PHYSICAL REVIEW VOLUME 81, NUMBER 5 MARCH 1, 1951

Neutron-Deuteron Scattering at High Energy

R. L. GLUCKSTERN* AND H. A. BETHE
Cornell University, Ithaca, New York
(Received October 20, 1950)

The neutron-deuteron total scattering cross section at high energy has been calculated by expressing it as a sum over two-particle states over a wide range of energy. A sum rule was then applied to give the total cross section, that is, the elastic *and* inelastic cross sections. The calculations were performed with a Serber type exchange force for the neutron-proton force in agreement with present neutron-proton scattering data and a neutron-neutron force of arbitrary depth and exchange character. The neutron-neutron well depth was then determined for an ordinary, Serber type and pure exchange force between the two neutrons by making use of the experimental neutron-deuteron total cross section. The neutron-deuteron elastic scattering was then calculated for the three different types of neutron-neutron exchange and was found to agree best with the experimental value for the Serber type exchange (no force in odd states). The cross section for production of low energy protons was also calculated and confirmed the Serber type exchange force between the two neutrons when compared with the experimental value. Finally, a qualitative discussion of the angular and energy distribution of the low and high energy protons was given.

I. INTRODUCTION

At first glance it appears likely that the neutron-deuteron scattering cross section can be given accurately as the sum of the individual neutron-proton and neutron-neutron cross sections for sufficiently high energy. However, as has been pointed out by Chew[1] and others,[2,3] the interference between the free particle collisions is appreciable at all energies, since collisions of small momentum transfer are favored. One must therefore correct the sum of the neutron-proton and neutron-neutron cross sections by the correctly evaluated interference term. It will be seen that another important correction arises from the fact that the neutron-proton cross section itself is affected when the Pauli principle is applied to the two neutrons. In the process of evaluating the total neutron-deuteron cross section, we shall determine the magnitude of both of these effects, as well as the influence of such factors as spin dependent and exchange type forces.

We shall assume that the neutron-proton interaction is a Serber force, in agreement with the interpretation of the present experimental evidence.[4] By comparison of the calculated and experimental results for neutron-deuteron scattering[5,6] we hope to derive some information concerning the magnitude and exchange character of the neutron-neutron force.

The calculations will be performed using the Born approximation; i.e., considering the interaction between the incident neutron and either particle in the deuteron as the perturbing potential. We shall express the total cross section as the sum of three terms: a neutron-proton cross section, a neutron-neutron cross section, and an interference term. If we can then identify the neutron-proton and neutron-neutron terms as those which would be obtained in the corresponding Born approximation calculations for the two-particle collisions, we can use the experimental cross sections wherever possible. In this way the calculated neutron-deuteron cross section will be more satisfactory than might be expected with the Born approximation.

In calculating the total neutron-deuteron cross section we shall need to sum over final states which are made up of two-particle states covering a wide range in energy. By rearranging the integrals we shall be able to express the cross section as a sum over the two-particle states; the use of sum rules will then give the total cross section without requiring the explicit calculation of the two-particle continuum states. However, the continuum states *will* be needed for the differential cross section.

In order to illustrate the method, we shall outline the calculation in Sec. II under the following assumptions. (a) There are no spin dependent forces. (b) There are no exchange forces. (c) The particles are all distinguishable; i.e., there is no Pauli principle. These assumptions will be removed in Sec. III. The calculations will be carried out only in the nonrelativistic case, although relativistic corrections start to be important at about 200 Mev. Tensor forces will not be included in the present paper.

II. ORDINARY SPIN-INDEPENDENT FORCES WITHOUT PAULI PRINCIPLE

Let $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ be the coordinates of the neutron and proton in the deuteron and the incident neutron, respectively. We shall transform to the coordinates $\mathbf{R}$, $\mathbf{x}$, and $\mathbf{r}$ as shown in Fig. 1.

$$\left.\begin{aligned}\mathbf{R}&=\tfrac{1}{3}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3)\\ \mathbf{x}&=\mathbf{r}_3-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)\\ \mathbf{r}&=\mathbf{r}_1-\mathbf{r}_2\end{aligned}\right\}. \quad (1) \qquad \left.\begin{aligned}\mathbf{r}_1&=\mathbf{R}-\tfrac{1}{3}\mathbf{x}+\tfrac{1}{2}\mathbf{r}\\ \mathbf{r}_2&=\mathbf{R}-\tfrac{1}{3}\mathbf{x}-\tfrac{1}{2}\mathbf{r}\\ \mathbf{r}_3&=\mathbf{R}+\tfrac{2}{3}\mathbf{x}\end{aligned}\right\}. \quad (2)$$

* Present address: Sloane Physics Laboratory, Yale University, New Haven, Connecticut.
[1] G. F. Chew, Phys. Rev. 74, 809 (1948).
[2] T. Y. Wu and J. Ashkin, Phys. Rev. 73, 986 (1948).
[3] F. de Hoffmann, Phys. Rev. 78, 216 (1950).
[4] R. S. Christian and E. W. Hart, Phys. Rev. 77, 441 (1950).
[5] Cook, McMillan, Peterson, and Sewell, Phys. Rev. 72, 1264 (1947); 75, 7 (1949).
[6] W. M. Powell, private communication.

The energy of the entire system is

$$\begin{aligned}\mathcal{E}&=\frac{(M\dot{\mathbf{r}}_1)^2}{2M}+\frac{(M\dot{\mathbf{r}}_2)^2}{2M}+\frac{(M\dot{\mathbf{r}}_3)^2}{2M}+V_{NP}(|\mathbf{r}_1-\mathbf{r}_2|)\\&\qquad+V_{NN}(|\mathbf{r}_1-\mathbf{r}_3|)+V_{NP}(|\mathbf{r}_3-\mathbf{r}_2|)\\&=\frac{(3M\dot{\mathbf{R}})^2}{2(3M)}+\frac{(\frac{2}{3}M\dot{\mathbf{x}})^2}{2(\frac{2}{3}M)}+\frac{(\frac{1}{2}M\dot{\mathbf{r}})^2}{2(\frac{1}{2}M)}+V_{NP}(r)\\&\qquad+V_{NN}(|\mathbf{x}-\tfrac{1}{2}\mathbf{r}|)+V_{NP}(|\mathbf{x}+\tfrac{1}{2}\mathbf{r}|),\end{aligned}\tag{3}$$

where the dots are time derivatives and the V's are the two-particle interaction potentials. If we now consider the last two terms in (3) as the perturbing potentials and eliminate the motion of the center of mass, we can write the initial state of the system, Ψ_i, as the product of a plane wave normalized in a box of volume L^3, and the ground state of the deuteron, $\psi_0(r)$.

$$\Psi_i=L^{-\frac{3}{2}}\exp(i\mathbf{k}\cdot\mathbf{x})\psi_0(r),\tag{4}$$

where $\hbar\mathbf{k}=\frac{2}{3}M\dot{\mathbf{x}}_i$. The total energy of the system in the laboratory system ($\dot{\mathbf{r}}_1=\dot{\mathbf{r}}_2=0$, $\dot{\mathbf{r}}_3=\dot{\mathbf{x}}_i$) is

$$E=\tfrac{1}{2}M\dot{\mathbf{r}}_3^2-\epsilon=\tfrac{1}{2}M\dot{\mathbf{x}}_i^2-\epsilon=(9\hbar^2k^2/8M)-\epsilon,\tag{5}$$

where ϵ is the binding energy of the deuteron.

We shall take for the asymptotic motion of the final state

$$\hbar\mathbf{k}'=\tfrac{2}{3}M\dot{\mathbf{x}}_f,\quad \hbar\mathbf{k}''=\tfrac{1}{2}M\dot{\mathbf{r}}_f,\tag{6}$$

but we must be careful in our choice of a wave function for the final state. Since the energy in the center-of-mass system is still quite large, there will always be at least one particle which may be taken as free after the collision, with a possible strong interaction between the other two. We shall therefore divide the momentum space for the final state into three regions:

(a) particle 3 free, interaction between 1 and 2 (near $\dot{\mathbf{r}}_1=\dot{\mathbf{r}}_2$; $\mathbf{k}''=0$),
(b) particle 1 free, interaction between 2 and 3 (near $\dot{\mathbf{r}}_2=\dot{\mathbf{r}}_3$; $\mathbf{k}'=\frac{2}{3}\mathbf{k}''$),
(c) particle 2 free, interaction between 1 and 3 (near $\dot{\mathbf{r}}_1=\dot{\mathbf{r}}_3$; $\mathbf{k}'=-\frac{2}{3}\mathbf{k}''$).

Since conservation of energy is given by[7]

$$k'^2+(4/3)k''^2=k^2,\tag{7}$$

region (a) can be taken, for example, as

$$0\leq k''/k\leq s\approx 0.6,\quad 1\geq k'/k\geq t\approx 0.7,\tag{8}$$

the exact values of s and t being unimportant. A similar choice for regions (b) and (c) is possible, but we shall show later that the boundary of the three regions has little bearing on the problem.

