# Radiative Processes in the Presence of Heavy Nuclei

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Calculations of the cross sections for bremsstrahlung and pair production are carried out without the use of the Born approximation, but by reducing the calculation of these cross sections to the calculation of the exact elastic scattering of electrons and positrons, the last quantities having been tabulated to some extent. Large deviations from the Born approximation results are obtained in the angular distributions of these radiative processes, but arguments are presented that the Born approximation should yield the integrated cross sections correctly at sufficiently high energies. The failure of the Born approximations for relativistic equations is explained in a purely classical way.

## I. INTRODUCTION

It is known that for radiative processes like bremsstrahlung and pair production in the presence of heavy nuclei, the application of the Born approximation may lead to some error. Exact calculations are very difficult to perform. However, it may be possible to express the cross sections for these processes in terms of the exact elastic scattering cross sections for a coulomb field, which has been tabulated to some extent. This idea is suggested by the usual perturbation calculation whereby the process is visualized as taking place in two steps, one of which is the scattering of the electron by the nucleus. In the following it is shown that this idea can be carried through in the high energy limit, and the calculation of the cross section for these radiative processes is reduced to the calculation of the electron and positron scattering amplitudes.

The results show rather large deviations from the Born approximation expressions for the angular distribution of these processes at the larger angles, but small deviation at the smaller angles where these processes are concentrated. Thus, the corrections to the integrated cross sections (that is, integrated over the angles) are small.

## II. THE HIGH ENERGY WAVE FUNCTION

It has been observed that for the scattering of electrons in the presence of heavy elements, it is not correct to apply the Born approximation to the Dirac equation. It has also been pointed out,[1] by investigating the behavior of the phase shifts at high energies, that this breakdown of the Born approximation is not peculiar to the coulomb field, but occurs whenever the potential is strong enough, and is an effect characteristic of relativistic equations.

We shall show that the failure of the Born approximation can be understood and can be corrected, at least theoretically, in a purely classical way. Let us consider a very high energy particle incident in the $z$ direction with velocity $v$ being scattered by a potential $V(\mathbf{r})$ that has no pole. The particle is given a transverse momentum $\Delta p$ perpendicular to the direction of incidence. If the energy is high enough, we may assume the path of the particle to be almost a straight line parallel to the $Z$ axis. If the particle enters at a distance $\rho$ from the $Z$ axis, then its transverse momentum when it reaches the point $(z, \rho)$ is given by,

$$\Delta p = -\int_{-\infty}^{z} (\partial V/\partial \rho) dt = -(\partial/\partial\rho)(1/v)\int_{-\infty}^{z} V(\rho, z)\cdot dz, \tag{1}$$

where in the integration, $V=V(\mathbf{r})=V(\rho, z)$, $r=(\rho^2+z^2)^{\frac{1}{2}}$ and the variable $\rho$ is held constant.

Now if the particle is nonrelativistic, as the energy increases, $v\rightarrow\infty$ and $\Delta p\rightarrow 0$; that is to say, the particle spends so little time near the scatterer that it gets no transverse momentum at all. However, if the particle is relativistic, then as the energy increases, $v\rightarrow c$ and the transverse momentum approaches the definite value of

$$\Delta p = -(1/c)\int_{-\infty}^{z} (\partial V/\partial\rho) dz. \tag{2}$$

This has the consequence that the wave function of the relativistic particle, forgetting for the moment the spin variable, cannot approach the form $\exp(ikz)$ at high energies, for this does not give the correct transverse momentum, which can be large if the potential is sufficiently strong. Instead, the wave function will have the form,

$$\psi \underset{k\rightarrow\infty}{\longrightarrow} e^{i(kz+u)}, \tag{3a}$$

$u$ being a function which is independent of the momentum $\hbar k$. If we choose $u$ as given by

$$u = -(1/\hbar c)\int_{-\infty}^{z} V(\rho, z)\cdot dz, \tag{3b}$$

then the expectation value of $\Delta p$ calculated from $\psi$ will agree with Eq. (2).

We now obtain the result indicated by Eq. (3) more formally. Let us treat at first, a particle which obeys

[1] G. Parzen, Phys. Rev. 80, 355 (1950).

the Klein-Gordon equation,

$$(\nabla^2+k^2-2EV+V^2)\psi=0, \tag{4}$$

where we put $\hbar=c=1$. Let $\psi=\exp[i(kz+u)]$, then we get the equation for $u$,

$$i\nabla^2u-\text{grad}^2u-2k(\partial u/\partial z)-2EV+V^2=0. \tag{5}$$

We now assume an expansion for $u$ in inverse powers of $k$,

$$u=u_0+u_1+u_2+\cdots, \tag{6}$$

where $u_0$ is independent of $k$, $u_1$, of order $1/k$, etc. Collecting the lowest power of $1/k$, we get

$$\partial u_0/\partial z=-E\cdot V/k, \tag{7}$$

so that neglecting terms of order $1/k$, we can write,

$$\psi=e^{i(kz+u)},\quad \partial u_0/\partial z=-V, \tag{8}$$

and

$$u_0=-\int_{-\infty}^{z}V(\rho,z)\cdot dz,$$

where $\rho$ is held constant in the integration. Equation (8) agrees with Eq. (3), if it is remembered that $\hbar=c=1$.

