and Jones[4] for a maxwellian gas and α independent of temperature, we then obtain the following values for the coefficients H, K, μ of the differential equation (3):

$$H/\alpha = 2.12\times 10^{-4}\ \text{g/sec};$$
$$K_c = 0.443\times 10^{-4}\ \text{g-cm/sec};$$
$$K_d = 2.28\times 10^{-4}\ \text{g-cm/sec};$$
$$K = K_c + K_d = 2.72\times 10^{-4}\ \text{g-cm/sec};$$
$$\mu = 0.487\times 10^{-3}\ \text{g/cm}.$$

These values are substituted in the solution (30) and a series of curves drawn by giving different values to α. Of these the curve for $\alpha = 0.014$ which fits best with the experimental points is reproduced in Fig. 1. This may be compared with the value 0.018 found by Waldmann[16] by another method.

In conclusion I wish to thank Professor M. N. Saha, F.R.S., for acquainting me with this subject and for his interest, Professor N. R. Sen for having kindly gone through the paper, Dr. U. C. Guha for his friendly cooperation in checking the calculations, and the National Institute of Sciences of India for the Fellowship.

[16] L. Waldmann, Z. Naturforsch. 1 (1946).

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On the Definition and Approximation of Feynman's Path Integrals

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A general and compact expression for Feynman's path integral has been obtained. A classical method is given for the computation of such expressions. The example of a Dirac particle in a constant external electromagnetic field is treated by this method.

I. INTRODUCTION

In order to treat problems involving action at a distance, Feynman has proposed a lagrangian form of quantum mechanics.[1] In this formulation the probability amplitude $K(x^B, x^A)$ for a particle to go from a space-time point x^A to a space-time point x^B is postulated to be given by an expression of the form:

$$K(x^B, x^A) = \int \exp(iS[x]/\hbar)d(\text{paths}), \qquad (1)$$

the integral being extended over all paths, $x(\tau)$ from x^A to x^B. In this paper we give a general and compact definition for this integral, and we give also a classical method for computing an approximate expression for it.

We make use of the following notation:

$x(\tau)$ is the parametric representation of a world line

$$x = x_\mu. \quad \mu = 1, 2, 3, 4. \quad \tau = \text{proper time}.$$

$\dot{x}(\tau) = dx(\tau)/d\tau$.
$\bar{x}(\tau)$ is the classical path.
$x^k \equiv x(\tau^k)$.
$\epsilon = \tau^{k+1} - \tau^k$.

$$S[x] = \int_{\tau^A}^{\tau} L\{x(\tau), \dot{x}(\tau)\}d\tau;$$

$S[x]$ is a functional of the function $x(\tau)$.
$\bar{S} = S[\bar{x}]$; $\bar{S}$ is the classical action.

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[1] R. P. Feynman, Revs. Modern Phys. 20, 367 (1948), hereafter called I. Following the suggestion made in paragraph 14 of I, we have defined a path $x(\tau)$ by four functions $x_\mu(\tau)$ of a parameter τ; the formulas of I are still valid, the quantities $\psi(x,\tau) = \exp(iMc/2\hbar)\psi(x)$ replacing the wave function $\psi(x)$. A proof of this fact is given in connection with the example studied below. For a more complete study of a formalism of relativistic quantum mechanics introducing the wave function $\psi(x,\tau)$, see E. C. G. Stueckelberg, Helv. Phys. Acta 14, 588 (1941), and 15, 23 (1942).

II. DEFINITION OF THE PATH INTEGRALS

In Feynmans' work[1] the definition of the path integrals involves an infinite product of "normalization factors." For his purposes Feynman determined these normalization factors in the cases in which the potential is velocity independent and gave their expressions in rectangular coordinates; and he indicated also the existence of a relationship between these factors and the action, S. We shall give here the general formula for the normalization factors valid for all actions and all frames of reference; moreover, we shall give a compact expression for the infinite product of the normalization factors. We shall give first the general formula for $K(x^{k+1}, x^k)$ for two points corresponding to an interval $\tau^{k+1} - \tau^k = \epsilon$ infinitesimally small; then we shall obtain $K(x^B, x^A)$ by iteration. (The essential formulas are given before their proofs.)

1 $$K(x^{k+1}, x^k) = \exp[(i/\hbar)\bar{S}(x^{k+1}, x^k)] \times (2\pi\hbar i)^{-\frac{1}{2}l}(\det_{\mu\nu} a^{\mu\nu}{}_{k+1,k})^{\frac{1}{2}}. \qquad (2)$$

Here l is the number of degrees of freedom ($l = 4$ in the actual case),

$$a^{\mu\nu}{}_{k+1,k} \equiv \partial^2\bar{S}/\partial x_\mu{}^{k+1}\partial x_\nu{}^k$$

$\det_{\mu\nu}$ means the determinant with respect to the indices μ and ν.

