rev. **79**, 407 (1950), and private communication.
[2] K. Brueckner, Phys. Rev. 79, 641 (1950), and private communication.
[3] G. F. Chew, Phys. Rev. **80**, 196 (1950).

for the high energy tail. Clearly this method ignores the possibility of the formation of a triton, but the results indicate that this process should be separable experimentally from the main body of the distribution.

II. DERIVATION OF THE SCATTERING MATRIX

The formal statement of the above assumptions and derivation of the scattering matrix will be carried out in terms of the R matrix notation.[4] The assumption that the two-nucleon cross section is known can be stated as knowledge of appropriate two-particle R matrices. These are then combined with the assumption that the third particle influences the reaction only by giving a deuteron momentum distribution to one of the interaction particles at some time previous to the production of the meson, to give an R matrix for the problem at hand.

In order to clarify the notation, consider the two-particle case briefly. Let the momentum and spin variables of the incident proton and struck neutron be denoted by ξ_0 and ξ_n, respectively, the final neutron variables by ξ_1 and ξ_2, and the positive meson variables by η^+. Then $R_{np}{}^+$, which describes the transformation of a proton ξ_0 and a neutron ξ_n into two neutrons ξ_1 and ξ_2 and a positive meson η^+, can be written as $(\xi_1\xi_2\eta^+|R_{np}{}^+|\xi_0\xi_n)$. Momentum conservation may be factored out giving

$$(\xi_1\xi_2\eta^+|R_{np}{}^+|\xi_0\xi_n)=\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{q}-\mathbf{K}_0-\mathbf{K}_n)\times(\tfrac{1}{2}(\mathbf{k}_1-\mathbf{k}_2),\sigma_1,\sigma_2,\mathbf{q}|r_{np}{}^+|\tfrac{1}{2}(\mathbf{k}_0-\mathbf{k}_n),\sigma_0,\sigma_n),\quad(1)$$

where the spin and momentum variables of the nucleons have been introduced explicitly. Note that this separation restricts us to treating the nucleons nonrelativistically throughout (except that the incident proton may be treated relativistically in calculating the energy and momentum available for the reaction; the treatment of the final nucleons can be shown to be a good approximation). The cross section for the production of positive mesons in an n–p collision is then to be written as

$$d\sigma_{np}{}^+=(2\pi/\hbar v_0)|r_{np}{}^+|^2\delta(E_1+E_2+E_q-E_0-E_n)\times[d\mathbf{k}'/(2\pi)^3][d\mathbf{q}/(2\pi)^3],\quad(2)$$

where $\mathbf{k}'=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$ and all other variables in $r_{np}{}^+$ are to be expressed in terms of $\mathbf{k}'$, $\mathbf{q}$ (and $\mathbf{k}_0$) by means of momentum conservation. That is, formally, $r_{np}{}^+$ appears in the cross section in the same way as a matrix element for the transition calculated in Born approximation; this analogy is useful in practice. In fact, the whole calculation is formally equivalent to a second-order perturbation calculation assuming that the nuclear forces and meson production arise from separate interaction terms in the hamiltonian and that the corresponding second-order calculation for the two-particle case gives the correct answer.

[4] C. Möller, Det. Kgl. Danske Vidensk. **XIII**, No. 1 (1946).

In terms of this notation the (unsymmetrized) R matrices for the deuteron problem under our assumptions may be written, for negatives:

$$(\xi_1\xi_2\xi_3\eta^-|R_{pd}{}^-|\xi_0\xi_D)=\sum_{\xi_0'\xi_n\xi_p}(\xi_1\xi_2\eta^-|R_{np}{}^-|\xi_0\xi_n)\delta_{\xi_p\xi_3}\times\psi_D(\xi_n\xi_p)\psi_{\xi_0}(\xi_0')\quad(3)$$

and for positives:

$$(\xi_1{}^{(n)}\xi_2{}^{(n)}\xi_3{}^{(p)}\eta^+|R_{pd}{}^+|\xi_0\xi_D)=\sum_{\xi_0'\xi_n\xi_p}\{\exp(i\delta_{np})\cdot(\xi_1\xi_2\eta^+|R_{np}{}^+|\xi_0\xi_n)\delta_{\xi_p\xi_3}+\exp(i\delta_{pp})\cdot(\xi_1{}^{(n)}\xi_3{}^{(p)}\eta^+|R_{pp}{}^+|\xi_0\xi_n)\delta_{\xi_n\xi_2}\}\times\psi_D(\xi_n\xi_p)\psi_{\xi_0}(\xi_0'),\quad(4)$$