The form of a Dirac wave function at high energies can be obtained by a similar procedure. We write $\psi$ as $\psi_\lambda=a_\lambda\exp[i(kz+u_\lambda)]$, $\lambda=1, 2, 3, 4$, where the $a_\lambda$ are constants, and substitute in the iterated Dirac equation,

$$(\nabla^2+k^2-2EV+V^2)\psi=i(\boldsymbol{\alpha}\cdot\text{grad}V)\psi. \tag{9}$$

If we expand $a_\lambda$ and $u_\lambda$ in powers of $1/k$,

$$\begin{aligned}a_\lambda&=a_\lambda^{(0)}+a_\lambda^{(1)}+\cdots\\ u_\lambda&=u_\lambda^{(0)}+u_\lambda^{(1)}+\cdots,\end{aligned} \tag{10}$$

then collecting the lowest power of $1/k$ gives

$$a_\lambda^{(0)}[(\partial u_\lambda^{(0)}/\partial z)+(EV/k)]=0. \tag{11}$$

For a plane wave in the $z$ direction, with the spin in the $+z$ direction, when $k\rightarrow\infty$, the $a_\lambda$ are given by

$$a_{\mathbf{k}0}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}; \tag{12}$$

then the solution of Eq. (11) with the correct limiting form is

$$\psi=a_{\mathbf{k}0}\exp[i(kz+u_0),$$
$$\partial u_0/\partial z=-V,$$

and

$$u_0=-\int_{-\infty}^{z}V(\rho,z)\cdot dz. \tag{13}$$

## III. SOME EXAMPLES OF THE HIGH ENERGY WAVE FUNCTION

Let us consider the gaussian potential $V=V_0\times\exp[-\alpha^2(\rho^2+z^2)]$, then

$$u_0=-V_0\exp(-\alpha^2\rho^2)\cdot\int_{-\infty}^{z}\exp(-\alpha^2z^2)\cdot dz. \tag{14}$$

The phase surfaces of $\psi$ given by $kz+u_0=\text{constant}$ are no longer plane surfaces, but have a slight curvature near the $Z$ axis, bulging out in $+Z$ direction for a repulsive potential and in the $-Z$ direction for an attractive potential. If we imagine the path of the particle to be perpendicular to these phase surfaces, the curvature of the surfaces will cause the path to deviate from a straight line showing the transverse momentum of Eq. (2).

Let us now consider the coulomb potential $V=Ze^2/(\rho^2+z^2)^{\frac{1}{2}}$. From $\partial u_0/\partial z=-V$, we get,

$$u_0=Ze^2\ln r(1-\cos\theta). \tag{15}$$

This will be recognized as the usual distortion of the plane wave obtained in coulomb scattering. In the derivation of Eq. (15), the indefinite integral was taken as the definite integral diverges. However, Eq. (15) does not give the entirely correct behavior of the coulomb wave function at high energies. Where our derivation breaks down can be easily seen from the classical treatment, where we assumed the path of the particle at high energies to be a straight line. The deviation of the path from a straight line for large impact parameters can be avoided by screening the coulomb field; but since the coulomb field has a pole at the origin, then no matter how high the energy, near the origin the field is strong enough to curve the particle path. We also know from the exact solutions of the Dirac equation for the coulomb field that the wave function has a singularity at the origin which Eq. (15) has not. The error in our power series expansion in $1k$ of $\psi$ probably lies in omitting the higher order terms which, though they go to zero for increasing $k$, have a singularity at the origin and are not negligible. Thus, we can say that Eq. (15) will represent the wave function well at large distances but not near the origin. The validity of Eq. (15) will be discussed further in Sec. V.

## IV. SOME CHECKS OF THE HIGH ENERGY WAVE FUNCTION

As a test of the wave function given by Eq. (3), we can consider what it means in terms of the phase shifts. Let us treat the simpler Klein-Gordon wave function in a central field. We expand $\psi$ in spherical harmonics,

$$\psi=\sum_l i^l(2l+1)\cdot e^{i\delta_l}\cdot R_l(r)\cdot p_l(\cos\theta), \tag{16}$$

where $R_l(r)$, the radical part of the wave function, has the asymptotic form,

$$R_l(r)\rightarrow(kr)^{-1}\cos[kr-\tfrac{1}{2}(l+1)\pi+\delta_l]. \tag{17}$$

Using the orthogonality property of $p_l(\cos\theta)$, we get

$$2\cdot i^l\cdot e^{i\delta_l}R_l(r)=\int_0^\pi e^{i(kr\cos\theta+u)}\cdot p_l(\cos\theta)\sin\theta\cdot d\theta. \quad (18)$$

Now let $r\to\infty$ in Eq. (18) and integrate the right side by parts, dropping terms of higher order than $1/r$, then,

$$2\cdot i^l\cdot e^{i\delta_l}(kr)^{-1}\cdot\cos(kr-\tfrac{1}{2}(l+1)\pi+\delta_l)$$
$$=(ikr)^{-1}e^{i[kr+u(0)]}-(-)^l e^{i[-kr+u(\pi)]}, \quad (19)$$

and thus we get for the phase shifts,

$$\delta_l=-\int_0^\infty V(r)\cdot dr. \quad (20)$$

It has been shown[2] that the phase shift $\delta_l$ approaches the quantity $\delta_\infty=-\int_0^\infty V\cdot dr$ when $k\to\infty$. Equation (20) shows that the use of our approximate wave function is in agreement with this result.

Another test is to consider the value of the wave function at the origin. According to Eq. (3),

$$\psi(0)=\exp\left(-i\int_{-\infty}^0 V\cdot dz\right), \quad (21)$$

and as

$$-\int_{-\infty}^0 V\cdot dz=-\int_0^\infty V\cdot dr=\delta_\infty,$$

then

$$\psi(0)=\exp(i\delta_\infty). \quad (22)$$

The mathematics will not be gone into here, but we can show that when $k\to\infty$, $\psi(0)$ does approach $\exp(i\delta\infty)$ if the potential function has no pole at $r=0$.