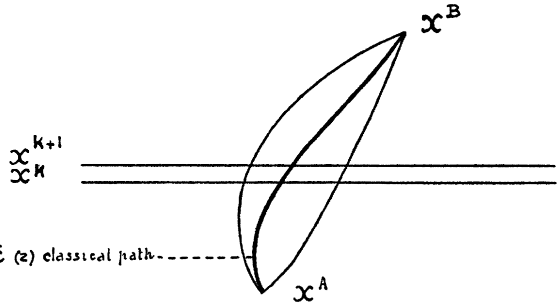

FIG. 1. A world line is defined by its parametric representation $x(\tau)$ the world line of the classical path is $\bar{x}(\tau)$.

Proof of formula (2). Feynman defined $K(x^{k+1}, x^k)$ as follows[1]

$$K(x^{k+1}, x^k)=\exp[i\bar{S}(x^{k+1}, x^k)/\hbar](1/\mathbf{c}^{k+1,k}) \quad (3)$$

and computed the normalization factor $\mathbf{c}^{k+1,k}$ for special cases. We shall determine $|\mathbf{c}^{k+1,k}|$ in a general way by the unitary condition,

$$\int_{-\infty}^{+\infty} K^*(x^{k+1}, x^k)K(x^{k+1}, x'^k)dx^{k+1}=\delta(x^k-x'^k). \quad (4)$$

By substitution of Eq. (3) into Eq. (4), one obtains:

$$\int_{-\infty}^{+\infty} \exp[-i[\bar{S}(x^{k+1}, x^k)-\bar{S}(x^{k+1}, x'^k)]/\hbar] \times|\mathbf{c}^{k+1,k}|^{-2}dx^{k+1}=\delta(x^k-x'^k). \quad (5)$$

The calculation of $\mathbf{c}^{k+1,k}$ goes then as follows. Let us expand $\bar{S}$ in a Taylor series:

$$\bar{S}(x^{k+1}, x^k)=\bar{S}(x^{k+1}, x^{k+1})+(x^k-x^{k+1})(\partial\bar{S}/\partial x^k)(x^{k+1}, x^{k+\zeta}) \quad 0\leqslant\zeta\leqslant 1. \quad (6)$$

By use of the classical relation:

$$\partial\bar{S}(x^{k+1}, x^k)/\partial x^k=p^{k+1,k}(x^{k+1}, x^k) \quad (7)$$

[the momentum $p^{k+1,k}$ is tangent to the classical path: $x^k\rightarrow x^{k+1}$ (Figs. 1 and 2) at the point x^k and is orientated in the same direction as the path], and by use of the following change of variable:

$$x^{k+1}\rightarrow p^{k+1,k} \quad (8)$$

$$dx^{k+1}\rightarrow J(x^{k+1}; p^{k+1,k})dp^{k+1,k}=|\det_{\mu\nu}a^{\mu\nu}{}_{k+1,k}|^{-1}dp^{k+1,k}, \quad (9)$$

(J is the jacobian associated with the change of variables (8)) Eq. (5) becomes:

$$\int_{-\infty}^{+\infty} \exp[-(i/\hbar)[(x^k-x'^k)p^{k+1,k} +0(x^k-x'^k)^2]]|\mathbf{c}^{k+1,k}|^{-2}|\det_{\mu\nu}a^{\mu\nu}{}_{k+1,k}|^{-1}dp^{k+1,k} =\delta(x^k-x'^k). \quad (10)$$

When the distances $|x^{k+1}-x^k|$ and $|x^{k+1}-x'^k|$ are smaller than $\epsilon\sqrt{2}$, the exponent in Eq. (10) reduces to its first term (in the limit $\epsilon\rightarrow 0$).

However, when the distances $|x^{k+1}-x^k|$ and $|x^{k+1}-x'^k|$ are larger than $\epsilon\sqrt{2}$ and when S does not contain $\dot{x}$ to powers larger than two,[2] Feynman has shown that the contributions of the corresponding actions cancel each other. We shall restrict ourselves to such actions. Consequently, in the limit $\epsilon\rightarrow 0$ Eq. (10) tends towards the following equation:

$$\int_{-\infty}^{+\infty} \exp[(i/\hbar)(x^k-x'^k)p^k]|\mathbf{c}^{k+1,k}|^{-2}|\det_{\mu\nu}a^{\mu\nu}{}_{k+1,k}|^{-1}dp^k =\delta(x^k-x'^k); \quad (11)$$

and the result is

$$|\mathbf{c}^{k+1,k}|^{-1}=h^{-2}|\det_{\mu\nu}a^{\mu\nu}{}_{k+1,k}|^{\frac{1}{2}}. \quad (12)$$