where $\psi_{\xi_0}(\xi_0')$ is the incident plane wave $\delta_{\xi_0\xi_0'}$, $\psi_D(\xi_n\xi_p)$ is the deuteron function $\chi_D(\sigma_n\sigma_p)g_0(\frac{1}{2}\mathbf{k}_n-\frac{1}{2}\mathbf{k}_p)\delta(\mathbf{k}_n+\mathbf{k}_p-2\mathbf{k}_c)$, and $2\mathbf{k}_c$ is the momentum of the deuteron in whatever coordinate system is chosen. These two equations contain the formal statement of the assumptions (a) that the third particle influences the production of the meson only through giving to the struck particle a deuteron momentum distribution at some time previous to the collision, and (b) that the three nucleons go directly into the final state $\xi_1\xi_2\xi_3$ without further interaction not contained in the two-nucleon $R_{np}{}^\pm$ and $R_{pp}{}^+$. According to the argument given in the introduction the interaction thus included should contain most of the influence of forces between the particles in the final state. Equation (3) is to be antisymmetrized in the three final neutrons $\xi_1\xi_2\xi_3$, and (4) in the two final neutrons; the corresponding r matrices are then to be squared, averaged over the initial proton and deuteron spin states, and summed over the final spin states. Note that the phase in the production of positives can be determined only from a specific meson theory so that the results will be uncertain by the amount of the interference term if empirical R matrices are used. The cross section is then given by

$$d\sigma_{pd}=(2\pi/kv_0)|r_{pd}|^2\delta(E_1+E_2+E_3+E_q-E_0-E_D)\rho_f,$$

where the variables appearing in r_{pd} and in the density of final states ρ_f are to be interpreted in terms of the new momentum conservation condition $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{q}-\mathbf{k}_0-2\mathbf{k}_c=0$ that arises from integrating the original $\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{q}-\mathbf{k}_0-\mathbf{k}_n)$ over $\mathbf{k}_n$ and $\mathbf{k}_p$.

Positive mesons can also be formed with the final nucleons coming off as a neutron and a deuteron instead of as two neutrons and a proton. According to the approximation being used here, this process will occur mainly when the positive meson is produced from the proton in the deuteron. This is the same process as that considered by Brueckner, Chew, and Hart[5] in proton-proton collisions, and can be included by calculating

$$(\xi_1\xi_{D'}\eta^+|R_{pd}{}^{+(d)}|\xi_0\xi_D)=\sum_{\xi_0'\xi_n\xi_p}(\xi_{D'}\eta^+|R_{pp}{}^{+(d)}|\xi_0\xi_p)\delta_{\xi_n\xi_1}\times\psi_D(\xi_n\xi_p)\psi_{\xi_0}(\xi_0').\quad(5)$$

[5] Brueckner, Chew, and Hart, Phys. Rev. (to be published).

Clearly this term is incoherent with processes that lead to one proton and two neutrons. The alternative case when one of the two neutrons produced in the n–p collisions picks up the original proton in the deuteron is neglected in this approximation. Equations (3)–(5) are the formal solution of the problem posed; given the R matrices for the production of mesons in two-nucleon collisions, which except for phase can be found empirically, they enable us to calculate the production of mesons by proton-deuteron collisions under the assumptions (a) and (b) above, and subject to the uncertainty in the sign and magnitude of the coefficient of the interference term in the production of positives introduced by the unknown phase $\exp[i(\delta_{pn}-\delta_{pp})]$.

III. RESULTS

The calculation of the matrix elements and phase space integrals is discussed in the appendix. Clearly (3), (4), and (5) imply a much more detailed knowledge of the production process than is at present available, so that simplifying assumptions have been introduced. These are general features found to hold for the four meson theories considered by Brueckner; namely, scalar, vector, and pseudoscalar with pseudoscalar and pseudovector coupling. These are that, near threshold, the two-particle matrix elements depend principally on the meson variables, the initial and final spins of the nucleons, and the magnitude of their relative momenta, and only weakly on the absolute magnitude or direction of the momenta. Consequently, when the two final nucleons are identical, their final state must be a singlet spin state. These restrictions arise from neglect of the final nucleon momenta relative to the initial momenta of the nucleons and neglecting the recoil of the nucleons due to the emission of the meson. The answers obtained by this approximation are in agreement with preliminary experiments, so these assumptions will be used here as a semi-empirical result.