## V. THE EXPRESSION FOR THE SCATTERING CROSS SECTION

For the Dirac equation, the scattering amplitude is given, in relativistic units $\hbar=c=1$, by[1]

$$f(\theta)=(E-\boldsymbol{\alpha}\cdot\mathbf{k}-\beta m)\cdot\int\exp(-i\mathbf{k}\cdot\mathbf{r})\cdot V\cdot\psi\cdot d\tau. \quad (23)$$

Putting $\psi=a_{\mathbf{k}_0}\exp(i\mathbf{k}_0\cdot\mathbf{r})$, a plane wave in the $\mathbf{k}_0$ direction, in Eq. (23) yields the Born approximation, which gives $f(\theta)$ correct to first order in $V$. However, there are terms which are of higher order in $V$ but do not decrease with increasing energy. That is to say, the Born approximation is the first term in an expansion in $V$ and not in $1/k$, so that it is not correct even at very high energies if the potential is strong enough.

If we substitute the wave function given by Eq. (13) into Eq. (23), we should get an expression which is correct at high energies,

$$f(\theta)=(E-\boldsymbol{\alpha}\cdot\mathbf{k}-\beta m)\cdot a_{\mathbf{k}_0}$$
$$\cdot\int\exp(-i\mathbf{k}\cdot\mathbf{r})\cdot V\cdot e^{i(kz+u)}\cdot d\tau; \quad (24)$$

[2] G. Parzen, reference 1. It was shown there for the Dirac equation, but it can be shown to be true for the Klein-Gordon equation as well.

and the cross section for an unpolarized beam is given by,

$$\sigma(\theta)=E^2(1-v^2\sin^2\tfrac{1}{2}\theta)\cdot|F|^2,$$

where

$$F=(2/4\pi)\cdot\int\exp(-i\mathbf{k}\cdot\mathbf{r})\cdot V\cdot e^{i(kz+u)}\cdot d\tau \quad (25)$$

and

$$u=\int_{-\infty}^z V(\rho\cdot z)\cdot dz.$$

The direct application of Eq. (25) is difficult, for the integral is not easily computed. However, the form is simple and some general relations can be obtained quickly.

By introducing an explicit representation for $\boldsymbol{\alpha}$ and $\beta$, and for $a_{\mathbf{k}_0}$, the spin or part of a plane wave in the $Z$ direction with spin in the $Z$ direction, we find[1] for $f_3(\theta,\varphi)$ and $f_4(\theta,\varphi)$,

$$f_3=\tfrac{1}{2}(E+m)(1+\cos\theta)\cdot F, \quad (26a)$$

$$f_4=\tfrac{1}{2}(E+m)\sin\theta\cdot F. \quad (26b)$$

Thus, at high energies,

$$|f_3/f_4|=\cot\tfrac{1}{2}\theta. \quad (27)$$

Relation (27) has the physical significance that, even after scattering, the spin lies along the direction of motion, and an unpolarized high energy beam remains unpolarized after scattering. In the case of the coulomb field, $f_3$ and $f_4$ have been calculated exactly by Mott.[3] His formulas in the limit $k\to\infty$ show that $|f_3/f_4|=\cot\frac{1}{2}\theta$. However, as the coulomb field has a pole at the origin, we could have expected Eq. (27) to hold only for the smaller angles.

A second immediate relation is the connection between Klein-Gordon scattering and Dirac scattering. The expression for the cross section $\sigma_{\mathrm{KG}}$ for a particle obeying the Klein-Gordon equation is, at high energies and if the potential is not singular at the origin,

$$\sigma_{\mathrm{KG}}=E^2\cdot|F|^2 \quad (28)$$

and comparing this with expression (25) for the Dirac scattering of an unpolarized beam, we get

$$\sigma_{\mathrm{D}}=(1-v^2\sin^2(\tfrac{1}{2}\theta)\cdot\sigma_{\mathrm{KG}}. \quad (29)$$

## VI. THE COULOMB FIELD

It was indicated above that Eq. (13) does not represent the coulomb wave function near the origin. The question arises now as to how well Eq. (25) will give the coulomb cross section. We were unable to do the integral involved, and thus we cannot make a direct comparison of Eq. (25) with the exact coulomb cross section as derived by Mott. However, the correctness of certain general features derivable from Eq. (25) seems to indicate it is fairly reliable for the coulomb

[3] N. F. Mott, Proc. Roy. Soc. (London) **A124**, 426 (1929).

field. Thus, the relation $|f_3|f_4|=\cot\frac{1}{2}\theta$ derived in Sec. IV does hold for the coulomb field. The wave function $\psi=a_{\mathbf{k}_0}\exp[i(kz+u)]$, with $u=Ze^2\ln r(1-\cos\theta)$ does represent the wave function correctly at large distances, and so we would expect Eq. (25) to give the scattering correctly at the smaller angles.

Another property of the exact coulomb cross section is that the ratio $\sigma/\sigma_R$, where $\sigma_R=(\csc^4\frac{1}{2}\theta)/k^2$ is essentially the Rutherford scattering cross section, is independent of energy past about 4 Mev for electrons.[4] According to Eq. (25) the energy dependence of $\sigma/\sigma_R$ is given by $(1-v^2\sin^2\frac{1}{2}\theta)$, which is independent of the energy except at angles very close to 180°.

In the following sections we will apply our approximate wave function as given by Eq. (13) to the calculation of bremsstrahlung and pair production in the presence of heavy nuclei. We shall not actually apply Eq. (13) to a coulomb field, but use the form of Eq. (13) to derive certain relationships.