Van Hove has shown that[3,4]

$$\mathbf{c}^{k+1,k}=\exp[i\pi l/4]|\mathbf{c}^{k+1,k}| \quad (13)$$

In the actions considered so far, all the off-diagonal terms vanish:

$$(\mathbf{c}^{k+1,k})^{-1}=\prod_\mu(ih)^{-\frac{1}{2}}a^{\mu\nu}{}_{k+1,k}\equiv\prod_\mu(c_\mu{}^{k+1,k})^{-1}. \quad (14)$$

One can readily check the value of $\mathbf{c}^{k+1,k}$ obtained by Feynman for actions expressed in cartesian coordinates where the potential is velocity independent; namely,

$$c_\mu{}^{k+1,k}=(ih\epsilon/Mc)^{\frac{1}{2}}. \quad (15)$$

② $K(x^B, x^A)$

$$=\lim_{n\rightarrow\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left[\frac{i}{\hbar}\cancel{\prod_{k=0}^{n-1}}\bar{S}(x^{k+1}, x^k)\right] \times J^{\frac{1}{2}}(x^0, p^{1,0}/ih\cdots p^{n,n-1}/ih; x^0\cdots x^n) \times\prod_{k=0}^{n} dx^k\delta(x^0-x^A)\delta(x^n-x^B). \quad (16)$$

Proof of Eq. (16). The straightforward iteration of Eq. (3) leads to the value of $K(x^B, x^A)$ given in **I**;

FIG. 2. Figure 2a shows a path defined by successive points and Fig. 2b shows the same path defined by successive tangents. Between two given points infinitisimally close to each other, the path followed by the particle is the classical path, i.e., a definite one; hence, it is possible to go from a point description to a tangent description of the path.

[2] The action functions studied so far do not involve $\dot{x}$ to powers larger than 2; nevertheless, such possibilities may be of interest. A more elaborate proof is then needed to determine $\mathbf{c}^{k+1,k}$ (see reference 4).

[3] We are greatly indebted to Dr. Van Hove for giving us formula 12 before publication and for very many helpful discussions in the course of this work. For a more general study of Eq. (2) we refer the reader to a work of Dr. Van Hove (unpublished as yet).

The square root of the determinant in formula (2) has appeared already in another connection in a work of Jordan, Physik **38**, 513 (1926), and a work of J. H. Van Vleck, Proc. Nat. Acad. Sci. **14**, 178 (1928).

[4] Equation (12) justifies Feynman's remark in **I**, reference 15; it enables us to generalize the very useful Eqs. (58) and (46) of **I**, one of which we shall need later on.

The definition of the hamiltonian:

$$H_k=(\partial/\partial\tau^k)[\bar{S}(x^{k+1}, x^k)-\ln\mathbf{c}^{k+1,k}]. \quad (A)$$

The equivalence of two functionals:

$$\partial f/\partial x^k\underset{S}{\leftrightarrow}-f(\partial/\partial x^k)\{[i\bar{S}(x^{k+1}, x^k)/\hbar] +[i\bar{S}(x^k, x^{k-1})/\hbar]-\ln\mathbf{c}^{k+1,k}\mathbf{c}^{k,k-1}\}. \quad (B)$$

namely,

$$K(x^B, x^A) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[\frac{i}{\hbar} \sum_{k=0}^{n-1} \bar{S}(x^{k+1}, x^k)\right]$$

$$\times \prod_{k=0}^{n-1} (\mathrm{c}^{k+1,k})^{-1} \prod_{k=0}^{n} dx^k \delta(x^0 - x^A)\delta(x^n - x^B). \quad (17)$$

The limits of integration of the x^k are fixed by the boundary conditions; for definiteness, we take them equal to $+\infty$ and $-\infty$.

We shall show that the untractable product $\prod_{k=0}^{n-1} (\mathrm{c}^{k+1,k})^{-1}$ is equal to the square root of a jacobian;[5] this jacobian corresponds to the following change in the description of a path, namely, the change from a tangent description to a point description.

Let us study the following change of $4(n+1)$ variables:

$$\begin{array}{c} x^0 \to x^0 \\ p^{1,0}(x^1, x^0)/ih \to x^1 \\ \cdots\cdots\cdots\cdots \\ p^{n,n-1}(x^n, x^{n-1})/ih \to x^n, \end{array} \quad (18)$$

where the $p^{k+1,k}$ are given by Eq. (7), to which corresponds the following jacobian:

$$J(x^0, p^{1,0}/ih \cdots p^{n,n-1}/ih; x^0 \cdots x^n)$$

$$= \begin{vmatrix} J(x^0; x^0) & & \\ J(p^{1,0}/ih; x^0) J(p^{1,0}/ih; x^1) & & 0 \\ & \ddots & \\ 0 & & J(p^{n,n-1}/ih; x^{n-1}) J(p^{n,n-1}/ih; x^n) \end{vmatrix}. \quad (19)$$