Under these restrictions, the transition rate for the production of negative mesons in p–d collisions is given by

$$
\begin{aligned}
w_{pd}^- = (2\pi/\hbar)\rho_f \times 1/6\{ & |M_{pn}^-(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^-)|^2 g_0^2(\mathbf{k}_3+\mathbf{k}_c) \\
& -M_{pn}^-(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^-)^* M_{pn}^-(|\mathbf{k}_2-\mathbf{k}_3|^2,\eta^-) \\
& \quad \times g_0^*(\mathbf{k}_3+\mathbf{k}_c)g_0(\mathbf{k}_1+\mathbf{k}_c) \\
& +|M_{pn}^-(|\mathbf{k}_2-\mathbf{k}_3|^2,\eta^-)|^2 g_0^2(\mathbf{k}_1+\mathbf{k}_c) \\
& -M_{pn}^-(|\mathbf{k}_2-\mathbf{k}_3|^2,\eta^-)^* M_{pn}^-(|\mathbf{k}_3-\mathbf{k}_1|^2,\eta^-) \\
& \quad \times g_0^*(\mathbf{k}_1+\mathbf{k}_c)g_0(\mathbf{k}_2+\mathbf{k}_c) \\
& +|M_{pn}^-(|\mathbf{k}_3-\mathbf{k}_1|^2,\eta^-)|^2 g_0^2(\mathbf{k}_2+\mathbf{k}_c) \\
& -M_{pn}^-(|\mathbf{k}_3-\mathbf{k}_1|^2,\eta^-)^* M_{pn}^-(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^-) \\
& \quad \times g_0^*(\mathbf{k}_2+\mathbf{k}_c)g_0(\mathbf{k}_3+\mathbf{k}_c)\}, \qquad (6)
\end{aligned}
$$

where M_{pn}^- is the two-nucleon matrix element for the production of negatives and g_0 is the momentum amplitude in the deuteron. The three positive terms give the distribution that would be produced in an n–p collision where the neutron had the momentum distribution in the deuteron. The negative terms are a correction taking account of those states which are excluded by the presence of the third particle. The assumption that M_{pn}^- is a constant gives the phase space for the problem less the exclusion principle correction. The meson distribution in the forward direction under this assumption is plotted in Fig. 1 to the same scale as the production of negative mesons in n–p collisions under the same assumption. The general character of this distribution persists when the forces between the two interacting nucleons in their final state are taken into account.

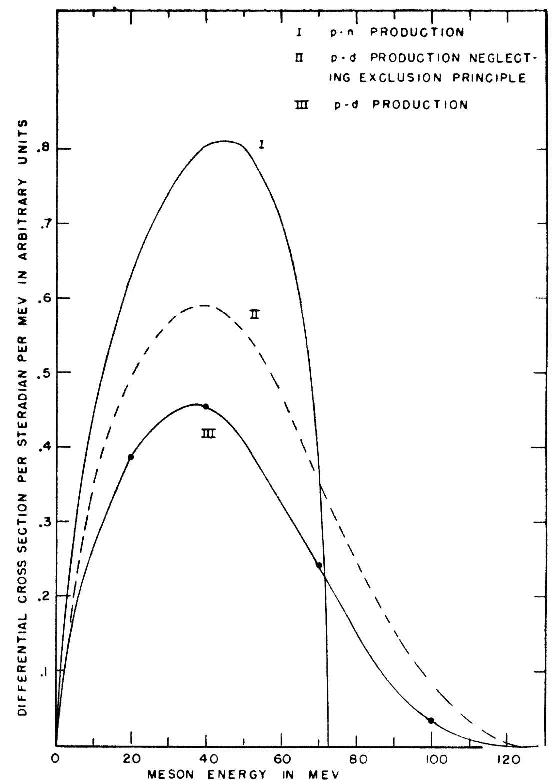

FIG. 1. Production of negative mesons in the forward direction by 345-Mev protons assuming a constant matrix element; coulomb forces neglected.

The experimental results on the production of positive mesons in proton-proton collisions[1] are clearly incompatible with the assumption that the matrix element is a constant. Brueckner, Chew, and Hart[5] have shown that the discrepancy can be removed by taking into account the interaction of the nucleons after the production of the meson. Essentially, this is found to introduce into the R matrix a factor $K_0/(\alpha_t^2+k_r^2)^{\frac{1}{2}}$ if the final state is a triplet, and $0.876\ K_0/(\alpha_s^2+k_r^2)^{\frac{1}{2}}$ if the final state is a singlet. $[\alpha^2=M|\epsilon|/\hbar^2$, where ϵ is the binding energy of the deuteron or of the virtual singlet level for triplet and singlet states, respectively, $\mathbf{k}_r=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$ is the relative momentum of the final nucleons, and K_0 is the inverse Compton wavelength of the meson $\mu c/\hbar$.] Since under our assumptions the final state in the production of negatives in a singlet, the result (relative to the two-nucleon production) is the same for all four theories and is given in Fig. 2. The chief effect on the deuteron spectrum (outside of the change in magnitude) of taking this effect into account is to shift the peak of the distribution from 40 Mev to 60 Mev. Coulomb forces between the two protons in the final state might be expected to wipe out the sharp peak in the two-particle distribution; but since little area is included under the peak, the smoothed-out deuteron distribution should be little affected by this correction.[6] Further, α_s is well known from p–p scattering, so that this result should be relatively trustworthy. In particular, it is not subject to the interference correction that appears in the production of positive mesons.