We can now write the final wave function in the three regions as

(a) $\Psi_f=L^{-\frac{3}{2}}\exp(i\mathbf{k}'\cdot\mathbf{x})\psi_{k''}(\mathbf{r})$ particle 3 free, (9a)

(b) $\Psi_f=L^{-\frac{3}{2}}\exp[i\mathbf{k}'''\cdot(\frac{3}{4}\mathbf{r}-\frac{1}{2}\mathbf{x})]\psi_{k^{iv}}(-\mathbf{x}-\frac{1}{2}\mathbf{r})$ particle 1 free, (9b)

(c) $\Psi_f=L^{-\frac{3}{2}}\exp[i\mathbf{k}^{v}\cdot(-\frac{3}{4}\mathbf{r}-\frac{1}{2}\mathbf{x})]\times\psi_{k^{vi}}(-\mathbf{x}+\frac{1}{2}\mathbf{r})$ particle 2 free, (9c)

where $\hbar\mathbf{k}'''$, $\hbar\mathbf{k}^{iv}$, $\hbar\mathbf{k}^{v}$, and $\hbar\mathbf{k}^{vi}$ are the momenta conjugate to the coordinates $\frac{3}{4}\mathbf{r}-\frac{1}{2}\mathbf{x}$, $-\mathbf{x}-\frac{1}{2}\mathbf{r}$, $-\frac{3}{4}\mathbf{r}-\frac{1}{2}\mathbf{x}$, and $-\mathbf{x}+\frac{1}{2}\mathbf{r}$, respectively. The wave functions, ψ_{k^l}, are two particle wave functions corresponding to relative momenta $\hbar k^l$ and may either represent the ground state (elastic collisions) or the continuum solutions of the two-particle problem. These are just the functions to which we hope to apply the sum rules.

Let us first consider a collision with values of $\mathbf{k}'$ and $\mathbf{k}''$ in region (a). The matrix element for the transition is

$$\begin{aligned}\mathfrak{M}&=\int\Psi_f^*V\Psi_i d\tau=L^{-3}\int\int\psi_{k''}^*(\mathbf{r})\psi_0(r)\\&\quad\times\exp[-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}]\{V_{NN}(|\mathbf{x}-\tfrac{1}{2}\mathbf{r}|)\\&\qquad+V_{NP}(|\mathbf{x}+\tfrac{1}{2}\mathbf{r}|)\}d\mathbf{r}d\mathbf{x}.\end{aligned}\tag{10}$$

If we make the change of variables $\mathbf{r}=\mathbf{r}$, $\mathbf{x}-\frac{1}{2}\mathbf{r}=\mathbf{y}$ in the V_{NN} term and $\mathbf{r}=\mathbf{r}$, $\mathbf{x}+\frac{1}{2}\mathbf{r}=\mathbf{y}$ in the V_{NP} term, the matrix element becomes

$$\begin{aligned}\mathfrak{M}=L^{-3}\Big\{&V_{NN}(q)\int\psi_{k''}^*(\mathbf{r})\psi_0(r)\exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\\&+V_{NP}(q)\int\psi_{k''}^*(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\Big\}\end{aligned}\tag{11}$$

with

$$V(q)=\int V(y)\exp(\pm i\mathbf{q}\cdot\mathbf{y})d\mathbf{y},\tag{12a}$$

$$\mathbf{q}=\mathbf{k}'-\mathbf{k}.\tag{12b}$$

The cross section is given by

$$d\sigma=L^3(2\pi/\hbar v)|\mathfrak{M}|^2\rho_E(\mathbf{k}',\mathbf{k}'')d\mathbf{k}'d\mathbf{k}'',\tag{13a}$$

where $v=|\dot{\mathbf{x}}_i|=(3\hbar k/2M)$ is the velocity of the incident particle in the laboratory system and $\rho_E(\mathbf{k}',\mathbf{k}'')d\mathbf{k}'d\mathbf{k}''$

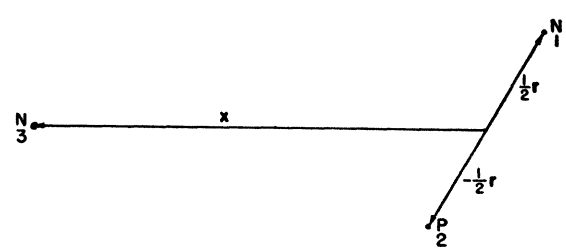

FIG. 1. Change of coordinates from $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ to $\mathbf{R}$, $\mathbf{x}$, $\mathbf{r}$.

[7] We shall later show that the binding energy can be neglected with respect to the incident energy in the conservation of energy. See Appendix D.

is the energy density of final states,

$$\rho_E(\mathbf{k}',\mathbf{k}'')d\mathbf{k}'d\mathbf{k}''=\left(\frac{L}{2\pi}\right)^3 k'^2\left(\frac{\partial k'}{\partial E}\right)_{k''} d\Omega'\left(\frac{L}{2\pi}\right)^3 d\mathbf{k}'';$$

$$\left(\frac{\partial E}{\partial k'}\right)_{k''}=\frac{3h^2k'}{2M}.$$

The cross section therefore becomes

$$\sigma=\int (M^2/9\pi^2h^4)(k'/k)d\Omega'(L/2\pi)^3d\mathbf{k}''|L^3\mathfrak{M}|^2. \quad (13b)$$

To evaluate the total cross section we must now integrate over all values of Ω', Ω'', and k'' for which energy is conserved, remembering however, that we have an upper limit sk on k'' which bounds region (a). The integrations on Ω' and Ω'' are over all solid angles. To obtain the total cross section for region (a) we shall hold k'' (and by (7), therefore k') fixed and change from the variable Ω' to q.

$$q^2=|\mathbf{k}'-\mathbf{k}|^2=k'^2+k^2-2kk'\cos\theta; \quad (14a)$$

$$-2qdq=2kk'd(\cos\theta)=kk'd\Omega'/\pi. \quad (14b)$$

The cross section for region (a) becomes, using the value of $\mathfrak{M}$ from (11),

$$\sigma_a=(2M^2/9\pi h^4k^2)\int_0^{sk}(L/2\pi)^3k''^2dk''\int d\Omega''$$
$$\times\int_{k-k'}^{k+k'}qdq\left|V_{NN}(q)\int\psi_{k''}^*(\mathbf{r})\psi_0(r)\exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\right.$$
$$\left.+V_{NP}(q)\int\psi_{k''}^*(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\right|^2. \quad (15)$$

The region of integration is shown in Fig. 2. It is bounded by the lines $k''=0$, $k''=sk$ and by the ellipse

$$(q-k)^2+(4/3)k''^2=k^2. \quad (16)$$

The point A represents a possible strong interaction between particle 3 and either 1 or 2, and must be avoided.

If we perform the integration first over the entire region between the two vertical lines $q=0$ and $q=2k$, and then subtract the integration over the shaded area, we obtain

$$\sigma_a=\sigma_{1a}+\sigma_{2a}+\sigma_{3a}-\sigma_a', \quad (17)$$

where

$$\sigma_{1a}=(2M^2/9\pi h^4k^2)$$
$$\int_0^{2k}qdqV_{NN}^2(q)\int_0^\infty(L/2\pi)^3d\mathbf{k}''$$
$$\times\left|\int\psi_{k''}^*(\mathbf{r})\psi_0(r)\exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\right|^2, \quad (18a)$$

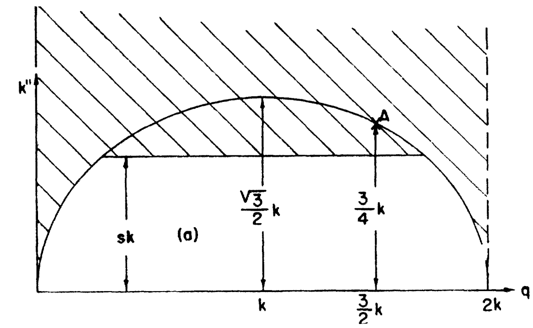

FIG. 2. Region of integration for I in the $k''q$ plane.

$$\sigma_{2a}=(2M^2/9\pi h^4k^2)$$
$$\times\int_0^{2k}qdqV_{NP}^2(q)\int_0^\infty(L/2\pi)^3d\mathbf{k}''$$
$$\times\left|\int\psi_{k''}^*(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\right|^2, \quad (18b)$$

$$\sigma_{3a}=2(2M^2/9\pi h^4k^2)$$
$$\times\int_0^{2k}qdqV_{NN}(q)V_{NP}(q)\int_0^\infty(L/2\pi)^3d\mathbf{k}''$$
$$\times Re\left\{\left(\int\psi_{k''}^*(\mathbf{r})\psi_0(r)\exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\right)^*\right.$$
$$\left.\times\int\psi_{k''}^*(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\right\}, \quad (18c)$$

$$\sigma_a'=(2M^2/9\pi h^4k^2)\int qdq\int(L/2\pi)^3d\mathbf{k}''|L^3\mathfrak{M}|^2. \quad (18d)$$

(shaded region in Fig. 2)

We have finally achieved what we set out to do: to express the total cross section in some way as the sum over final states for the *two-particle* wave functions. The $\int_0^\infty(L/2\pi)^3d\mathbf{k}''$ represents just such a sum, and we can now apply a sum rule to evaluate σ_{1a}, σ_{2a}, and σ_{3a}, which, as we shall show, represent the major portion of σ_a. We shall use

$$\sum_f(\psi_f,M\psi_0)^*(\psi_f,N\psi_0)=\sum_f {}_0M_f^*{}_fN_0={}_0(M^*N)_0, \quad (19)$$

where M and N are hermitian operators, and

$${}_2M_1\equiv\int\psi_2^*(\mathbf{r})M\psi_1(\mathbf{r})d\mathbf{r}\equiv(\psi_2,M\psi_1).$$

σ_{1a}, σ_{2a}, and σ_{3a} therefore become[8]

[8] We have assumed in applying the sum rule that for fixed q, M and N do not depend on the final state. However, it appears that they do, since they seem to depend on the direction as well as on the magnitude of $\mathbf{q}$. But by performing the angular integration on $\mathbf{k}''$ first, one can see that the integrals σ_{1a}, σ_{2a}, and σ_{3a} are functions only of the magnitude of $\mathbf{q}$, since there is no preferred direction possible.