This compound determinant is equal to the product of its diagonal elementary determinants:

$$J(x^0, p^{1,0}/ih \cdots p^{n,n-1}/ih; x^0 \cdots x^n) = \prod_{k=0}^{n-1} J(p^{k+1,k}/ih; x^k). \quad (20)$$

According to Eqs. (8) and (12),

$$J^{\frac{1}{2}}(x^0, p^{1,0}/ih \cdots p^{n,n-1}/ih; x^0 \cdots x^n) = \prod_{k=0}^{n-1} (\mathrm{c}^{k+1,k})^{-1}. \quad (21)$$

3 $$K(x^B, x^A) = \int \exp[iS[x(\tau)]/\hbar] \cancel{J^{\frac{1}{2}}}$$
$$\times J^{\frac{1}{2}}[p(\tau)/ih; x(\tau)] d[x(\tau)], \quad (22)$$

where the integral is extended over all allowed paths, $x(\tau)$, from x^A to x^B. Equation (22) is merely the transcription of Eq. (16), where the notion of a functional as the limit of a function of n variables when $n \to \infty$ has been introduced. Incidentally, Eq. (22) is manifestly dimensionless.

We have thus obtained a mathematical expression for Eq. (1); consequently, we hope to be able to study more simply and more deeply both the mathematical and physical questions raised in connection with Eq. (1).[5] In particular, on the one hand, expression (1) gives more information than does the S-matrix $K(+\infty, -\infty)$; on the other hand, it is possible to choose actions for which Eq. (1) is not equivalent to the detailed information given by the differential equation formalism $K(x^{k+1}, x^k)$, and hence it might be possible to avoid difficulties pertaining to the differential equations and yet answer questions for which the S-matrix seems to be lacking the information.

III. AN APPROXIMATE EXPRESSION FOR $K(x^B, x^A)$

So far, when K cannot be computed exactly, an approximate expression for it is obtained by perturbation method. We shall give, now, a different method for the computation of an approximate expression, K_a, for the exact expression, K; the method is quite general and leads to a result such that $K - K_a = 0(\hbar)$.

A *Rule*:

Let S_a be the first two terms of the Taylor expansion of S around its extremal value. K_a is obtained by substitution of S_a for S in the equation of definition of K, Eq. (22) or (16).

B *Method*:

Let us first write Eq. (22) explicitly for K_a. Let $\bar{x}(\tau)$ be the function which makes $S[x]$ minima, namely, the classical path,

$$\delta S[\bar{x}] = 0 \quad (23)$$

$$S[\bar{x}] = \bar{S}(x^B, x^A). \quad (24)$$

Set

$$x(\tau) = \bar{x}(\tau) + \chi(\tau); \quad (25)$$

it follows

$$\chi(\tau^A) = \chi(\tau^B) = 0. \quad (26)$$

The Taylor expansion of S around its extremal value is given by:

$$\begin{array}{l} S[x] = S[\bar{x}] + (1/2\,!)\delta^2 S[\bar{x}] + (1/3\,!)\delta^3 S[\bar{x}] + \cdots \\ S_a[x] \equiv S[\bar{x}] + (1/2\,!)\delta^2 S[\bar{x}]. \end{array} \quad (27)$$

or more briefly,

$$S_a \equiv \bar{S} + (1/2!)\delta^2 \bar{S} \quad (28)$$

$$\delta^2 \bar{S} = \int_{\tau^A}^{\tau^B} [(\partial^2 L/\partial x_\mu \partial x_\nu)|_{x=\bar{x}} \chi_\mu \chi_\nu + 2(\partial^2 L/\partial x_\mu \partial \dot{x}_\nu)|_{x=\bar{x}} \chi_\mu \dot{\chi}_\nu + (\partial^2 L/\partial \dot{x}_\mu \partial \dot{x}_\nu)|_{x=\bar{x}} \dot{\chi}_\mu \dot{\chi}_\nu] d\tau. \quad (29)$$

The second variation $\delta^2 \bar{S}$ is quadratic in the function of integration, χ.

Set

$$J[p_a(\tau)/ih; x(\tau)] \equiv J_a. \quad (30)$$

J_a is independent of χ and involves only the second functional derivative of S_a. The second functional derivative of $\bar{S}$ is zero and that of $\delta^2 S$ is a constant with respect to the function of integration.

$$K_a = \exp[i\bar{S}/\hbar] \int_{x^A}^{x^B} \exp[i\delta^2 \bar{S}/2\hbar] J_a^{\frac{1}{2}} d[x]. \quad (31)$$

[5] We are happy to acknowledge a very interesting discussion with Professor von Neuman in connection with this point.