The production of positives is more complicated in that a deuteron may appear as one of the final particles, that the final state may be singlet, triplet, or a mixture depending upon the theory, and that positives produced

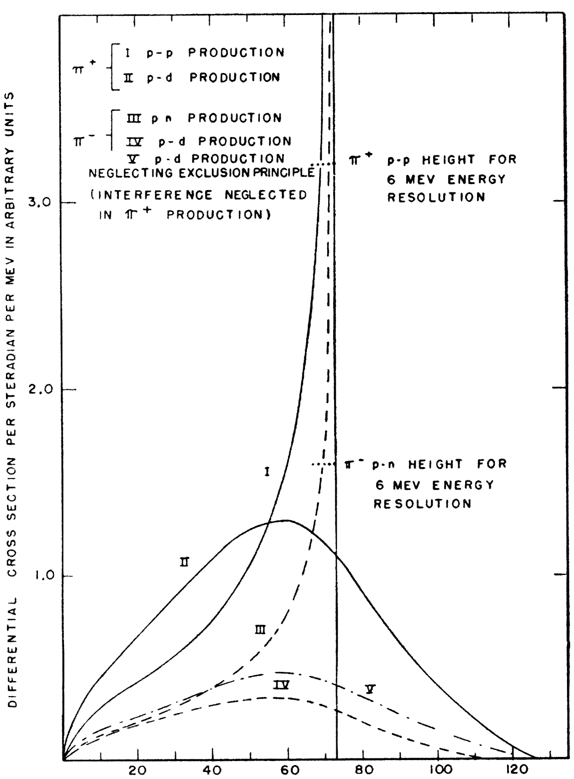

FIG. 2. Production of negative and positive mesons in the forward direction by 345-Mev protons leading to a singlet final state; interference term neglected; p–n and p–p matrix elements assumed equal.

[6] This correction has been calculated by K. Watson and is indeed small.

from the neutron may interfere with positives produced from the proton with an undetermined phase. The general formula for the case of a deuteron appearing as a final particle has already been given; the calculation is straightforward and leads to the result given in the Appendix. For the rest of the cross section the transition probability is

$$\begin{aligned}w_{pd}{}^+=(2\pi/\hbar)\rho_f\{&|M_{pn}{}^+(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^+)|^2g_0{}^2(\mathbf{k}_3+\mathbf{k}_c)\\&+|M_{pp}{}^+(|\mathbf{k}_1-\mathbf{k}_3|^2,\eta^+)|^2g_0{}^2(\mathbf{k}_2+\mathbf{k}_c)\\&+|M_{pp}{}^+(|\mathbf{k}_2-\mathbf{k}_3|^2,\eta^+)|^2g_0{}^2(\mathbf{k}_1+\mathbf{k}_c)\\&-M_{pp}{}^+(|\mathbf{k}_1-\mathbf{k}_3|^2,\eta^+)^*M_{pp}(|\mathbf{k}_2-\mathbf{k}_3|^2,\eta^+)\\&\qquad\times g_0{}^*(\mathbf{k}_2+\mathbf{k}_c)g_0(\mathbf{k}_1+\mathbf{k}_c)\\&+A\cos(\delta_{pn}-\delta_{pp})M_{pn}{}^+\\&\qquad\times(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^+)^*g_0{}^*(\mathbf{k}_3+\mathbf{k}_c)\\&\qquad\times[M_{pp}{}^+(|\mathbf{k}_1-\mathbf{k}_3|^2,\eta^+)g_0(\mathbf{k}_2+\mathbf{k}_c)\\&\qquad+M_{pp}{}^+(|\mathbf{k}_2-\mathbf{k}_3|^2,\eta^+)]g_0(\mathbf{k}_1+\mathbf{k}_c)]\},\end{aligned}\tag{7}$$

where $A\cos(\delta_{pn}-\delta_{pp})$ depends upon the theory. Note that for equal p–n and p–p matrix elements the exclusion correction is one-third the correction in the case of negatives. This is a specific example of a general argument given by Chew and Steinberger[7] to show that the exclusion correction will increase the positive negative ratio for the production of positive mesons by protons in complex nuclei.