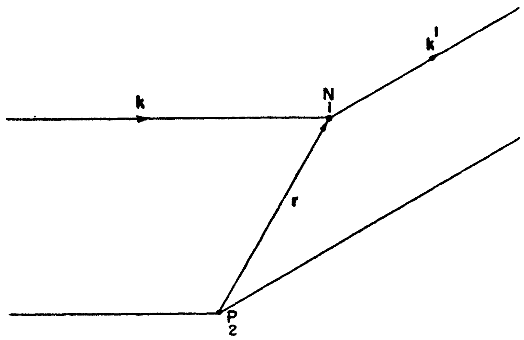

FIG. 3. Superposition of the two-particle scattered waves.

$$\sigma_{1a}=(2M^2/9\pi\hbar^4k^2)\int_0^{2k} qdqV_{NN}^2(q)\int\psi_0^2(r)d\mathbf{r}, \qquad (20a)$$

$$\sigma_{2a}=(2M^2/9\pi\hbar^4k^2)\int_0^{2k} qdqV_{NP}^2(q)\int\psi_0^2(r)d\mathbf{r}, \qquad (20b)$$

$$\sigma_{3a}=2(2M^2/9\pi\hbar^4k^2)\int_0^{2k} qdqV_{NN}(q)V_{NP}(q)$$
$$\times Re\left\{\int\psi_0^2(r)\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\right\}. \qquad (20c)$$

The integration over the shaded region in Fig. 2 is mostly for large k''. In this region, we shall therefore replace $\psi_{k''}^*(\mathbf{r})$ by $L^{-\frac{3}{2}}\exp(-i\mathbf{k}''\cdot\mathbf{r})$, its asymptotic form for large k''. This approximation will be justified in Appendix E.

We have seen that the application of the sum rule to σ_{1a}, σ_{2a}, and σ_{3a} gives a result independent of the form of the $\psi_{k''}(\mathbf{r})$. The only requisite is that the $\psi_{k''}(\mathbf{r})$ form a *complete* set of eigenfunctions, a condition which we satisfy by extending the k'' integration up to ∞ and by including the ground state in the case of triplet interaction. Exactly the same result would have been obtained if we had used the set of plane waves $L^{-\frac{3}{2}}\exp(i\mathbf{k}''\cdot\mathbf{r})$ for the $\psi_{k''}(\mathbf{r})$. We therefore arrive at the result that we can obtain the cross section σ_a by using *plane waves* for the two-particle wave functions in the original region of integration [region (a)]. This is quite startling, since it is in just this region that the two-particle wave functions are *not* plane waves. The reason for this surprising result is that the contribution of the elastic scattering is just canceled by the corrections to the plane waves which must be applied for the low energy free two-particle states. The elastic scattering in a sense "robs" the neighboring low energy two-particle states.

It now appears that the total cross section is a more fundamental quantity than the inelastic cross section, and calculations should be directed accordingly. To be sure, the total cross section is made up of an elastic and an inelastic part, but these complement one another in such a way that the two-particle wave functions approximate a complete set only if both parts are taken together.

In a similar manner the same result can be obtained for regions (b) and (c). It now becomes clear that the total cross section can be obtained without splitting the region of integration by using plane waves throughout, that is, in the entire half-ellipse in Fig. 2.

Of course, in the calculation of the angular and energy distributions of either neutrons or protons, a better approximation to the wave function, $\psi_{k''}(\mathbf{r})$ will be necessary, since there are no sums over two particle states. This matter will be treated in Sec. VI.

We shall now evaluate the total cross section by replacing $\psi_{k''}^*(\mathbf{r})$ by the plane wave

$$\psi_{k''}(\mathbf{r})=L^{-\frac{3}{2}}\exp(i\mathbf{k}''\cdot\mathbf{r}),$$

as we have just discussed, and by extending the region of integration for (15) to the half ellipse in Fig. 2. If we define the momentum distribution (Fourier transform) of the final state as

$$\varphi(z)=\int\exp(i\mathbf{z}\cdot\mathbf{r})\psi_0(r)d\mathbf{r}, \qquad (21)$$

we obtain for the total cross section

$$\sigma=(2M^2/9\pi\hbar^4k^2)\int_0^{2k} qdq\int(1/2\pi)^3d\mathbf{k}''$$
$$\times|V_{NN}(q)\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)+V_{NP}(q)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|)|^2. \qquad (22)$$

If we neglect the contribution above the ellipse, we obtain the expressions in (20) combined to give

$$\sigma'=(2M^2/9\pi\hbar^4k^2)\int_0^{2k} qdq\int d\mathbf{r}$$
$$\times|V_{NN}(q)+V_{NP}(q)\exp(i\mathbf{q}\cdot\mathbf{r})|^2\psi_0^2(r). \qquad (23)$$

We shall now find it possible to give a simple physical picture to both (22) and (23).

Let us consider the wave scattered by each of two particles, separated by a displacement, $\mathbf{r}$, as in Fig. 3. The amplitude for transferring a momentum $\mathbf{q}$ is proportional to $V(q)$ for each particle, and the difference in phase for the two waves is $\mathbf{k}'\cdot\mathbf{r}-\mathbf{k}\cdot\mathbf{r}=\mathbf{q}\cdot\mathbf{r}$. Since the probability of finding the deuteron with a relative separation $\mathbf{r}$ is $\psi_0^2(r)d\mathbf{r}$, the cross section for this calculation will be proportional to

$$\int d\mathbf{r}|V_{NN}(q)+V_{NP}(q)\exp(i\mathbf{q}\cdot\mathbf{r})|^2\psi_0^2(r),$$

which when integrated over angle with appropriate factors gives the expression (23) for σ'. The inaccuracy in this expression then lies in the assumption that the particle not struck remains stationary.

If instead we consider the deuteron as a superposition of plane wave states of momenta $\mathbf{z}$ with amplitude $\varphi(z)$, the amplitude for scattering from the state given in Fig. 4(a) in the laboratory system to the state in Fig. 4(c) is proportional to $V_{NN}(q)\varphi(z)$. However, we may also obtain the final state in Fig. 4(c) by starting with the state in Fig. 4(b) with an amplitude for this transition proportional to $V_{NP}(q)\varphi(|\mathbf{z}-\mathbf{q}|)$. If we now let the momentum of the struck neutron and proton relative to their center of mass be $\mathbf{k}''$, we have $\mathbf{z}=\mathbf{k}''+\frac{1}{2}\mathbf{q}$. The cross section for this calculation will therefore be proportional to

$$|V_{NN}(q)\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)+V_{NP}(q)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|)|^2,$$

which, when integrated over all values of $\mathbf{q}$ and $\mathbf{z}$ that conserve the over-all energy, leads to the expression (22) for σ. The reason for the validity of this picture is that we have justified using plane waves to obtain the *total* cross section.

Before calculating the value of these integrals, we shall remove the simplifying assumptions used in this section.

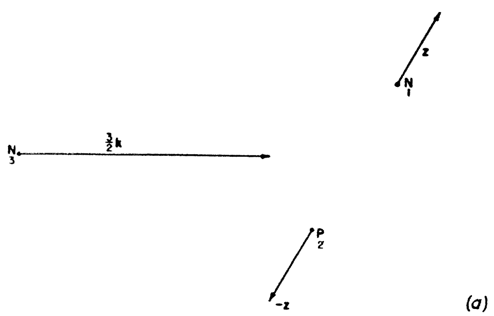

FIG. 4a. Momentum state before a neutron-neutron collision.

III. SPIN DEPENDENT, EXCHANGE FORCES WITH PAULI PRINCIPLE

Since the neutrons are indistinguishable, we must apply the operator $(1-P_{13}Q_{13})/\sqrt{2}$ to both the initial and final state wave functions, where P and Q exchange the space and spin coordinates, respectively. The matrix element will then be

$$\mathfrak{M}=\tfrac{1}{2}(\{1-P_{13}Q_{13}\}\Psi_f\chi_f,\quad V\{1-P_{13}Q_{13}\}\Psi_i\chi_i),\quad (24)$$

where χ_i and χ_f are the initial and final spin wave functions. The perturbation V must be chosen differently for the direct and antisymmetrized terms; it must be taken as the interaction between the particle described as free in the Ψ_i or $P_{13}\Psi_i$ and the other two. This gives

$$\mathfrak{M}=\tfrac{1}{2}(\{1-P_{13}Q_{13}\}\Psi_f\chi_f,\ \{1-P_{13}Q_{13}\}\{V_{NN}{}^{13}+V_{NP}{}^{23}\}\Psi_i\chi_i).$$

Since $1-P_{13}Q_{13}$ is hermitian and

$$(1-P_{13}Q_{13})^2=2(1-P_{13}Q_{13}),$$

we obtain for the matrix element

$$\mathfrak{M}=(\Psi_f\chi_f,\ \{1-P_{13}Q_{13}\}\{V_{NN}{}^{13}+V_{NP}{}^{23}\}\Psi_i\chi_i).\quad (25)$$

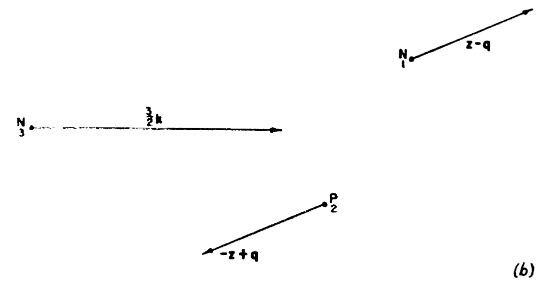

FIG. 4b. Momentum state before a neutron-proton collision.