The two circumstances $\delta^2\bar{S}$ quadratic in x, and J_a independent of x, make the computation K_a always possible.

We shall now compute:[6]

$$\int_{xA}^{xB} \exp[i\delta^2\bar{S}/2\hbar]J_a^{\frac{1}{2}}d[x]\equiv A_a. \quad (32)$$

Though it should be possible to evaluate A_a from Eq. (32), we resort to the limiting definition of functional integrals [Eq. (16)]. The calculation of A_a amounts, then, to the evaluation of the flattening of successive gaussian curves; more precisely, we establish a recurrence formula giving the result of $k+1$ integrations A_a^{k+1} in terms of the result of k integrations A_a^k. In the limit $n\rightarrow\infty$, $A_a^n\rightarrow A_a$.

Because of Eq. (29) A_a can be written as follows:

$$A_a=\lim_{n\rightarrow\infty}\int\cdots\int \exp\left[\frac{i}{\hbar}\sum_{k=0}^{n}\alpha^k(x^k-\beta^k)^2\right]\prod_{k=1}^{n-1}dx^k/\mathbf{c}_a. \quad (33)$$

$\mathbf{c}_a$ is the common value of $\mathbf{c}_a^{k+1,k}$ when the interval $\tau^B-\tau^A$ is divided in equal parts; $\mathbf{c}_a^{k+1,k}$ stands to J_a as $\mathbf{c}^{k+1,k}$ to $J\cdot\beta^k$ is a linear homogeneous function of x^{k+1} α^k is obtain by a recurrence formula when one writes $\delta^2\bar{S}$ as a sum of successive squares $(x^l-\beta^l)^2$ (successive means increasing value of l). This recurrence formula is particularly easy to find because, x^0 and x^n being zero, the first term has the same structure as the following ones. α^k is function of α^{k-1}, the coefficients of x^k and $\dot{x}^k$, i.e., x^A, x^B, τ^k, ϵ, and the constants appearing in S: it is important to notice that α^k is real and independent of $\hbar$. The result A_a^{k+1} of $(k+1)$ integration is equal to

$$A_a^{k+1}=A_a^k\Pi_\mu 2\pi\hbar i/\alpha_\mu^{k+1}c_{a\mu}. \quad (34)$$

As $A_a^{k+1}\rightarrow A_a^k$ in the limit $n\rightarrow\infty$, a natural change of variable is

$$\alpha_\mu^{k+1}=\lambda_\mu^{k+1}+2\pi\hbar i/c_{a\mu}, \quad (35)$$

where λ_μ^{k+1} is given by an equation of the following structure:

$$\lambda_\mu^{k+1}-\lambda_\mu^k=\epsilon f_\mu(\lambda^k,\tau^k,x^A,x^B)+0(\epsilon), \quad (36)$$

and the limit,[7]

$$d\lambda_\mu/d\tau=f_\mu(\lambda,\tau,x^A,x^B), \quad (37)$$

the constant of integration is determined by Eq. (35) for $k+1=0$. Consequently, in the limit, Eq. (34) is written

$$dA_a/d\tau=A_aF(\tau,x^A,x^B)$$
$$A_a=\text{const}\exp\left[\int_{\tau A}^{\tau B}Fd\tau\right]. \quad (38)$$

The constant of integration is determined by the condition

$$A_a(\epsilon)=\mathbf{c}_a^{-1}. \quad (39)$$

C *Nature of the approximation:*

This approximation is, in functional analysis, the analog to the approximation of the osculatrix parabola in function theory (see, for instance, the Darwin-Fowler method in statistical mechanics). We shall show that K_a is equal to the value obtained by the WKB method, when the lagrangian formulation reduces to the hamiltonian formulation of quantum mechanics; that is, when hamiltonian equations of motion exist. In the WKB method,[8] the wave function ψ is written in the form:

$$\psi=\exp\{(i/\hbar)[S^0+(\hbar S^1/i)+0(\hbar)]\}, \quad (40)$$

and approximated by

$$\psi_a=\exp[iS^0/\hbar]\exp[S^1]. \quad (41)$$

We shall show that $\psi_a=K_a$.

As is well known, $S^0=\bar{S}$; we shall show that A_a, like $\exp[S^1]$, is real and independent of $\hbar$ up to a multiplicative constant,[9] this constant is the same for A_a and $\exp[S^1]$; moreover, $K-K_a=0(\hbar)$.

(a) A_a is real and independent of $\hbar$. One observes from Eq. (14) and (34) that the result of each integration is independent of i and $\hbar$; moreover, as $x^A=x^B=0$, there is no phase term left after integrations (33).

(b) $\exp[S^1]$ and A_a are real and independent of $\hbar$ up to a constant of integration: this constant is determined by the same condition, namely, the normalization condition (39) and (4).