For the final state a singlet spin state (scalar or vector mesons, $A=1$), under the usual assumption that the n–p and p–p scattering lengths are equal, the production of positives is also given in Fig. 2. This curve (II) is made up of three parts, which can be determined from the curves already given for the production of negative mesons (IV and V). The distribution of positives produced by a collision of the incident proton with the neutron will be the same as the distribution of negative (V) times the ratio of $n+p\rightarrow\pi^+$ to $n+p\rightarrow\pi^-$ cross sections. The distribution of positives produced in collisions between the incident proton and the proton in the deuteron will again have the same form but will be multiplied by the ratio of $p+p\rightarrow\pi^+$ to $p+n\rightarrow\pi^+$ cross sections. (If the p–n and p–p matrix elements are equal this ratio will be two, since either proton can give a positive meson in the p–p case.) The exclusion principle correction is the same as that taking (V) into (IV) times one-half the ratio of p–p to p–n cross sections. Note that in the case plotted, ($\sigma_{pn}{}^+/\sigma_{pn}{}^-=1$ and $\sigma_{pp}{}^+/\sigma_{pn}{}^+=2$), this exclusion correction is only one-third as large compared to the total cross section as is the exclusion correction in the production of negatives. Under this assumption that the p–n and p–p matrix elements are equal, the positive-negative ratio for mesons of 60 Mev in the forward direction is 3.84 ± 0.84. The uncertainty

[7] G. F. Chew and J. Steinberger, Phys. Rev. 78, 497 (1950).

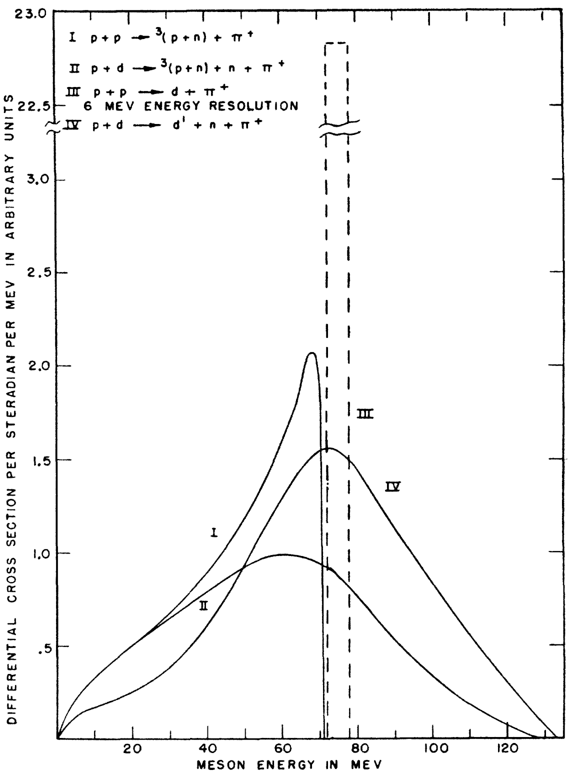

FIG. 3. Production of positive mesons in the forward direction by 345-Mev protons striking free protons or protons bound in deuterons leading to a triplet final state.

is due to the possible interference between positives produced from the neutron and those produced from the proton in the deuteron.

For the final state a triplet (pseudoscalar theory with pseudovector coupling or $\frac{2}{3}$ of the time with pseudoscalar coupling) the cross section is greatly increased by the reformation of a deuteron in the final state. The distribution of positives produced from the proton is much the same as when the final state is a singlet, but the case when the deuteron reappears as one of the final particles has comparable cross section. The comparison of these two parts of the cross section is given in Fig. 3. Note that the value of the measured p-d cross section relative to the cross section measured for the production of positives in p-p collisions will depend critically on the energy resolution of the apparatus used in the latter experiment. (The ratio of the deuteron peak to the peak of the continuum distribution is given by $4\pi E_d/\Delta E$ where E_d is the binding energy of the deuteron and ΔE is the energy resolution in the center of mass system.) The total cross section in the forward direction (again assuming n-p and p-p matrix elements equal, and neglecting interference) is plotted in Fig. 4. The positive-negative ratio in the forward direction at 60 Mev is 8.22 ± 0.84. This large increase in the positive-negative ratio when the final state is a triplet is, of course, due to the added cross section resulting from the reformation of the deuteron. This increase is clearly much larger than the uncertainty due to the interference term. Hence, if the matrix element for p-n production were known to be approximately equal to that for p-p production, the experimental value for this positive-negative ratio would determine fairly clearly whether or not the reformation of a deuteron occurs appreciably in p-p meson production. Conversely, if the formation of a deuteron could be demonstrated in meson production from hydrogen, this ratio would give a fair idea as to the ratio of p-n to p-p matrix elements (assuming the p-n matrix element the same for both positive and negative meson production).