We shall choose a linear combination of Wigner, Majorana, Bartlett, and Heisenberg forces for the neutron-proton interaction. For the neutron-neutron interaction only the Wigner and Majorana forces are necessary.

$$V_{NP}{}^{23}=V_{NP}(r_{23})(w_p+m_pP_{23}+b_pQ_{23}+h_pP_{23}Q_{23}),\quad (26a)$$

$$V_{NN}{}^{13}=V_{NN}(r_{13})(w_n+m_nP_{13}).\quad (26b)$$

There are eight linearly independent and orthogonal spin functions which may be written as

$$\chi_1=a_1a_2a_3,\quad (27a)$$

$$\chi_2=(1/\sqrt{3})(a_1a_2b_3+a_1b_2a_3+b_1a_2a_3),\quad (27b)$$

$$\chi_3=(1/\sqrt{3})(b_1b_2a_3+b_1a_2b_3+a_1b_2b_3),\quad (27c)$$

$$\chi_4=b_1b_2b_3,\quad (27d)$$

$$\chi_5=[1/(6)^{\frac{1}{2}}](2a_1a_2b_3-a_1b_2a_3-b_1a_2a_3),\quad (27e)$$

$$\chi_6=[1/(6)^{\frac{1}{2}}](2b_1b_2a_3-b_1a_2b_3-a_1b_2b_3),\quad (27f)$$

$$\chi_7=(1/\sqrt{2})(a_1b_2a_3-b_1a_2a_3),\quad (27g)$$

$$\chi_8=(1/\sqrt{2})(b_1a_2b_3-a_1b_2b_3).\quad (27h)$$

The first four are quartet spin functions corresponding to a total spin of $\frac{3}{2}$. The other four are states of total spin $\frac{1}{2}$, the first two of which are symmetric and the last two antisymmetric in the two particles 1 and 2, originally in the deuteron.

The initial state is clearly either a quartet or a doublet symmetric state with a relative frequency of occurrence of two to one. For the quartet states all the spin

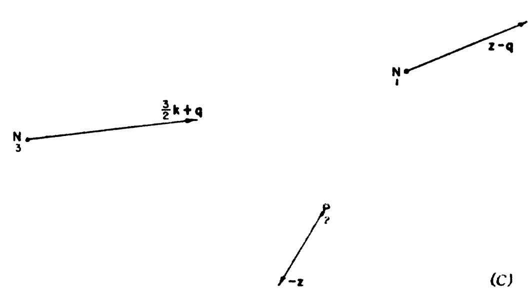

FIG. 4c. Momentum state after either collision with a momentum transfer **q**.

exchange operators leave the state unchanged. The final spin function must therefore be the same as the initial one to give a nonvanishing matrix element. For the doublet symmetric state, χ_5,[9] the spin exchange operators Q give a mixture of χ_5 and χ_7. The final spin function can therefore be either χ_5 or χ_7.

If we put the expressions (26) for V_{NP} and V_{NN} into $\mathfrak{M}$ given in (25), and use the initial state Ψ_i given in (4), we obtain

$$\begin{aligned}L^3\mathfrak{M}=(\Psi_f,\ \{&a_1V(|\mathbf{x}-\tfrac{1}{2}\mathbf{r}|)\exp(i\mathbf{k}\cdot\mathbf{x})\psi_0(r)\\ &+a_2V(|\mathbf{x}-\tfrac{1}{2}\mathbf{r}|)\exp[i\mathbf{k}\cdot(\tfrac{3}{4}\mathbf{r}-\tfrac{1}{2}\mathbf{x})]\psi_0(|\mathbf{x}+\tfrac{1}{2}\mathbf{r}|)\\ &+a_3V(|\mathbf{x}+\tfrac{1}{2}\mathbf{r}|)\exp(i\mathbf{k}\cdot\mathbf{x})\psi_0(r)\\ &+a_4V(|\mathbf{x}+\tfrac{1}{2}\mathbf{r}|)\exp[i\mathbf{k}\cdot(-\tfrac{3}{4}\mathbf{r}-\tfrac{1}{2}\mathbf{x})]\psi_0(|\mathbf{x}-\tfrac{1}{2}\mathbf{r}|)\\ &+a_5V(r)\exp[i\mathbf{k}\cdot(\tfrac{3}{4}\mathbf{r}-\tfrac{1}{2}\mathbf{x})]\psi_0(|\mathbf{x}+\tfrac{1}{2}\mathbf{r}|)\\ &+a_6V(r)\exp[i\mathbf{k}\cdot(-\tfrac{3}{4}\mathbf{r}-\tfrac{1}{2}\mathbf{x})]\psi_0(|\mathbf{x}-\tfrac{1}{2}\mathbf{r}|)\}).\end{aligned}\tag{28}$$

We have used

$$\begin{aligned}&P_{12}\mathbf{x}=\mathbf{x}, && P_{12}\mathbf{r}=-\mathbf{r},\\ &P_{23}\mathbf{x}=-\tfrac{3}{4}\mathbf{r}-\tfrac{1}{2}\mathbf{x}, && P_{23}\mathbf{r}=-\mathbf{x}+\tfrac{1}{2}\mathbf{r},\\ &P_{13}\mathbf{x}=\tfrac{3}{4}\mathbf{r}-\tfrac{1}{2}\mathbf{x}, && P_{13}\mathbf{r}=-\mathbf{x}-\tfrac{1}{2}\mathbf{r},\end{aligned}\tag{29}$$

TABLE I. Values of the coefficients a_l for the quartet, doublet symmetric, and doublet antisymmetric matrix elements in (33) and (35).

	Quartet	Doublet χ_5	Doublet χ_7
a_1	w_n-m_n	$w_n+\frac{1}{2}m_n$	$\frac{1}{2}m_n\sqrt{3}$
a_2	$-w_n+m_n$	$\frac{1}{2}w_n+m_n$	$\frac{1}{2}w_n\sqrt{3}$
a_3	w_p+b_p	$w_p-\frac{1}{2}b_p$	$\frac{1}{2}b_p\sqrt{3}$
a_4	m_p+h_p	$m_p-\frac{1}{2}h_p$	$\frac{1}{2}h_p\sqrt{3}$
a_5	$-w_p-b_p$	$\frac{1}{2}w_p+\frac{1}{2}b_p$	$\frac{1}{2}(w_p-b_p)\sqrt{3}$
a_6	$-m_p-h_p$	$\frac{1}{2}m_p+\frac{1}{2}h_p$	$\frac{1}{2}(m_p-h_p)\sqrt{3}$

$$\begin{aligned}&Q_{12}\chi_1=\chi_1,\quad Q_{12}\chi_5=\chi_5,\quad Q_{12}\chi_7=-\chi_7,\\ &Q_{23}\chi_1=\chi_1,\quad Q_{23}\chi_5=-\tfrac{1}{2}\chi_5+\tfrac{1}{2}\sqrt{3}\chi_7,\\ &\qquad\qquad Q_{23}\chi_7=\tfrac{1}{2}\sqrt{3}\chi_5+\tfrac{1}{2}\chi_7,\\ &Q_{13}\chi_1=\chi_1,\quad Q_{13}\chi_5=-\tfrac{1}{2}\chi_5-\tfrac{1}{2}\sqrt{3}\chi_7,\\ &\qquad\qquad Q_{13}\chi_7=-\tfrac{1}{2}\sqrt{3}\chi_5+\tfrac{1}{2}\chi_7.\end{aligned}\tag{30}$$

The a's[10] are given in Table I. The total cross section can now be obtained as

$$\sigma_{TOT}=\tfrac{2}{3}\sigma_{QUAR}+\tfrac{1}{3}(\sigma_{DOUB\ SYM}+\sigma_{DOUB\ ANTISYM}),\tag{31}$$

where the σ's on the right side of (31) are given by (13b) and (14b) except for a factor $\frac{1}{2}$ due to the identity of the two neutrons in the final state;[11] for example,

$$\sigma_{QUAR}=(M^2/9\pi\hbar^4k^2)\int_0^{\frac{1}{2}k\sqrt{3}}(L/2\pi)^3k''^2dk''\times\int d\Omega''\int_{k-k'}^{k+k'}qdq|L^3\mathfrak{M}_{QUAR}|^2,\tag{32}$$

where $\mathfrak{M}_{QUAR}$ is obtained from (28) by using the quartet values for the a's in Table I.