## VII. BREMSSTRAHLUNG

In the usual perturbation treatment of bremsstrahlung, it is considered as a two-step process, the emission of a photon and scattering by the nucleus. The breakdown of the Born approximation in treating bremsstrahlung may be thought of as stemming from the application of the Born approximation to the scattering by the nucleus. This indicates that it may be possible to express the bremsstrahlung cross section in terms of the exact coulomb cross section, which has been tabulated fairly well by Bartlett and Watson,[5] and by McKinley and Feshbach.[4]

In order to formulate this idea mathematically, let us set up the exact expression for the bremsstrahlung cross section. Let $\psi(\mathbf{r};\mathbf{p}_0)$ be the wave function for an electron, which is incident with momentum $\mathbf{p}_0$ in the field of a potential $V(\mathbf{r})$. We can write this wave function as[1]

$$\psi(\mathbf{r};\mathbf{p}_0)=a_{\mathbf{p}_0}\exp(i\mathbf{p}_0\cdot\mathbf{r})+\phi(\mathbf{r},\mathbf{p}_0), \quad (30)$$

where

$$\phi(\mathbf{r},\mathbf{p}_0)=-\frac{1}{4\pi}(E_0-\boldsymbol{\alpha}\cdot\mathbf{p}-\beta m)\times\int\frac{\exp(ip_0|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}V(\mathbf{r}')\psi(\mathbf{r}')\cdot d\tau', \quad (31)$$

and $E_0=(p_0^2+m^2)^{\frac{1}{2}}$, $\mathbf{p}=-i\,\text{grad}$.

Using the Fourier integral representation of $\exp[ip_0\times|\mathbf{r}-\mathbf{r}'|]/|\mathbf{r}-\mathbf{r}'|$,

$$\frac{\exp(ip_0|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{(2\pi)^3}\int d\mathbf{s}\cdot\exp[i\mathbf{s}\cdot(\mathbf{r}-\mathbf{r}')]\cdot\frac{4\pi}{s^2-p_0^2}, \quad (32)$$

we can write $\phi(\mathbf{r},\mathbf{p}_0)$ as,

$$\phi(\mathbf{r},\mathbf{p}_0)=\frac{1}{(2\pi)^3}\cdot\int d\mathbf{s}\exp(i\mathbf{s}\cdot\mathbf{r})\frac{4\pi}{s^2-p_0^2}\cdot f(\mathbf{s},\mathbf{p}_0), \quad (33)$$

where

$$f(\mathbf{s},\mathbf{p}_0)=-\frac{1}{4\pi}(E_0-\boldsymbol{\alpha}\cdot\mathbf{s}-\beta m)\cdot\int\exp(-i\mathbf{s}\cdot\mathbf{r})V\psi d\tau \quad (34)$$

is a generalization of the scattering amplitude. If $|\mathbf{s}|=p_0$, then $f(\mathbf{s},\mathbf{p}_0)$ is the elastic scattering amplitude for an electron incident in the $\mathbf{p}_0$ direction to be scattered by the potential into the $\mathbf{s}$ direction.

The calculation of the bremsstrahlung cross section for an electron incident with momentum $\mathbf{p}_0$ and scattered to momentum $\mathbf{p}$ while emitting a photon with momentum $\mathbf{k}$ requires the calculation of the matrix element,

$$M=\int\psi^*(\mathbf{r},\mathbf{p})\cdot\exp(-i\mathbf{k}\cdot\mathbf{r})\cdot\alpha_\lambda\cdot\psi(\mathbf{r},\mathbf{p}_0)d\tau, \quad (35)$$

where $\alpha_\lambda$ is the component of the Dirac $\boldsymbol{\alpha}$-matrix along a polarization direction. Using expression (31) for $\psi(\mathbf{r},\mathbf{p})$ and the Fourier integral expression for $\phi(\mathbf{r},\mathbf{p})$, we can write for $M$,

$$M=[4\pi/(\mathbf{p}_0-\mathbf{k})^2-\mathbf{p}^2]f^*(\mathbf{p}_0-\mathbf{k},\mathbf{p})\cdot\alpha_\lambda a_{\mathbf{p}_0}+[4\pi/(\mathbf{k}+\mathbf{p})^2-\mathbf{p}_0^2]a_{\mathbf{p}}^*\alpha_\lambda f(\mathbf{k}+\mathbf{p},\mathbf{p}_0)+\int d\mathbf{r}\cdot\phi^*(\mathbf{r},\mathbf{p})\exp(-i\mathbf{k}\cdot\mathbf{r})\cdot\alpha_\lambda\phi(\mathbf{r},\mathbf{p}_0). \quad (36)$$

At high energies, assuming $k$, $p$, and $p_0$ are of the same order of magnitude, we can show that the third term in Eq. (36) is of order $m/p_0$ compared with the other two. To compare these terms, substitute into the integrals the asymptotic form of $\phi(\mathbf{r},\mathbf{p}_0)$, $f(\theta)\exp(ip_0r)/r$.

The evaluation of Eq. (36) at high energies reduces them to evaluating $f(\mathbf{s},\mathbf{p}_0)$ at high energies. We will show that $f(\mathbf{s},\mathbf{p}_0)$ can be evaluated in terms of the elastic scattering amplitude.