This problem was suggested to me by Professor Chew, and he has been most liberal with advice and help during the investigation. Keith Brueckner's parallel investigation of the fundamental two-nucleon problem has been drawn on throughout, and his advice was most helpful. Much is owed to the criticism and encouragement of Professor Robert Serber.

APPENDIX

A. Calculation of the Matrix Elements

The general expression (3) is specialized by the assumption that R_{np}^- depends only on σ_1, σ_2, $|\mathbf{k}_1-\mathbf{k}_2|^2$, and η^- in the final state, σ_0 and σ_n in the initial state, and is antisymmetric in σ_1 and σ_2. Performing the integrations, antisymmetrizing in the final

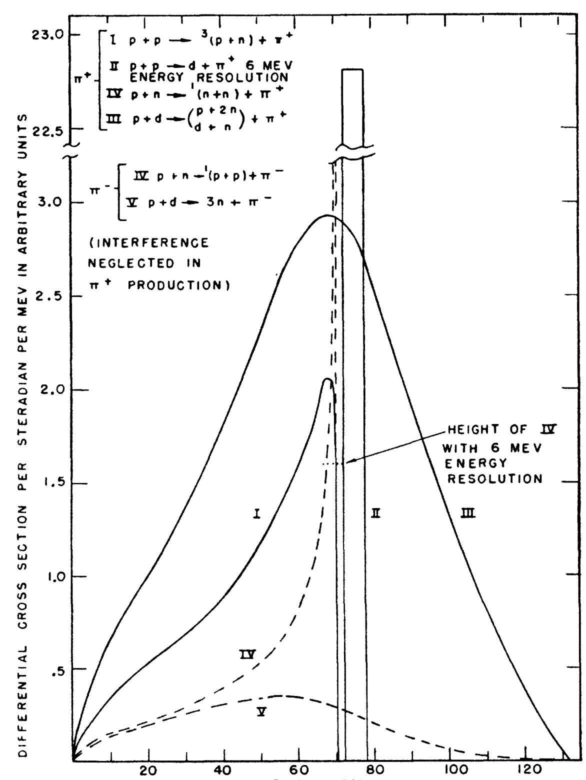

FIG. 4. Production of positive and negative mesons in the forward direction by 345-Mev protons leading to a triplet final state when the two final particles are not identical; interference term neglected.

three protons, and taking out the delta-function giving momentum conservation gives

$$r_{pd}^{-}=\sum_{\sigma_n}(\sigma_1\sigma_2|\mathbf{k}_1-\mathbf{k}_2|^2\eta^-|r_{np}^-|\sigma_0\sigma_n)\chi_D(\sigma_n\sigma_3)g_0(\mathbf{k}_3+\mathbf{k}_c)$$
$$-(\sigma_1\sigma_3|\mathbf{k}_1-\mathbf{k}_3|^2\eta^-|r_{np}^-|\sigma_0\sigma_n)\chi_D(\sigma_n\sigma_2)g_0(\mathbf{k}_2+\mathbf{k}_c)$$
$$-(\sigma_3\sigma_2|\mathbf{k}_3-\mathbf{k}_2|^2\eta^-|r_{np}^-|\sigma_0\sigma_n)\chi_D(\sigma_n\sigma_1)g_0(\mathbf{k}_1+\mathbf{k}_c). \quad (8)$$

Since we are interested only in the meson distribution, the result will be integrated over all nucleon momenta; hence the matrix element that will appear in the cross section may be written

$$|r_{pd}^-|^2=\left(\frac{1}{3!}\right)\times\left(\frac{1}{6}\right)\times 3\sum_{\substack{\sigma_0\sigma_n\sigma_n'D\\ \sigma_1\sigma_2\sigma_3}} g_0^*(\mathbf{k}_3+\mathbf{k}_0)F^*(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^-)$$
$$\times\chi_D^{*2}(\sigma_n'\sigma_3)(\sigma_1\sigma_2|r_{np}^-|\sigma_0\sigma_n')^*\{(\sigma_1\sigma_2|r_{np}^-|\sigma_0\sigma_n)$$
$$\times\chi_D(\sigma_n\sigma_3)F(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^-)g_0(\mathbf{k}_3+\mathbf{k}_c)$$
$$-2(\sigma_3\sigma_2|r_{np}^-|\sigma_0\sigma_n)\chi_D(\sigma_n\sigma_1)$$
$$\times F(|\mathbf{k}_3-\mathbf{k}_2|^2,\eta^-)g_0(\mathbf{k}_1+\mathbf{k}_c)\}, \quad (9)$$