In considering the final state we shall, as before, divide the $\mathbf{k}'$, $\mathbf{k}''$ space into three regions. In the region in which particle 3 can be considered free the final state is given by (9a). The matrix element, after appropriate changes of variables, becomes

$$L^3\mathfrak{M}=\sum_{l=1}^{6}a_l\int\psi_{k''}^*(\mathbf{r})F_l(\mathbf{r})d\mathbf{r},\tag{33}$$

where

$$F_1(\mathbf{r})=\psi_0(r)\exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})V(q),\tag{34a}$$

$$F_2(\mathbf{r})=\exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})\int d\mathbf{y}V(y)\psi_0(|\mathbf{y}+\mathbf{r}|)\times\exp[-i\mathbf{y}\cdot(\mathbf{q}+\tfrac{3}{2}\mathbf{k})],\tag{34b}$$

$$F_3(\mathbf{r})=\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})V(q)=F_1^*(\mathbf{r}),\tag{34c}$$

$$F_4(\mathbf{r})=\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})\int d\mathbf{y}V(y)\psi_0(|\mathbf{y}-\mathbf{r}|)\times\exp[-i\mathbf{y}\cdot(\mathbf{q}+\tfrac{3}{2}\mathbf{k})]=F_2^*(\mathbf{r}),\tag{34d}$$

$$F_5(\mathbf{r})=V(r)\exp[\tfrac{1}{2}i\mathbf{r}\cdot(\mathbf{q}+3\mathbf{k})]\varphi(|\mathbf{q}+\tfrac{3}{2}\mathbf{k}|),\tag{34e}$$

$$F_6(\mathbf{r})=V(r)\exp[-\tfrac{1}{2}i\mathbf{r}\cdot(\mathbf{q}+3\mathbf{k})]\varphi(|\mathbf{q}+\tfrac{3}{2}\mathbf{k}|)=F_5^*(\mathbf{r}).\tag{34f}$$

When we now take $|L^3\mathfrak{M}|^2$ we obtain 36 terms of the type

$$\int\psi_{k''}^*(\mathbf{r})F_l(\mathbf{r})d\mathbf{r}\int\psi_{k''}(\mathbf{r})F_{l'}^*(\mathbf{r})d\mathbf{r}.$$

As before, we can apply a sum rule in evaluating the total cross section. The procedure is similar to that in the preceding section and justifies the use of plane waves for the $\psi_k''(\mathbf{r})$ throughout the elliptical region in Fig. 2.[12]

[9] χ_6 and χ_8 differ from χ_5 and χ_7 by interchanging a and b so that they need not be considered separately in obtaining the matrix element.

[10] A sample calculation for the doublet χ_5 a's is given in Appendix F.

[11] As in neutron-neutron scattering.

[12] One important difference is that the $F_l(\mathbf{r})$ are not really independent of the final state. In the preceding section we considered only the terms F_1 and F_3 and we were able to show that the variation of the direction of $\mathbf{q}$ over the final states had no effect, but the argument is not valid here since the other F's depend on $\mathbf{q}$. After the Ω'' and $\mathbf{r}$ integrations are performed the terms will still depend on the angle between $\mathbf{q}$ and $\mathbf{k}$ which depends on *both* q and k'', and can be expressed, with the use of (12b) and (7), as

$$\mathbf{k}\cdot\mathbf{q}=\tfrac{1}{2}(k'^2-k^2-q^2)=-\tfrac{1}{4}q^2-\tfrac{2}{3}k''^2.$$

However we now see that, for small k'', where the sum rules are necessary, the angle between $\mathbf{q}$ and $\mathbf{k}$ is reasonably constant and

If we now set

$$\psi_{k''}^*(\mathbf{r})=L^{-\frac{3}{2}}\exp(-i\mathbf{k}''\cdot\mathbf{r})$$

in the matrix element (33), we obtain

$$L^{9/2}\mathfrak{M}=\sum_{l=1}^{6}a_lG_l, \tag{35}$$

where

$$G_1=V_{NN}^{O;\,X,\,A.S.}(q)\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)=V(q)\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|), \tag{36a}$$

$$G_2=V_{NN}^{X;\,O,\,A.S.}(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}-\mathbf{k}''|)\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)=V(q')\varphi(|\mathbf{k}^{iv}+\tfrac{1}{2}\mathbf{q}'|), \tag{36b}$$

$$G_3=V_{NP}^{O}(q)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|)=V(q)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|), \tag{36c}$$

$$G_4=V_{NP}^{X}(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}+\mathbf{k}''|)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|)=V(q'')\varphi(|\mathbf{k}^{vi}+\tfrac{1}{2}\mathbf{q}''|), \tag{36d}$$

$$G_5=V_{NP}^{O,\,A.S.}(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}-\mathbf{k}''|)\varphi(|\tfrac{3}{2}\mathbf{k}+\mathbf{q}|)=V(q')\varphi(|\mathbf{k}^{iv}-\tfrac{1}{2}\mathbf{q}'|), \tag{36e}$$

$$G_6=V_{NP}^{X,\,A.S.}(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}+\mathbf{k}''|)\varphi(|\tfrac{3}{2}\mathbf{k}+\mathbf{q}|)=V(q'')\varphi(|\mathbf{k}^{vi}-\tfrac{1}{2}\mathbf{q}''|), \tag{36f}$$

where the subscripts and superscripts on the $V_b^{c,a}$ are included for clarity, and describe (a) the type of force (O, ordinary, or exchange) between (b) the two particles interacting and (c) whether or not the term arises from the antisymmetrized part of the initial wave function. G_1 and G_2 can arise in two different ways; these are separated by semicolons. The G's have also been expressed in terms of the momentum vectors $\mathbf{k}'''=\mathbf{q}+\mathbf{k}'$, $\mathbf{k}^{iv}$, $\mathbf{k}^{v}=\mathbf{q}''+\mathbf{k}$, $\mathbf{k}^{vi}$ obtained from a cyclic permutation of the three particles. A little thought will show that the forms of the G's are necessarily the same as in the $\mathbf{k}'$, $\mathbf{k}''$ space, since the space exchange operators giving rise to the terms G_2, G_4, G_5, and G_6 do nothing more than permute (and invert the "sense" of) the three particles. In addition the density of final states can be expressed in terms of the permuted momentum vectors since

$$\frac{d\mathbf{k}'d\mathbf{k}''}{dE}=\frac{d\mathbf{k}'''d\mathbf{k}^{iv}}{dE}=\frac{d\mathbf{k}^{v}d\mathbf{k}^{vi}}{dE}.$$

If we now put the matrix element (35) into the cross section (32), we find we have $6+\frac{1}{2}(6\cdot5)=21$ integrals to evaluate. However, because of the symmetry mentioned in the previous paragraph all these integrals are not distinct. In fact, there are only five different integrals

$$I=\int G_1^2=\int G_2^2=\int G_3^2=\int G_4^2=\int G_5^2=\int G_6^2, \tag{37a}$$

$$II=\int G_1G_3=\int G_2G_5=\int G_4G_6, \tag{37b}$$

$$III=\int G_1G_2=\int G_3G_4=\int G_5G_6, \tag{37c}$$

$$IV=\int G_1G_4=\int G_1G_5=\int G_2G_3=\int G_2G_6=\int G_3G_6=\int G_4G_5, \tag{37d}$$

$$V=\int G_2G_4=\int G_1G_6=\int G_3G_5, \tag{37e}$$

$$I=\int_0^{2k}qdqV^2(q)\int_0^Q k''^2dk''\int d\Omega''\varphi^2(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|), \tag{38a}$$

$$II=\int_0^{2k}qdqV^2(q)\int_0^Q k''^2dk''\int d\Omega''\times\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|), \tag{38b}$$

$$III=\int_0^{2k}qdqV(q)\int_0^Q k''^2dk''\int d\Omega''\times V(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}-\mathbf{k}''|)\varphi^2(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|), \tag{38c}$$

$$IV=\int_0^{2k}qdqV(q)\int_0^Q k''^2dk''\int d\Omega''\times V(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}-\mathbf{k}''|)\times\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|), \tag{38d}$$

$$V=\int_0^{2k}qdqV(q)\int_0^Q k''^2dk''\int d\Omega''\times V(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}+\mathbf{k}''|)V(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}-\mathbf{k}''|)\times\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|), \tag{38e}$$

where

$$Q^2=\tfrac{3}{4}q(2k-q). \tag{39}$$

Detailed evaluation of the integrals I–V is carried out in the Appendices A, B, and C. However, there are several qualitative features which may be discussed without specifying a particular potential. If we change

$\mathbf{q}$ may therefore be considered fixed. In addition it can be seen from (36) that the vicinity $k''=0$ contributes only very slightly in all but the first and third terms.

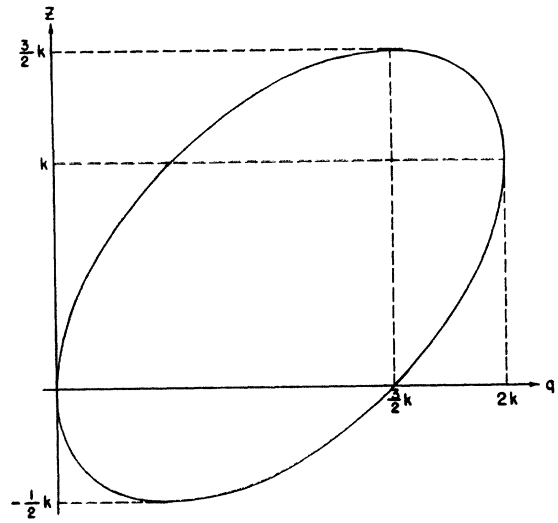

FIG. 5. Region of integration for I in the zq plane.

the variable Ω'' to $z=|\mathbf{k}''+\frac{1}{2}\mathbf{q}|$ as in the appendix, we can express I as

$$I=2\pi\int_0^{2k} dqV^2(q)\int_{\frac{1}{2}q-Q}^{\frac{1}{2}q+Q} z\varphi^2(z)dz \times\{zq-(q^2+z^2-\tfrac{3}{2}qk)\}. \quad (40)$$

The region of integration is shown in Fig. 5. Since the deuteron is large compared with the wavelength of the incident particle, $\varphi(z)$ will be important only for small z. We can obtain a rough approximation to the integral at high energy by integration from $-\infty$ to ∞ on z wherever the z integration goes through $z=0$, that is, for $0\leq q\leq\frac{3}{2}k$ and using zero for the z integral otherwise; i.e., for $q>\frac{3}{2}k$. If we do this, only the even part of the z integrand will contribute and we will obtain for this approximation to I

$$I_0=2\pi\int_0^{\frac{3}{2}k} qdqV^2(q)\int_{-\infty}^{\infty} z^2\varphi^2(z)dz. \quad (41)$$

Since

$$\int_{-\infty}^{\infty} z^2\varphi^2(z)dz=(1/2\pi)\int_{\text{all space}} \varphi^2(z)d\mathbf{z}$$

and

$$\int \varphi^2(z)d\mathbf{z}=(2\pi)^3\int \psi_0{}^2(r)d\mathbf{r}=(2\pi)^3$$

because of closure, I becomes

$$I_0=(2\pi)^3\int_0^{\frac{3}{2}k} qdqV^2(q). \quad (42)$$

As can be seen from (20a, b) this is just what was obtained when we neglected the contribution of the region above the ellipse in Fig. 2 and used the sum rule result exclusively. One slight change is that the upper limit on the q integration has been changed from $2k$ to the more accurate value $\frac{3}{2}k$. This agrees with the *two-particle* picture where the maximum momentum transfer $\frac{3}{2}k$ is obtained in a head-on collision between the incident particle and one of the particles in the deuteron.