Using our approximate wave function (13), we can write $f(\mathbf{s},\mathbf{p}_0)$ as,

$$f(\mathbf{s},\mathbf{p}_0)=\tfrac{1}{2}(E_0-\boldsymbol{\alpha}\cdot\mathbf{s}-\beta m)\cdot a_{\mathbf{p}_0}\cdot F(\mathbf{s},\mathbf{p}_0), \quad (37a)$$

where

$$F(\mathbf{s},\mathbf{p}_0)=(-2/4\pi)\int\exp(-i\mathbf{s}\cdot\mathbf{r})\cdot V(\mathbf{r})\cdot\exp i[\mathbf{p}_0\cdot\mathbf{r}+u(\mathbf{r},\mathbf{p}_0)]d\tau. \quad (37b)$$

And the function $u(\mathbf{r},\mathbf{p}_0)$ depends only on the direction of $\mathbf{p}_0$, not on the energy of the particle. Now $F(\mathbf{s},\mathbf{p}_0)$ by (37b) depends only on $\mathbf{q}=\mathbf{p}_0-\mathbf{s}$ and on $\alpha$ the angle between $\mathbf{q}$ and $\mathbf{p}_0$. But the quantity $F(\mathbf{s},\mathbf{p}_0)$, by Eq. (25), is also involved in the calculation of the elastic scattering cross section. In this case, when $|\mathbf{s}|=p_0$, we can write $F(\mathbf{s},\mathbf{p}_0)=F(\theta,E_0)$, $\theta$ being the angle of scattering, $E_0$ the energy of the incident par-

[4] W. A. McKinley, Jr., and H. Feshbach, Phys. Rev. 74, 1759 (1948).

[5] J. H. Bartlett and R. E. Watson, Proc. Am. Acad. Arts Sci. 74, 53 (1940).

ticle, and $q=2p_0 \sin\frac{1}{2}\theta$. We can evidently relate $F(\mathbf{s}, \mathbf{p}_0)$ to $F(\theta', E_0')$ for any $\mathbf{s}$, and $\mathbf{p}_0$, by choosing $\theta'$ and $E_0'$ so as to get the same $\mathbf{q}$ and $\alpha$. Thus,

$$F(\mathbf{s}, \mathbf{p}_0)=F(\theta', E_0')$$

if

$$|\mathbf{q}|=|\mathbf{p}_0-\mathbf{s}|=2p_0' \sin\tfrac{1}{2}\theta' \tag{38}$$

and

$$\tfrac{1}{2}\theta'=\tfrac{1}{2}\pi-\alpha,$$

where $\alpha$ is the angle between $\mathbf{p}_0$ and $\mathbf{q}$.

However, if it so happens that $\alpha$ is larger than 90°, then this situation cannot be duplicated in elastic scattering where $\alpha$ is always less than 90°. By taking the conjugate of Eq. (37b) we can change the direction of $\mathbf{q}$ to $-\mathbf{q}$, which will make an angle of less than 90° with $\mathbf{p}_0$; but we have also changed the sign of $u(\mathbf{r}, \mathbf{p}_0)$, which means we must relate $F(\mathbf{s}, \mathbf{p}_0)$ to positron scattering rather than to electron scattering. So, in general, if $\alpha \leq 90°$,

$$F(\mathbf{s}, \mathbf{p}_0)=F_{\mathrm{el}}(\theta', E_0'), \tag{39a}$$

where

$$\theta'/2=\tfrac{1}{2}\pi-\alpha,$$

and

$$q=2p_0' \cos\alpha.$$

If $\alpha \geq 90°$

$$F^*(\mathbf{s}, \mathbf{p}_0)=F_{pos}(\theta', E_0'),$$

where

$$\theta'/2=\tfrac{1}{2}\pi-\alpha, \tag{39b}$$

and

$$q=2p_0' \cos\alpha.$$

$F_{\mathrm{el}}(\theta', E_0')$ and $F_{pos}(\theta', E_0')$ are the functions involved, according to Eq. (25), in the cross sections for electron and positron scattering, respectively.

Thus, we have reduced the calculation of the bremsstrahlung matrix element $M$ to calculating the elastic scattering amplitudes of both positrons and electrons. To put our result in a form where it can be easily compared with the Bethe-Heitler formula, we note the relationship,

$$\sum_{p'}(a_{\mathbf{p}'}a_{\mathbf{p}'}^*/E_0-E_{p'}) =(1/E_0^2-E_{p'}^2)(E_0-\boldsymbol{\alpha}\cdot\mathbf{p}'-\beta m) \tag{40}$$

where $\sum_{p'}$ means summing over the 4 free particle state with the same momentum $\mathbf{p}'$. Now substitute Eqs. (40) and (37) into Eq. (36): we get,

$$M=-(4\pi/2)\{\sum_{p'}[(a_{\mathbf{p}}^*a_{\mathbf{p}'})(a_{\mathbf{p}'}^*\alpha_\lambda a_{\mathbf{p}_0})\cdot F^*(p_0-\mathbf{k}, \mathbf{p})/(E_p-E_{p'})]+\sum_{p''}[(a_{\mathbf{p}}^*\alpha_\lambda \alpha_{\mathbf{p}''})(a_{\mathbf{p}''}^*a_{\mathbf{p}_0})F(\mathbf{p}+\mathbf{k}, \mathbf{p}_0)/(E_{p_0}-E_{p''})]\}, \tag{41}$$

where

$$\mathbf{p}'=\mathbf{p}_0-\mathbf{k}, \quad \text{and} \quad \mathbf{p}''=\mathbf{p}+\mathbf{k}.$$

Now let,

$$F(\mathbf{s}, \mathbf{p}_0)=(2Ze^2/q^2)H(\mathbf{s}, \mathbf{p}_0), \tag{42}$$

where

$$\mathbf{q}=\mathbf{p}_0-\mathbf{s}$$

and $2Ze^2/q^2$ is the Born approximation result for $F(\mathbf{s}, \mathbf{p}_0)$. Thus, putting $H(\mathbf{s}, \mathbf{p}_0)=1$ in what follows will bring us back to the Born approximation and the Bethe-Heitler formula.