where (as in the case in the theories considered by Brueckner) the momentum dependence can be factored out as $F(|\mathbf{k}_i-\mathbf{k}_j|^2,\eta^-)$, the $1/3!$ is the phase space factor for three identical particles, and the $\frac{1}{6}$ comes from the average over the six initial spin states. Since

$$\sum_{D,\sigma_3}\chi_D^*(\sigma_n'\sigma_3)\chi_D(\sigma_n\sigma_3)=\tfrac{3}{2}\delta_{\sigma_n\sigma_n'} \quad (10)$$

the diagonal term reduces to

$$\tfrac{1}{2}g_0^2(\mathbf{k}_3+\mathbf{k}_c)\times\tfrac{1}{4}\sum_{\substack{\sigma_0\sigma_n\\ \sigma_1\sigma_2}}|F(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^-)(\sigma_1\sigma_2|r_{np}^-|\sigma_0\sigma_n)|^2$$
$$=\tfrac{1}{2}g_0^2(\mathbf{k}_3+\mathbf{k}_c)|M_{np}^-(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^-)|^2, \quad (11)$$

i.e., simply to $\frac{1}{2}$ the matrix element (squared) for negative production in p–n collisions times the momentum distribution in the deuteron. This is the term to be expected for direct production from the neutron. (The $\frac{1}{2}$ occurs in the two particle case also as the phase space factor for two identical particles.) Similarly, the spin sum for the exclusion correction gives

$$\tfrac{1}{2}g_0^*(\mathbf{k}_3+\mathbf{k}_c)M_{pn}^-(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^-)$$
$$\times M_{pn}^-(|\mathbf{k}_1-\mathbf{k}_2|^2,\eta^-)^*M_{pn}^-(|\mathbf{k}_2-\mathbf{k}_3|^2,\eta^-), \quad (12)$$

so that the exclusion correction is 100 percent when the three final particles all have the same momenta, as it should be. Combining these and including the cyclic permutations of 1, 2, 3 gives the result (6) already quoted. The matrix element for positives, (7), is obtained in the same way. Here, however, no simplification such as the final state being always a singlet occurs and different theories can give different results for the coefficient of the interference term $A\cos(\delta_{pn}-\delta_{pp})$.

B. Phase Space Integrations

The canonical variables picked were r, the internal coordinate of the deuteron, y, the distance between the incident proton and the center of mass of the deuteron, and x, the center-of-mass coordinate for the three particles, with conjugate momenta $\hbar\mathbf{k}_r$, $\hbar\mathbf{k}_y$, and $\hbar\mathbf{k}_x$. The initial kinetic energy T_0 and momentum $\hbar\mathbf{k}_0$ of the incident proton must be calculated relativistically, but in the deuteron case it was found that the final nucleons may be treated as nonrelativistic without greatly altering the meson distribution. (This is to be contrasted with the two-nucleon case where the final nucleons may be treated as nonrelativistic in the center-of-mass system, but the result must be transformed relativistically to the laboratory system.) Conservation of momentum requires that $\mathbf{k}_x+\mathbf{q}=\mathbf{k}_0$, where $\hbar\mathbf{q}$ is the meson momentum and conservation of energy gives

$$k_r=\hbar^{-1}[MT_0-ME_d-ME_q-\tfrac{1}{6}\hbar^2|\mathbf{k}_0-\mathbf{q}|^2-\tfrac{3}{4}\hbar^2k_y^2]^{\frac{1}{2}}$$
$$\equiv[g-\tfrac{3}{4}k_y^2]^{\frac{1}{2}}, \quad (13)$$

where $E_q=(\mu^2c^4+\hbar^2q^2c^2)^{\frac{1}{2}}$ is the meson energy and E_d the deuteron binding energy. For a constant matrix element the diagonal term leads to the integral

$$\int\rho_f g_0^2(k_1)=\frac{d\mathbf{q}}{(2\pi)^3}\int_0^{(4g/3)^{\frac{1}{2}}}\frac{M[g-\frac{3}{4}k_y^2]^{\frac{1}{2}}k_y^2dk_y}{2\hbar^2(2\pi)^3}$$
$$\times\int_{-1}^{3}\frac{d\mu_y}{(2\pi)^3}8\pi^2\frac{8\pi\alpha N^2}{(\alpha^2+\frac{1}{9}k_x^2+k_y^2+\frac{2}{3}k_xk_y\mu_y)^2}, \quad (14)$$