(c) $K-K_a=0(\hbar)$. Set:

$$K\equiv\exp[i\bar{S}/\hbar]^A \quad (42)$$

and compare A with A_a. Make the change of variable:

$$\chi\rightarrow\hbar^{\frac{1}{2}}y \quad (43)$$

which implies $\beta\rightarrow\hbar^{\frac{1}{2}}b$, b being independent of $\hbar$. Then,

$$A^1=A^0\int\exp[i\alpha^1(y^1-b^1)^2][1+i\hbar^{\frac{1}{2}}(1/3!)\delta^3S(y^0,y^1)+0(\hbar)]dy^1/\hbar^2\mathbf{c}^{1,0}. \quad (44)$$

If we assume that S does not contain $\dot{x}$ to powers larger than 2, δ^3S is proportional to ϵ; hence,

$$A^2=A^0[A_a^1+\epsilon 0(\hbar)f(\tau^1)]. \quad (45)$$

By iteration and keeping at each step only the term of lower degree in $\hbar$, one gets

$$A=A_a\left[1+0(\hbar)\int_{\tau A}^{\tau B}f(\tau)d\tau\right]. \quad (46)$$

Consequently, the method given here for computing an approximate expression for K is in the lagrangian formalism, the equivalent of the WKB method.[10]

Remark: When the lagrangian is a linear, bilinear, or quadratic function x_μ and $\dot{x}_\mu$, $S_a=S$; hence, $K_a=K$. In this case A_a is function of τ alone: the coefficients of x_μ and $\dot{x}_\mu$ are constant and Eq. (37) becomes $d\lambda_\mu/d\tau=f_\mu(\lambda)$. Incidentally, it is then not necessary to solve the equation of motion in order to write $\delta^2\bar{S}$ explicitly, Similarly, it can be shown directly that when $\psi=\psi_a$. $\exp[S^1]$ is a function of τ alone.

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[6] The computation of the path integral for exponential of quadratic functions has already been given [R. P. Feynman, Ann Arbor, Summer Symposium, 1949]. The method described here is that used by Feynman in his thesis (Princeton University, 1942) in the study of a forced harmonic oscillator.

At this stage, the order of the variables x^k (order according to increasing or decreasing k) is irrelevant, but the transcription of the results in terms of operators of the usual quantum mechanics requires such an ordering. For this reason, we prefer the method here described to other methods often used in statistical mechanics.

[7] In the case $L=(M\dot{x}^2/2c)-V(x)$ for instance, Eq. (37) becomes:

$$d\lambda/d\tau=(2c\lambda^2/M)-\tfrac{1}{2}\partial^2V/\partial x^2|_{x=\bar{x}(\tau)} \quad \text{(Riccati equation).}$$

[8] See, for instance W. Pauli, *Handbuch der Physik* 2 Aufl. Band 24, 1 Teil (Verlag. Julius Springer, Berlin, 1933), p. 166.

[9] In the WKB method, $\exp[S']$ is given as the solution of a differential equation (namely the continuity equation). It is not possible to see directly which differential equation is satisfied by A_a for the following reason. In Eq. (32) x^A and x^B are involved; hence, the knowledge of the system at time z is not sufficient to determine the knowledge of the system at time $\tau+d\tau$.

[10] In spite of the fact that with the change of variable (43), the ratio of the $(n+1)$th variation to the nth variation goes to zero with $\hbar$, we are not in a position to ascertain that each successive approximation in the Taylor expansion of S corresponds to the successive approximations of the WKB method.

As a working example, we treat here the motion of a Dirac particle in a constant external electromagnetic field. The choice of this example does not imply that Feynman's formalism is restricted to problems pertinent to the first quantization.

The action S for such a system is

$$S=\int[(Mc/2)(\dot{x})^2+(e/2c)(\gamma\cdot\dot{x})(\gamma\cdot\mathfrak{A})]d\tau, \tag{47}$$

where $\mathfrak{A}$ is the four-potential vector of the electromagnetic field and γ are the 4×4 Dirac matrices. The fulfillment of the two following conditions justifies Eq. (47).

(a) $\delta S=0$ gives, in spite of the γ-matrices, the Lorentz equations.

(b) the corresponding wave function is the Dirac wave functions, more precisely the product $\exp[iMc\tau/2\hbar]$ by the square of the Dirac equation.