We have obtained our results thus far without the use of an explicit potential. However, we must now choose a particular form for the potential and for the corresponding deuteron ground-state wave function. We shall use the Yukawa interaction[13]

$$V(r)=V_0e^{-\mu r}/\mu r \quad (43)$$

with $V_0=67.8$ Mev, $\mu^{-1}=1.18\times10^{-13}$ cm. For the ground state $\psi_0(r)$ we shall use the Hulthén wave function

$$r\psi_0(r)=N(e^{-\alpha r}-e^{-\beta r}) \quad (44)$$

with $N^2=\alpha\beta(\beta+\alpha)/2\pi(\beta-\alpha)^2$, $\alpha=(M\epsilon/\hbar^2)^{\frac{1}{2}}$, $\beta/\alpha=7$. The value of 7 for β/α differs from Chew's value of $5\frac{1}{2}$ and represents an average of the values obtained[14] by using a variational principle on the binding energy, calculating the effective range[15,16] in terms of β/α, satisfying the Schroedinger equation at $r=0$, and using a variational principle on the wave function in momentum space. These criteria give values for β/α of 6.8, 6.9, 7.3, and 7.1, respectively.

We shall consider an incident energy of 90 Mev, for which $k^{-1}=0.72\times10^{-13}$ cm. The latest value of the binding energy[17] is 2.226 Mev and leads to the deuteron radius $\alpha^{-1}=4.314\times10^{-13}$ cm, and $\beta^{-1}=\alpha^{-1}/7=0.616\times10^{-13}$ cm.

Using the Yukawa potential (43), we obtain for I_0, the sum rule approximation to I,

$$I_0=(2\pi)^3(4\pi V_0/\mu)^2(1/2\mu^2)\{1-(1+9k^2/4\mu^2)^{-1}\}. \quad (45)$$

The exact evaluation of the integral I in the appendix shows that the value obtained for I is 14 percent smaller than the value obtained for I_0 at an incident energy of 90 Mev. The main contribution to the 14 percent comes from the fact that we have extended the lower limit on the z integration to $-\infty$. If instead we use

$$\int_{-\frac{1}{2}k}^{\infty} z^2\varphi^2(z)dz$$

in (41), our result for I will be only $4\frac{1}{2}$ percent too small.

A similar phenomenon occurs in the interference term II. If we change variables from k'' and Ω'' to $r_1=|\mathbf{k}''+\frac{1}{2}\mathbf{q}|$ and $r_2=|\mathbf{k}''-\frac{1}{2}\mathbf{q}|$ as in Appendix B, we

[13] These are the values used by Chew (footnote 1) and represent the best fit with the Berkeley data without tensor forces.
[14] E. E. Salpeter, private communication.
[15] G. F. Chew and M. L. Goldberger, Phys. Rev. 77, 470 (1950).
[16] H. A. Bethe and C. Longmire, Phys. Rev. 77, 647 (1950).
[17] R. C. Mobley and R. A. Laubenstein, Phys. Rev. 80, 309 (1950).

obtain

$$II=2\pi\int_0^{2k} dqV^2(q)\int_{\frac{1}{2}q-Q}^{\frac{1}{2}q+Q} r_1\varphi(r_1)dr_1 \times\int_{q-r_1}^{(3qk-r_1^2-q^2)^{\frac{1}{2}}} r_2\varphi(r_2)dr_2. \quad (46)$$

We have shown in Appendix B that at high energy II reduces to

$$II_0=(2\pi)^3\int_0^{\frac{3}{2}} qdqV^2(q)\int\psi_0^2(r)\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}, \quad (47)$$

which is the result obtained in (23) for the interference term evaluated directly from the sum rule, independent of the explicit form of the potential.

The exact evaluation of II in the appendix shows that II is smaller than II_0 by 19 percent at 90 Mev. This is somewhat larger than the error in I_0 but is due to approximately the same cause.

The relative value of I and II at 90 Mev is

$$II=0.46I. \quad (48)$$

This implies quite a large interference term and is surprising in view of the high energy. Because of the factor $V^2(q)$ in the integrands, the most important momentum transfers are of order of magnitude $q\sim\mu$. Since the most important values of r in (47) are $r^{-1}\sim 2\alpha$, the most likely phase factor will be $\exp(i\mu/2\alpha)$. Since μ is *not* much larger than $2\alpha[(\mu/2\alpha)=1.8]$, the interference term will *not* be small. Moreover, the ratio, II/I, will not decrease appreciably with increasing energy. This can be seen from (42) and (47), since neither I_0 or II_0 are appreciably affected by raising the upper limit on q. Of course, we have omitted the constant factor, which is inversely proportional to the energy in each of these terms. However, this does not affect their ratio.

The evaluation of III is considerably more difficult and has been done only approximately in the appendix. The value obtained is

$$III=0.43I, \quad (49)$$

which is quite large. This integral represents interference between direct and exchange *two*-particle scattering (G_1G_2, G_3G_4, G_5G_6) and is similar to the corresponding term arising in either neutron-proton or neutron-neutron scattering. It need not be calculated accurately, since it can be obtained from the two-body scattering experiments. We shall later group it with terms which will represent the experimentally observed neutron-neutron and neutron-proton cross sections.

The integrals IV and V are also evaluated approximately and represent more complicated interference terms. Their values are

$$IV=0.11I, \quad (50)$$

$$V=0.03I. \quad (51)$$

We are now in a position to evaluate the total cross section. In order to do so, we must specify the exchange character and spin dependence of both the neutron-proton and neutron-neutron forces. For the neutron-proton interaction we shall use the spin-dependent Serber force

$$V(r)=V_0{}^T\frac{e^{-\mu r}}{\mu r}\left(\frac{1+P}{2}\right)\left(\frac{1+\eta}{2}+\frac{1-\eta}{2}Q\right), \quad (52)$$

where $\eta=V_0{}^S/V_0{}^T=46.5/67.8=0.686$ and for the neutron-neutron force

$$V(r)=V_0{}^T(e^{-\mu r}/\mu r)\xi[\tfrac{1}{2}(1+\delta)+\tfrac{1}{2}(1-\delta)P], \quad (53)$$

where δ describes the space exchange character of the force.[18] We therefore have

$$\left.\begin{array}{ll} w_p=m_p=\frac{1}{4}(1+\eta), & b_p=h_p=\frac{1}{4}(1-\eta) \\ w_n=\frac{1}{2}(1+\delta)\xi, & m_n=\frac{1}{2}(1-\delta)\xi \end{array}\right\}, \quad (54)$$

so that the a's take on the values in Table II. If we now use (31), (32), and (35–38), we obtain for the total cross section

$$\sigma_{TOT}=(\sigma_{NP}+\sigma_{INTERF}+\sigma_{NN}), \quad (55)$$

where

$$\sigma_{NP}=(M^2/9\pi\hbar^4k^2)(1/2\pi)^3\{\tfrac{1}{4}(3+\eta^2)(I+III) -\tfrac{1}{16}(5-2\eta+\eta^2)(II+2IV+V)\}, \quad (56a)$$

$$\sigma_{INTERF}=(M^2/9\pi\hbar^4k^2)(1/2\pi)^3\{\tfrac{1}{4}(1+\eta) \times(II+2IV+V)+\tfrac{1}{4}\delta(5+\eta)(II-V)\}, \quad (56b)$$

$$\sigma_{NN}=(M^2/9\pi\hbar^4k^2)(1/2\pi)^3 \times\{\tfrac{1}{2}(I+III)+\tfrac{3}{2}\delta^2(I-III)\}. \quad (56c)$$

We have used the subscripts NP, $INTERF$, and NN to correspond to the source of these terms as evidenced in the particular power of ξ in each term.

If we had made separate calculations of free neutron-proton and neutron-neutron cross sections at the same incident energy using the Born approximation, we would have obtained

$$\sigma_{NP}{}^{\text{free}}=(M^2/9\pi\hbar^4k^2)\tfrac{1}{4}(3+\eta^2)\left\{\int_0^{\frac{3}{2}k} qdqV^2(q) +\int_0^{\frac{3}{2}k} qdqV(q)V(|\tfrac{3}{2}\mathbf{k}+\mathbf{q}|)\right\}, \quad (57a)$$

TABLE II. Values of the coefficients a_i for the neutron-proton force in (52) and the neutron-neutron force in (53).