With Eq. (42) and the matrix element (41), we can write the bremsstrahlung cross section as,

$$\sigma(\theta, \varphi, \theta_0)\cdot d\Omega_k d\Omega=(Z^2e^4pEE_0k\cdot d\Omega_k d\Omega/137\pi^2p_0q^4) \times|\sum\{[(a_{\mathbf{p}}^*a_{\mathbf{p}'})(a_{\mathbf{p}'}^*\alpha_\lambda a_{\mathbf{p}_0})/(E_p-E_{p'})] \cdot H^*(\mathbf{p}_0-\mathbf{k}, \mathbf{p})+[(a_{\mathbf{p}}^*\alpha_\lambda a_{\mathbf{p}''})(a_{\mathbf{p}''}^*a_{\mathbf{p}_0})/(E_{p_0}-E_{p''})] H(\mathbf{p}+\mathbf{k}, \mathbf{p}_0)\}|^2, \tag{43}$$

where $d\Omega$ and $d\Omega_k$ are the solid angles of scattered electron and emitted photon, respectively. Expression (43) differs from the Bethe-Heitler formula[6] by the appearance of the two $H(\mathbf{s}, \mathbf{p}_0)$ factors which are equal to unity in the Born approximation. If we perform the indicated sum,[7] we get the result for the bremsstrahlung cross section,

$$d\phi=(Z^2e^4\cdot p\cdot dk\cdot \sin\theta d\theta d\varphi\cdot \sin\theta_0 d\theta_0/2\pi\cdot 137p_0kq^4) \times\{Q(\theta, \varphi, \theta_0)|H(\mathbf{p}_0-\mathbf{k}, \mathbf{p})|^2+R(\theta, \varphi; \theta_0) \times|H(\mathbf{p}+\mathbf{k}, \mathbf{p}_0)|^2+E(\theta, \varphi, \theta_0)[H^*(\mathbf{p}_0-\mathbf{k}, \mathbf{p}) \cdot H(\mathbf{p}+\mathbf{k}, \mathbf{p}_0)+\text{c.c.}]\}. \tag{44}$$

Here $\theta$, $\theta_0$, and $\varphi$ are as defined in Heitler.[6]

$$Q=\frac{p_0^2 \sin^2\theta_0}{(E_0-p_0 \cos\theta_0)^2}(4E^2-q^2)+\frac{2k}{E_0-p_0 \cos\theta_0} \times[p_0^2 \sin^2\theta_0-p_0p \sin\theta \sin\theta_0 \cos\varphi +k(E+p \cos\theta)] \tag{45a}$$

$$R=\frac{p^2 \sin^2\theta}{(E-p \cos\theta)^2}(4E_0^2-q^2)-\frac{2k}{(E-p \cos\theta)}[p^2 \sin^2\theta -p_0p \sin\theta \sin\theta_0 \cos\varphi-k(E_0+p_0 \cos\theta_0)] \tag{45b}$$

$$S=\frac{1}{(E-p \cos\theta)(E_0-p_0 \cos\theta_0)}\{p_0p \sin\theta \sin\theta_0 \cos\varphi \times[q^2-3k^2-4E_0E+k(p_0 \cos\theta_0-p \cos\theta)]-2m^2k^2\} +k\left\{\frac{p^2 \sin^2\theta}{E-p \cos\theta}-\frac{p_0^2 \sin^2\theta_0}{E_0-p_0 \cos\theta_0}\right\}, \tag{45c}$$

where we may note that

$$Q+R+2S=T \tag{45d}$$

[6] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 164.

[7] I would like to thank Dr. S. C. Wright for his aid in breaking down the bremsstrahlung formula.

and

$$T=\left[\frac{p^2\sin^2\theta}{(E-p\cos\theta)^2}(4E_0{}^2-q^2)+\frac{p_0{}^2\sin^2\theta_0}{(E_0-p_0\cos\theta_0)^2}(4E^2-q^2)\right.$$
$$-\frac{2pp_0\sin\theta\sin\theta_0\cos\varphi}{(E-p\cos\theta)(E_0-p_0\cos\theta_0)}(4E_0E-q^2+2k^2)$$
$$\left.+2k^2\frac{p^2\sin^2\theta+p_0{}^2\sin^2\theta_0}{(E-p\cos\theta)(E_0-p_0\cos\theta_0)}\right],\quad(45e)$$

and is the expression that occurs in the Bethe-Heitler formula. Thus, putting $H(\mathbf{p}_0-\mathbf{k},\mathbf{p})=H(\mathbf{p}+\mathbf{k},\mathbf{p}_0)=1$ in Eq. (44) will yield the Bethe-Heitler formula.

Assuming the correctness of applying relation (13) to our problem, the only situation in which expression (44) may break down is if in calculating $H(\mathbf{s},\mathbf{p}_0)$, the energy $E_0'$ of the equivalent elastic scattering is small compared with the rest mass. Since $p_0'=q/2\cos\alpha$, this situation can only arise if $q\ll m$, that is, for small $\theta$ and $\theta_0$, and also if $\cos\alpha$ is not too small. For small $\theta$ and $\theta_0$ we can write as an order of magnitude relation,

$$\tan\alpha\simeq\frac{k\theta_0+p(\theta+\theta_0)}{p_0-p-k+\frac{1}{2}k\theta_0{}^2+\frac{1}{2}p(\theta+\theta_0)^2}\qquad(46)$$

and $q\simeq p_0\theta$. If $p_0\sim p\sim k$, then $\tan\alpha\sim p_0{}^2\theta_0/m^2$ and $E_0'\sim m$ if $\theta$, $\theta_0\sim(m/p_0)^{1/2}$. In the exceptional case when $p\ll m$, then $E_0'\sim m$ if $\theta$, $\theta_0\sim m/p_0$. Formula (44) will give large deviations from the Bethe-Heitler formula, as much as a factor of five, at the larger angles, but gives very small deviations at the smaller angles $\theta\sim\theta_0\sim m/p_0$ where most of the bremsstrahlung occurs.