The procedure to obtain the wave function corresponding to a given action is established in paragraph 6 of **I** when the paths are defined by three space functions (x_1, x_2, x_3) of the time, x_4. The transposition of this procedure when the paths are defined by four space-time functions x_μ of the proper time τ leads to the following results:

$$\psi(x^{k+1}, \tau^{k+1})=\int \exp[(i/\hbar)\bar{S}(x^{k+1}, x^k)]J^{\frac{1}{2}}\psi(x^k, \tau^k)dx^k. \tag{48}$$

By expanding both sides of Eq. (48) and equating terms of two first-order in $(\tau^{k+1}-\tau^k)$, one obtains

$$\psi(x^{k+1}, \tau)=\psi(x^{k+1}, \tau)$$

$$(2iMc/\hbar)\partial\psi/\partial\tau=\{\gamma_\mu[(\partial/\partial x_\mu)-(ie/\hbar c)\mathfrak{A}_\mu]\}^2\psi; \tag{49}$$

hence,

$$\psi(x, \tau)=\exp[iMc\tau/2\hbar]\psi(x). \tag{50}$$

A supplementary condition is necessary to eliminate the unwanted solutions introduced by the use of the square of the Dirac operator instead of the Dirac operator itself.

We shall proceed to evaluate K; as $K=K_a$, we can compute K by the method given in part $\boxed{\mathbf{B}}$ of this section. The following equations are merely the values of the quantities defined there when the action is given by Eq. (47).

$$\begin{aligned}\delta^2\bar{S}(x^{k+1}, x^k)=\epsilon\big[Mc\{&[(x_\mu{}^{k+1}-x_\mu{}^k)/\epsilon]\\ &+2e(x_\nu{}^{k+1}-x_\mu{}^k)x_\rho{}^k(\partial\mathfrak{A}_\mu/\partial x_\rho)/\epsilon c\}\\ &+(e/c)\gamma_\mu\gamma_\nu\{[(x_\mu{}^{k+1}-x_\mu{}^k)x_\rho{}^k(\partial\mathfrak{A}^\nu/\partial x_\rho)/\epsilon]\\ &\qquad -x_\rho{}^{k+1}(\partial\mathfrak{A}^\nu/\partial x_\rho)(x_\mu{}^{k+1}-x_\mu{}^k)/\epsilon\}\big],\end{aligned} \tag{29a}$$

$$c_\mu=(2\pi\hbar i\epsilon/Mc)^{\frac{1}{2}}. \tag{15a}$$

With the use of Eq. (24) of **I**, we can write

$$[(x_\mu{}^{k+1}-x_\mu{}^k)x_\rho{}^k/\epsilon]-x_\rho{}^{k+1}(x_\mu{}^{k+1}-x_\mu{}^k)/\epsilon=\hbar\delta_{\mu\rho}/iM.$$

Thus, in Eq. (22) the terms involving γ's can be taken out of the integral[11]

$$K_D=MK_{KG}$$

where K_D and K_{KG} means the amplitude probability for a Dirac and a Klein-Gordon particle respectively.

$$\begin{aligned}M&=\exp[(e/2Mc)\gamma_\mu\gamma_\nu I_{\mu\nu}(A, B)]\\ &=\cos[(e/2Mc)\{\Sigma_{\mu\nu}[I_{\mu\nu}(A, B)]^2\}^{\frac{1}{2}}]\\ &\qquad+[\gamma_\mu\gamma_\nu I_{\mu\nu}(A, B)/\{\Sigma_{\mu\nu}[I_{\mu\nu}(A, B)]^2\}^{\frac{1}{2}}]\\ &\qquad\qquad\cdot\sin[(e/2Mc)\{\Sigma_{\mu\nu}[I_{\mu\nu}(A, B)]^2\}^{\frac{1}{2}}],\end{aligned}$$

$$I_{\mu\nu}(A, B)=\int_A^{\tau B} F_{\mu\nu}d\tau.$$

For simplicity, let us consider a constant electric field, $\mathcal{E}$, the problem being not essentially more difficult for a constant arbitrary field. Set

$$\mathfrak{A}_3=-i\mathcal{E}x_4 \quad \mathfrak{A}_1=\mathfrak{A}_2=\mathfrak{A}_4=0$$

Then, the coefficients α^k introduced in Eq. (33) are

$$\alpha_\mu{}^k=Mc/\epsilon-M^2c^2/4\epsilon^2\alpha_\mu{}^{k-1}-e\mathcal{E}/4c\alpha_\mu{}^{k-1} \quad \mu=3, 4$$

$$\alpha_\nu{}^k=(Mc/\epsilon)-(Mc/2\epsilon)^2(\alpha_\nu{}^{k-1})^{-1}; \quad \nu=1, 2$$

hence,

$$\begin{aligned}d\lambda_\mu/d\tau&=-(2\lambda_\mu{}^2/Mc)-(e\mathcal{E})^2/8Mc^3\\ d\lambda_\nu/d\tau&=-2\lambda_\nu{}^2/Mc.\end{aligned} \tag{37a}$$