	Quartet	Doublet χ_5	Doublet χ_7
a_1	$\delta\xi$	$\xi(3+\delta)/4$	$\xi(1-\delta)\sqrt{3}/4$
a_2	$-\delta\xi$	$\xi(3-\delta)/4$	$\xi(1+\delta)\sqrt{3}/4$
a_3	$\frac{1}{2}$	$(1+3\eta)/8$	$(1-\eta)\sqrt{3}/8$
a_4	$\frac{1}{2}$	$(1+3\eta)/8$	$(1-\eta)\sqrt{3}/8$
a_5	$-\frac{1}{2}$	$\frac{1}{4}$	$\eta\sqrt{3}/4$
a_6	$-\frac{1}{2}$	$\frac{1}{4}$	$\eta\sqrt{3}/4$

[18] δ is the relative magnitude of the neutron-neutron force in odd and even l states.

$$\sigma_{NN}{}^{\text{free}}=(M^2/9\pi\hbar^4k^2)\xi^2\left\{\tfrac{1}{2}(1+3\delta^2)\int_0^{\frac{3}{2}k}qdqV^2(q)\right.$$

$$\left.+\tfrac{1}{2}(1-3\delta^2)\int_0^{\frac{3}{2}k}qdqV(q)V(|\tfrac{3}{2}\mathbf{k}+\mathbf{q}|)\right\}. \quad (57b)$$

Since the integrals I and III are given by (38a) and (38c) as

$$I=\int qdqV^2(q)\int\varphi^2(z)d\mathbf{z}, \quad (58a)$$

$$III=\int qdqV(q)\int V(|\tfrac{3}{2}\mathbf{k}+\mathbf{q}-\mathbf{z}|)\varphi^2(z)d\mathbf{z} \quad (58b)$$

integrated over appropriate values for $\mathbf{z}$, where $\mathbf{z}=\mathbf{k}''+\frac{1}{2}\mathbf{q}$, and since

$$\int_{\text{all } \mathbf{z}}\varphi^2(z)d\mathbf{z}=(2\pi)^3,$$

we see that σ_{TOT} contains terms which are very similar to the two-body cross sections. There are therefore three corrections to considering the total cross section as the sum of the two-body cross sections.

(a) The integrals must be performed over only those momentum states of the deuteron permitted by conservation of energy.

(b) The interference term must be taken into account.

(c) There is a correction to the neutron-proton cross section which arises from the interference between the exchange scattering of the direct and antisymmetrized neutrons by the proton. This is the second term in (56a) and is considerably smaller than the first.

Since the Born approximation for the total scattering gives a result which is a corrected sum of the two-particle cross sections calculated with the Born approximation, it seems reasonable that the *exact* total cross section would give the sum of the *exact* two-particle cross sections, corrected in a similar way. If this is so, we ought to use the measured two-particle cross sections rather than those calculated by Born approximation when evaluating the total scattering. An argument which may make this more plausible is the following.

Consider the next order Born approximation, that is, double scattering in the potential wells of the two nucleons. This may take place either twice in the neutron well, once in each well, or twice in the proton well. Since the radius of the deuteron is considerably larger than the range of the forces, the term corresponding to successive scattering by the neutron and the proton will be smaller than the term corresponding to a double scattering in either well, and will be omitted. If we also neglect successive scattering in higher orders, then, roughly speaking, the amplitude for the total scattering is just the sum of the first, second, and higher order scatterings in the neutron well and the first, second, and higher order scattering in the proton well, that is, the sum of the true neutron-neutron and neutron-proton scattering amplitudes (with appropriate phase factors). We may therefore feel justified in replacing the appropriate terms in (56) by the observed values wherever possible. Although this treatment is far from rigorous,[19] it should be a considerable improvement over using the Born approximation directly.

Let us therefore adjust the constant in (56) so that $\sigma_{NP}{}^{\text{obs}}$ is 83 mb, the observed neutron-proton cross section[5] where $\sigma_{NP}=\sigma_{NP}{}^{\text{obs}}-\Delta\sigma_{NP}$. The corresponding neutron-deuteron cross section[5] is 117 mb. Since we know all the quantities in σ_{TOT} except δ and ξ, we obtain the following relation between δ and ξ:

$$[1-0.15]+\xi[0.24+0.49\delta]+\xi^2[0.58+0.69\delta^2]=117/83.$$

If we set $\delta=+1$, 0, -1 corresponding to ordinary, Serber, and pure exchange forces, respectively, for the neutron-neutron interaction, we can determine ξ and therefore the relative value of each term. The results are given in Table III.

It may be noticed from (56a, b) that the form of the interference term for $\delta=0$ is the same as the correction for the neutron-proton cross section. As we mentioned before, the reason for this is that the neutron-proton correction term is mainly an interference term between the exchange scattering of the direct and antisymmetrized neutrons by the proton, a term which has exactly the same form as the interference between the incident neutron scattered by the neutron and by the proton. The similarity of these two terms can be seen more easily if we antisymmetrize the final state instead of the initial state. A neutron-proton exchange collision will then give rise to a proton in the forward direction and two slowly moving neutrons. If we now antisymmetrize this final state in the two neutrons, we see that the two terms represent very similar states and therefore will interfere. This interference will therefore be analogous to the ordinary interference between the small momentum transfer neutron-neutron and neutron-proton collisions. The similarity of these two terms in magnitude is most likely accidental, since they depend differently on η and ξ. We evaluated the form of these two terms for a neutron-proton force of arbitrary

TABLE III. Division of σ_{ND} between σ_{NP}, σ_{INTERF}, and σ_{NN} as a function of the exchange character of the neutron-neutron force.

δ	1	0	-1
N-N force	1	$\frac{1}{2}(1+P)$	P
$\xi=V_{NN}/V_{NP}{}^T$	0.44	0.80	0.77
σ_{NP}	71 mb	71 mb	71 mb
σ_{INTERF}	26 mb	16 mb	-16 mb
σ_{NN}	20 mb	30 mb	62 mb
σ_{TOT}	117 mb	117 mb	117 mb

[19] Perhaps the most important error is due to the phase factors in the exact scattering amplitudes, which cannot easily be determined from experiment.

exchange character and found that only in the case of a Serber force for both interactions is the form of the two correction terms, σ_{INTERF} and $\Delta\sigma_{NP}$, the same.

If we are to choose between the values of $+1$, 0, and -1 for δ, we must make use of other measured values, such as the elastic scattering and the angular and energy distributions. For this reason we have repeated Chew's calculation of the elastic scattering,[1] with arbitrary δ and ξ. This work is discussed in the next section.

IV. ELASTIC SCATTERING

The elastic scattering is obtained from (31), (32), and (33) by replacing the two-particle wave function $\psi_{k''}(\mathbf{r})$ by the ground state $\psi_0(r)$. In addition, the integration over $\mathbf{k}''$ reduces to the single state $\psi_0(r)$, so that $(L/2\pi)^3\int d\mathbf{k}''$ in (32) is replaced by unity. We then obtain for the elastic cross section[20]

$$\sigma_{\mathrm{el}}=(2M^2/9\pi\hbar^4k^2)\int_0^{2k} qdq\left|\sum_{l=1}^{6} a_lH_l\right|^2, \tag{59}$$

where

$$H_1=H_3\equiv I_1=V(q)\int d\mathbf{r}\psi_0^2(r)\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r}), \tag{60a}$$

$$H_2=H_4\equiv I_2=\int d\mathbf{r}\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})\int d\mathbf{y}V(y)\times\psi_0(|\mathbf{y}-\mathbf{r}|)\exp[-i\mathbf{y}\cdot(\mathbf{q}+\tfrac{3}{2}\mathbf{k})], \tag{60b}$$

$$H_5=H_6\equiv I_3=\varphi(|\mathbf{q}+\tfrac{3}{2}\mathbf{k}|)\int d\mathbf{r}\psi_0(r)V(r)\times\exp[\tfrac{1}{2}i(\mathbf{q}+3\mathbf{k})\cdot\mathbf{r}], \tag{60c}$$

and the a_l's are given as before in Table I. The integrals I_1, I_2, and I_3 correspond to Chew's[1] notation for the elastic cross section.

With our assumptions for $V(r)$ and $\psi_0(r)$ as stated in (43) and (44), I_1 and I_3 can be calculated explicitly, but I_2 must be evaluated numerically. The term I_1 represents a small momentum transfer to either the neutron or proton in the deuteron and is therefore the main term, giving a large forward peak for the incident neutrons. I_2 and I_3 represent the effects of antisymmetrization and exchange forces and include the possibility of "pick-up" processes, that is, collisions in which the incident neutron forms a deuteron by "picking up" the proton in the initial deuteron, giving a small backward peak for the free neutron.

In our sum over final spins, it must be remembered that the deuteron exists only in a triplet spin state. The final spin must be either a quartet or doublet symmetric spin state, and therefore the doublet antisymmetric term in (31) will be absent. The elastic cross section is then given by

$$\sigma_{\mathrm{el}}=\tfrac{2}{3}\sigma_{\mathrm{el\ quar}}+\tfrac{1}{3}\sigma_{\mathrm{el\ doub}}, \tag{61}$$

where $\sigma_{\mathrm{el\ quar}}$ and $\sigma_{\mathrm{el\ doub}}$ are given by (59) with the appropriate a_l's.