where the deuteron momentum distribution has been taken to be $(8\pi\alpha N^2)^{\frac{1}{2}}/(\alpha^2+k^2)$. The calculation has also been carried through for the more reasonable deuteron wave function $[\exp(-\alpha r)-\exp(-\beta r)]/r$ with $\beta/\alpha=6$. The only important change this makes over the wave function $\exp(-\alpha r)/r$ is that N^2 changes from unity to $6(6+1)/(6-1)^2=1.68$. The angular integration is elementary and the branch points of the final integral allow it to be replaced by a contour and done by residues giving

$$\int\rho_f g_0^2(k_1)=\frac{d\mathbf{q}}{(2\pi)^3}\left(\frac{3^{\frac{1}{2}}MN^2\alpha}{\hbar^2}\right)\left\{\frac{A[r-\rho]^{\frac{1}{2}}+B[r+\rho]^{\frac{1}{2}}}{2^{\frac{1}{2}}AB}-1\right\}, \quad (15)$$

where

$$A^2=3\alpha^2/4K_0^2;\quad B^2=k_x^2/12K_0^2;\quad G=g/K_0^2;$$
$$\rho=G+A^2-B^2;\quad r=[\rho^2+4A^2B^2]^{\frac{1}{2}}$$

and K_0 is the meson inverse Compton wavelength. The exclusion principle correction reduces to

$$\frac{d\mathbf{q}}{(2\pi)^3}\left(\frac{3MN^2\alpha}{2\pi\hbar^2}\right)\left(\frac{K_0}{k_x}\right)\int_0^{(4G/3)^{\frac{1}{2}}}xdx\int_{\frac{1}{3}(k_x/K_0)-\frac{1}{2}x}^{\frac{1}{3}(k_x/K_0)+\frac{1}{2}x}dy$$
$$\times\frac{K_0^2\ln\left[\dfrac{\alpha^2+(K_0y+k_r)^2}{\alpha^2+(K_0y-k_r)^2}\right]}{\alpha^2+K_0^2y^2+k_r^2}, \quad (16)$$

where $k_r^2=K_0^2(4/3G-x)$. This integral was evaluated numerically.

The diagonal term integral that occurs when the forces between the particles in the final state are taken into account differs only by the factor $K_0^2/(\alpha^2+k_r^2)$ in the final integration, so that this integral may again be done by residues giving

$$\int\frac{\rho_f g_0^2(k_1)K_0^2}{\alpha_i^2+\frac{1}{4}|\mathbf{k}_2-\mathbf{k}_3|^2}=\frac{d\mathbf{q}}{(2\pi)^3}\left(\frac{3^{\frac{1}{2}}MN^2\alpha K_0^2}{\hbar^2[(\rho+A_i^2)^2+4A^2B^2]}\right)$$
$$\times\{(\rho+A_i^2/2^{\frac{1}{2}}AB)[A(r-\rho)^{\frac{1}{2}}+B(r+\rho)^{\frac{1}{2}}]$$
$$+(1/2^{\frac{1}{2}})[B(r-\rho)^{\frac{1}{2}}-A(r+\rho)^{\frac{1}{2}}]-A_i(G+A_i^2)^{\frac{1}{2}}\}, \quad (17)$$

where $A_i^2=\alpha_s^2/K_0^2$ or α_t^2/K_0^2 depending upon whether the final state is a singlet or a triplet. The corresponding exclusion principle correction was not calculated but was estimated to be approximately the same percent of the diagonal term at the same meson energy as the corresponding correction for the constant matrix element case.

When one of the final particles is a deuteron, energy and momentum conservation give for the density of final states

$$\rho_f=\left(\frac{2M}{3\hbar^2}\right)k_y\frac{d\Omega_y}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3},\ \text{where } k_y=(2/3^{\frac{1}{2}})(g+\alpha_t^2)^{\frac{1}{2}}. \quad (18)$$

The angular integration is elementary, giving

$$\int d\Omega_y g_0^2(k_1)=9\pi N^2/4[(\rho+A_t^2)^2+4A^2B^2]^2. \quad (19)$$

The ratio of matrix elements for the two-particle case is given by Bruckner as

$$\left|\frac{M_{pp}^{+(d)}}{M_{pp}^+(\alpha^2+k_r^2)^{\frac{1}{2}}}\right|^2=\frac{(2\pi)^3[(\alpha/2\pi)^{\frac{1}{2}}K_0]^2}{K_0^2}=4\pi^2\alpha, \quad (20)$$

so that this result can be immediately related to our previous formulas.