As an example, let us consider the case when $\theta_0=0$ and $\theta\gg m/p_0$. In this simple case, $H(\mathbf{p}_0-\mathbf{k},\mathbf{p})=H(\theta',E')$ where $\theta'=\theta$ and $E'=E$ to terms of order $m/p_0$, and $H(\mathbf{p}+\mathbf{k},\mathbf{p}_0)=H(\theta'',E'')$ where $\theta''=\theta$ and $E''=E_0$. Since $H(\theta,E)$ is independent of energy above 4 Mev for electrons,[4] we can write at $\theta_0=0$, $\theta\gg m/p_0$,

$$H(\mathbf{p}_0-\mathbf{k},\mathbf{p})=H(\mathbf{p}+\mathbf{k},\mathbf{p}_0)=H_{\mathrm{el}}(\theta).\qquad(47)$$

Thus, the deviation may be expressed as

$$d\phi/d\phi_{\mathrm{BH}}=|H_{\mathrm{el}}(\theta)|^2,\qquad(48)$$

where $d\phi_{\mathrm{BH}}$ stands for the Bethe-Heitler cross section. $|H_{\mathrm{el}}(\theta)|^2$ is plotted in Fig. 1 for atomic number $Z=82.2$, and one can see clearly the large deviation for large $\theta$ when $\theta_0=0$. The neglect of screening should not be important in a result like Eq. (48), for we have restricted ourselves to the larger angles where the screening should have little effect.

Although the Born approximation leads to large errors in the angular distribution of the bremsstrahlung, this is not so after the cross section has been integrated over the angles $\theta$, $\theta_0$, and $\varphi$. Our formula (44) shows that the integrated cross section as obtained by the Born approximation has a percentage error which is at most

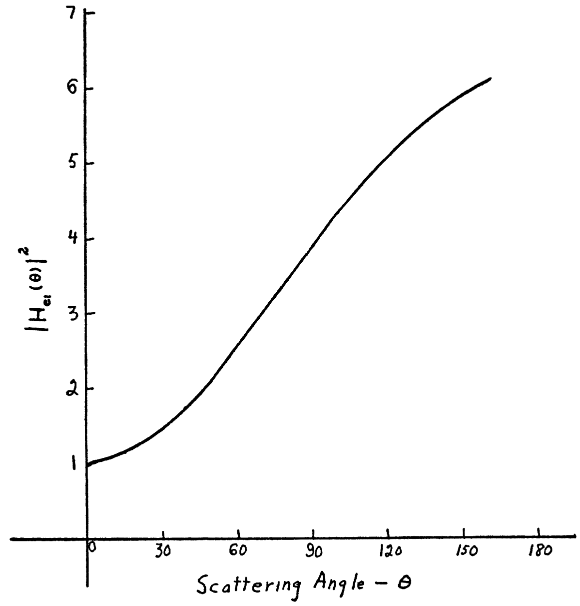


FIG. 1. $|H_{\mathrm{el}}(\theta)|^2$, the ratio of the exact coulomb cross section to the Born approximation cross section, plotted as a function of the scattering angle $\theta$ for $z=82.2$ at high energies.

the order of $(m/p_0)^{1/2}$. This results from the fact that the percentage deviation expected at angles $\theta\sim\theta_0\sim m/p_0$, where the bremsstrahlung is greatest, is only of the order of $m/p_0$. We placed the error at $(m/p_0)^{1/2}$ instead of $(m/p_0)$, because our expression (44) does not hold for angles smaller than $\sim(m/p_0)^{1/2}$.

We have not taken screening into account. However, one might expect this result, that the Born approximation leads to little error in the integrated cross section at high energies, to carry over when screening is included. For, as the above calculation indicates, the Born approximation will give the small angle bremsstrahlung correctly and thus also the integrated cross section.

In Table I, we have tabulated $H(\theta)$, so that using relations (39), (42), and (44), the bremsstrahlung cross section can be calculated at different values of $\theta$, $\theta_0$, and $\varphi$. $H(\theta)$ is independent of energy, for sufficiently high energies and is related to the function[4,5] $G(\theta)$ which occurs in the calculation of coulomb scattering

TABLE I. $H_{\mathrm{el}}(\theta)$ is the ratio of the exact scattering amplitude to the Born approximation expression for the scattering amplitude of electrons in a coulomb field where $Z=82.2$ and $Ze^2/\hbar c=0.6$ $\cdot H_{\mathrm{pos}}(\theta)$ is the corresponding quantity for positron scattering. For high enough energies, $H(\theta)$ is independent of energy.

| | $H_{\mathrm{el}}(\theta)$ | | $H_{\mathrm{pos}}(\theta)$ | |
|---|---|---|---|---|
| $\theta$ | Real | Imaginary | Real | Imaginary |
| 15° | −0.211 | −1.05 | | |
| 30° | 0.820 | −0.890 | −0.26 | −0.92 |
| 45° | 1.34 | −0.389 | −0.62 | −0.69 |
| 60° | 1.60 | 0.128 | −0.82 | −0.47 |
| 90° | 1.72 | 0.994 | | |
| 120° | 1.68 | 1.54 | | |
| 150° | 1.56 | 1.89 | | |

by the equation,

$$H(\theta)=(2\hbar c/Ze^2)\cdot\tan^2\tfrac{1}{2}\theta\cdot G(\theta). \qquad (49)$$