As is already known,[12] one finds that in the coordinates 3, 4, the particle behaves like a complex harmonic oscillator of frequency $2\omega=e\mathcal{E}/Mc^2$ and in the coordinates 1, 2, it behaves like a free particle:

$$\begin{aligned}K_{KG}(x^B, x^A)=&[\omega/2\pi\hbar i\sin\omega(\tau^B-\tau^A)]\\ &\times[Mc/2\pi\hbar i(\tau^B-\tau^A)]\exp[i\bar{S}/\hbar].\end{aligned}$$

The classical action $\bar{S}$ is equal to

$$\begin{aligned}(Mc\tau/2)&+[Mc(\omega x_3{}^A-i\dot{x}_4{}^A)(-\dot{x}_3{}^A\cosh\alpha\tau-i\dot{x}_4{}^A\sinh\alpha\tau)/\omega]\\ &+Mc\{-i\dot{x}_3{}^A\dot{x}_4{}^A\cosh 2\alpha\tau\\ &\qquad-\tfrac{1}{2}[(\dot{x}_3{}^A)^2-(\dot{x}_4{}^A)^2]\sinh 2\alpha\tau+\tau\}/2\omega,\end{aligned}$$

taken between the limits $\tau=\tau^A$ and $\tau=\tau^B$. τ can be eliminated from K with the help of the following relation:

$$\omega(\tau-\tau^A)=\sinh^{-1}[-i\omega(x_4-x_4{}^A)-\dot{x}_3{}^A]-\sinh^{-1}(-\dot{x}_3{}^A).$$

$\dot{x}_3{}^A$ and $\dot{x}_4{}^A$ can be expressed in terms of $(dx_3/dx_4)^A=x_3{}'^A$, which in turn can be expressed in terms of $x_3{}^B$ and $x_4{}^B$ by the following equations:

$$\dot{x}_3{}^A=x_3{}'^A[1+(x_3{}'^A)^2]^{-\frac{1}{2}}$$

$$\dot{x}_4{}^A=[1+(x_3{}'^A)^2]^{-\frac{1}{2}}$$

$$\begin{aligned}\{\omega x_3{}^B-\omega x_3{}^A&+i[1+(x_3{}'^A)^2]^{-\frac{1}{2}}\}^2\\ &+\{\omega x_4{}^B-\omega x_4{}^A-x_3{}'^A[1+(x_3{}'^A)^2]^{-\frac{1}{2}}\}^2-1=0\end{aligned}$$

Strueckelberg[13] and Feynman[14] have shown independently that the motion of a positive electron is the same as the motion of a negative electron going backwards in time. In a constant field, $x_4-x_4{}^A$ is always of the same sign as $\dot{x}_4{}^A$, the x_4 direction of the path cannot be reversed and there is no pair creation or pair annihilation. But if there is a potential difference (in time or space) equal to a least $2Mc^2$ within a distance equal to h/Mc^2 (time) or h/Mc (space), it has been shown that the present formalism describes adequately the phenomenon of pair creation and annihilation by a time potential barrier[13] or by a space potential barrier;[15] then the classical path is reflected at a time potential barrier or refracted with reversal of time at a space potential barrier.

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[11] If that part of the exponent which is under the integral sign is a matrix, one can break the integrals into integrals over scalars; for instance,

$$\int \exp[i\Gamma_\mu f^\mu[x]]J^{\frac{1}{2}}d[x].$$

where the Γ_μ's are products of γ's such that

$$\Gamma_\mu\Gamma_\nu+\Gamma_\nu\Gamma_\mu=\pm 2\delta_{\mu\nu}$$

is equal to

$$\begin{aligned}\int&{\textstyle\genfrac{}{}{0pt}{}{\cosh}{\cos}}(\Sigma_\mu f^{\mu 2}[x])^{\frac{1}{2}}J^{\frac{1}{2}}d[x]\\ &+\Gamma_\mu\int\{f^\mu[x]/(\Sigma_\mu f^{\mu 2}[x])^{\frac{1}{2}}\}{\textstyle\genfrac{}{}{0pt}{}{\sinh}{\sin}}(\Sigma_\mu f^{\mu 2}[x])^{\frac{1}{2}}J^{\frac{1}{2}}d[x].\end{aligned}$$

This last expression does not involve Γ's in the integrand though it might be hard to evaluate.

It is a pleasure to thank Dr. Bruria Kaufmann on this point. See also O. Klein, Z. Physik **80**, 792 (1933). The author has enjoyed several interesting conversations with Professor Klein on this subject.

[12] See, for instance, L. Landau, Physik **64**, 629 (1930), for the case of a constant magnetic field; thus, the indices 1, 2 replace the indices 3, 4 and vice versa.

[13] See references given in footnote 1.

[14] R. P. Feynman, Phys. Rev. **76**, 749 (1949).

[15] R. P. Feynman, Phys. Rev. **74**, 943 (1948).