Table I is calculated for $Z=82.2$, $Ze^2/\hbar c=0.6$. For electron scattering, $H_{el}(\theta)$ was calculated by extending the results of Bartlett and Watson for 3.35 $mc^2$ energy and $Z=80$ to higher energies. For positron scattering, the tables of McKinley and Feshbach were used. Since these tables give results with an error of order of 10 to 15 percent, the results for angles above 60° turned out not to be meaningful. For 60° and less, the results given have an error of 10 to 15 percent. For angles smaller than 15°, $H(\theta)$ can be calculated from the small angle formula of Bartlett and Watson.

## VIII. PAIR PRODUCTION

Here, as when one applies the Born approximation, the calculation of pair production is identical with that of bremsstrahlung. Therefore, only the results will be given here. The result for the angular pair-production cross section is

$$\begin{aligned} d\phi= &-(Z^2e^4p_+p_-dE_+\sin\theta_+\sin\theta_-d\theta_+d\theta_-d\varphi_+/2\pi\cdot 137k^3q^4) \\ &\times\{Q(\theta_+,\varphi_+,\theta_-)|H(\mathbf{p}_--\mathbf{k},-\mathbf{p}_+)|^2+R(\theta_+,\varphi_+,\theta_-) \\ &\times|H(-\mathbf{p}_++\mathbf{k},\mathbf{p}_-)|^2+S(\theta_+,\varphi_+,\theta_-) \\ &\times[H^*(\mathbf{p}_--\mathbf{p},-\mathbf{p}_+)\cdot H(-\mathbf{p}_++\mathbf{k},\mathbf{p}_-)+\text{c.c.}]\}, \end{aligned} \qquad (50)$$

where the same notation is used as by Heitler.

$$\begin{aligned} Q=&\frac{p_-^2\sin^2\theta_-}{(E_--p_-\cos\theta_-)^2}(4E_+^2-q^2)+\frac{2k}{E_--p_-\cos\theta_-} \\ &\times[p_-^2\sin^2\theta_-+p_+p_-\sin\theta_+\sin\theta_-\cos\varphi_+ \\ &-k(E_++p_+\cos\theta_+)] \end{aligned} \qquad (51a)$$

$$\begin{aligned} R=&\frac{p_+^2\sin^2\theta_+}{(E_+-p_+\cos\theta_+)^2}(4E_-^2-q^2)+\frac{2k}{E_+-p_+\cos\theta_+} \\ &\times[p_+^2\sin^2\theta_++p_-p_+\sin\theta_+\sin\theta_-\cos\varphi_+ \\ &-k(E_-+p_-\cos\theta_-)] \end{aligned} \qquad (51b)$$

$$\begin{aligned} S=&-\frac{1}{(E_--p_-\cos\theta_-)(E_+-p_+\cos\theta_+)} \\ &\times\{-p_-p_+\sin\theta_-\sin\theta_+\cos\varphi_+[q^2-3k^2+4E_-E_+ \\ &+k(p_-\cos\theta_-+p_+\cos\theta_+)-2m^2k^2\} \\ &-k\left\{\frac{p_+^2\sin^2\theta_+}{E_+-p_+\cos\theta_+}+\frac{p_-^2\sin^2\theta_-}{E_--p_-\cos\theta_-}\right\}, \end{aligned} \qquad (51c)$$

where again we may note that

$$Q+R+2S=T \qquad (51d)$$

and

$$\begin{aligned} T=&\frac{p_+^2\sin^2\theta_+}{(E_+-p_+\cos\theta_+)^2}(4E_-^2-q^2) \\ &+\frac{p_-^2\sin^2\theta_-}{(E_--p_-\cos\theta_-)^2}(4E_+^2-q^2) \\ &+\frac{2p_+p_-\sin\theta_+\sin\theta_-\cos\varphi_+}{(E_+-p_+\cos\theta_+)(E_--p_-\cos\theta_-)}(4E_+E_-+q^2-2k^2) \\ &-2k^2\frac{p_-^2\sin^2\theta_-+p_+^2\sin^2\theta_+}{(E_--p_-\cos\theta_-)(E_+-p_+\cos\theta_+)} \end{aligned} \qquad (51e)$$

and is the expression that occurs in the Born approximation formula.[8]

In the past experiments on high energy bremsstrahlung and pair production in the presence of heavy nuclei, only the integrated cross sections were measured. The deviations from the Born approximation obtained were of the order of 10 percent at energies of the order of 20 Mev. Of course, one must take screening into account in calculating the integrated cross section. However, as mentioned above, our calculations lead us to believe that when screening is included, we would expect the Born approximation to be in error by at most of the order of $(m/p_0)^{\frac{1}{2}}$. This is not in contradiction with the measured deviations.

To repeat, the one weakness in the above treatment seems to be the application of our high energy wave function [Eq. (13)] to the coulomb field which has a pole at the origin. However, the general nature of our results should still be valid in any case. The basis for our remarks concerning the integrated cross sections is not affected as our high energy wave function should give small angle scattering correctly. The large deviations from the Born approximation results obtained at the larger angles should still be expected, although they may not be given entirely correctly by our formulas.

In Eq. (35), $\psi^*(\mathbf{r},\mathbf{p})$ should really have an ingoing scattered wave. However our results are independent of this point.

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[8] Reference 6, p. 196.