We have calculated I_1 and I_3 explicitly and an approximate expression for I_2 assuming the range of nuclear forces much smaller than the radius of the deuteron[21] $(1/\mu\ll 1/\alpha)$. The integrations in (59) have been performed numerically and the final result for the elastic cross section can be obtained in a form similar to (56). If we use the values of δ and ξ in Table III, we obtain the values for σ_{el} of 80, 60, 30 mb for the direct, Serber, and pure exchange neutron-neutron forces, respectively. Since the measured value for the elastic cross section is[22] 48 mb, it appears a Serber type exchange force between the two neutrons is not inconsistent.

It may seem surprising that the elastic scattering is such a large fraction of the total. As we have previously pointed out, this is due to the fact that small momentum transfers $(q\sim\mu)$, independent of energy, are favored at high energies. For this reason the ratio of the elastic to the total cross section should approach a constant as the energy is increased. The value of this constant is related to the probability of the deuteron remaining bound after absorbing a momentum transfer of order $q\sim\mu$.

The wide variation of the elastic cross section with the exchange character of the neutron-neutron force is due in part to the fact that an exchange may result in the wrong spin state for the proton and the exchanged neutron, thus causing them to separate. It also seems that the interference is destructive when there is a neutron-neutron exchange. This is similar to what happens in the total scattering where the interference term (Table III) is negative for a pure exchange neutron-neutron force.

V. LOW ENERGY PROTON COMPONENT

The cross section for the production of protons of low energy has recently been measured[23] in order to obtain additional information about the neutron-neutron interaction. It was thought that low energy protons result most probably from neutron-neutron collisions which break up the deuteron. We shall calculate the low energy proton cross section by first calculating the *total* cross section for low energy free and bound protons.

This cross section for the production of slow protons will necessarily include those that remain bound in elastic collisions. For this reason, we must evaluate the elastic cross section for slow deuterons and subtract it from the calculated total low energy proton cross section. This will give us what we are after, namely, the cross section for the production of low energy *free* protons.

Protons resulting from neutron-deuteron scattering

[20] The factor $\frac{1}{2}$ (footnote 11) is no longer present, since the neutron and deuteron in the final state are distinguishable.

[21] This is admittedly not too good, but we shall only be interested in the qualitative features of the results.

[22] W. Powell (private communication).

[23] W. Powell, Phys. Rev. **79**, 219 (1950).

fall roughly into two groups: a high energy group which results from small momentum transfer neutron-proton exchange collisions, and a low energy group which results from all low momentum transfer collisions except the neutron-proton exchange. These groups are quite well separated, so that the total cross section for the production of protons in the low energy group will involve a sum over a wide range of final two-particle states for all terms except those due to neutron-proton exchange collisions, namely, F_4 and F_6 in (34). This can also be seen from (36) in which the final momentum of the proton in the *laboratory* system is $\mathbf{z}=\mathbf{k}''+\frac{1}{2}\mathbf{q}$. For small $\mathbf{z}$ all the terms except G_4 and G_6 can be large when integrated over $\mathbf{q}$. G_1 and G_3 are large near $\mathbf{q}=0$, and G_2 and G_5 are large near $\mathbf{q}=-\frac{3}{2}\mathbf{k}$. We shall therefore set $G_4=G_6=0$ in (35) to evaluate the low energy proton component. Using the values of δ and ξ in Table I we obtain 66, 69, 75 mb for the direct, Serber, and pure exchange neutron-neutron forces, respectively.

As we previously mentioned, the part of the elastic cross section giving rise to low energy deuterons must be subtracted. In order to do this we must confine the integrations in (59) to values of q for which the deuteron has low energy. Since the momentum of the deuteron in the laboratory system after the collision is just $\mathbf{q}$, we can restrict the range of integration in q from 0 to k. In this way, the backward peaks for I_2 and I_3 mentioned in Sec. IV are omitted. The values of the integrals in (60) change somewhat and lead to cross sections for the collisions resulting in low energy deuterons of 65, 50, 25 mb for the direct, Serber, and pure exchange neutron-neutron forces, respectively.

If we now subtract the cross sections just evaluated, we obtain 1, 19, 50 mb for the cross section for the production of low energy protons for the direct, Serber, and pure exchange neutron-neutron forces, respectively.

The experimental value reported for this cross section is 6 mb in the backward direction. For lack of more detailed information we can assume that, since the low energy protons are more or less spherically symmetric, the cross section in all directions[24] is 12 mb. This evidence is therefore once again not inconsistent with a Serber type exchange force between the two neutrons. The discrepancy may be accounted for by the fact that there are a considerable number of protons of quite small energy which may not have been counted.

It may be worthwhile pointing out that the low energy proton cross section is not directly connected with the neutron-neutron cross section. The bound states play an important part and must be included in the calculation.

[24] Since the system as a whole has forward motion, there will probably be more protons forward than backward. This will raise the 12 mb into closer agreement with the calculated value for a Serber exchange neutron-neutron force. We have investigated the approximate angular distribution of low energy protons in the next section.

VI. ENERGY DISTRIBUTION OF PROTONS AND NEUTRONS AT VARIOUS ANGLES

In order to calculate the energy distribution of protons or neutrons at various angles, we can no longer use plane waves for the $\psi_{k''}(\mathbf{r})$; there is no complete sum over final states to which we may apply a sum rule. We must instead find a more accurate expression for $\psi_{k''}(\mathbf{r})$ to use in the integrations.

Before investigating the problem in detail we shall be able to make a rough guess as to the nature of the angular and energy distribution from the qualitative features already discussed. There will be two main groups of protons: a group at low energy resulting from neutron-neutron and ordinary neutron-proton collisions (favoring small momentum transfers) which break up the deuteron, and a group of high energy protons which result from neutron-proton exchange collisions. The low energy group will be roughly isotropic, since the momentum distribution of the particles in the deuteron is isotropic; but the high energy proton group will be confined to the forward direction as in the scattering of neutrons by protons.

In order to set up the expressions for the angular distribution, we must transform the momenta $\mathbf{k}'$ and $\mathbf{k}''$ to the laboratory system. We shall outline the calculation for the more easily measurable proton distribution.

For the high energy proton group, it will be more convenient to use the description of the final state in which particle 2 is free, namely, (9c) in terms of the quantities $\mathbf{k}^{v}$ and $\mathbf{k}^{vi}$. The angular distribution of the low energy group will require a description of the final state in which either particle 1 or 3 is free.

If we now put the final state (9c) into (28), the matrix element $\mathfrak{M}$ in (33) becomes

$$L^3\mathfrak{M}=\sum_{l=1}^{6} a_l \int \psi_{k^{vi}}^*(\mathbf{r})F_l'(\mathbf{r})d\mathbf{r}, \tag{62}$$

where

$$F_1'(\mathbf{r})=V(r)\exp[\tfrac{1}{2}i\mathbf{r}\cdot(\mathbf{q}+3\mathbf{k})]\varphi(|\mathbf{q}''+\tfrac{3}{2}\mathbf{k}|), \tag{63a}$$

$$F_2'(\mathbf{r})=V(r)\exp[-\tfrac{1}{2}i\mathbf{r}\cdot(\mathbf{q}+3\mathbf{k})]\varphi(|\mathbf{q}''+\tfrac{3}{2}\mathbf{k}|), \tag{63b}$$

$$F_3'(\mathbf{r})=\exp[-\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r}]\int d\mathbf{y}V(y)\psi_0(|\mathbf{y}+\mathbf{r}|) \times\exp[-i\mathbf{y}\cdot(\mathbf{q}''+\tfrac{3}{2}\mathbf{k})], \tag{63c}$$

$$F_4'(\mathbf{r})=\psi_0(r)\exp(-\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})V(q''), \tag{63d}$$

$$F_5'(\mathbf{r})=\exp[\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r}]\int d\mathbf{y}V(y)\psi_0(|\mathbf{y}-\mathbf{r}|) \times\exp[-i\mathbf{y}\cdot(\mathbf{q}''+\tfrac{3}{2}\mathbf{k})], \tag{63e}$$

$$F_6'(\mathbf{r})=\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})V(q''). \tag{63f}$$

As mentioned previously, these expressions can be obtained directly by a proper cyclic transformation of the relative coordinates. The similarity between (63) and (34) is evident.

The relative importance of the various terms in (63) can be obtained *qualitatively* by using the plane wave expression for $\psi_{k^{vi}}(\mathbf{r})$, that is by using

$$\psi_{k^{vi}}^*(\mathbf{r})=L^{-\frac{3}{2}}\exp(-i\mathbf{k}^{vi}\cdot\mathbf{r}).$$

The matrix element (62) can now be written as

$$L^{9/2}\mathfrak{M}=\sum_1^6 a_l G_l', \tag{64}$$

where

$$G_1'=\varphi(|\mathbf{q}''+\tfrac{3}{2}\mathbf{k}|)\overset{O;\,X,\,A.S.}{V_{NN}}(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}''-\mathbf{k}^{vi}|), \tag{65a}$$

$$G_2'=\varphi(|\mathbf{q}''+\tfrac{3}{2}\mathbf{k}|)\overset{X;\,A.S.}{V_{NN}}(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}''+\mathbf{k}^{vi}|), \tag{65b}$$

$$G_3'=\varphi(|\mathbf{k}^{vi}+\tfrac{1}{2}\mathbf{q}''|)\overset{O}{V_{NP}}(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}''-\mathbf{k}^{vi}|), \tag{65c}$$

$$G_4'=\varphi(|\mathbf{k}^{vi}+\tfrac{1}{2}\mathbf{q}''|)\overset{X}{V_{NP}}(q''), \tag{65d}$$

$$G_5'=\varphi(|\mathbf{k}^{vi}-\tfrac{1}{2}\mathbf{q}''|)\overset{A.S.}{V_{NP}}(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}''+\mathbf{k}^{vi}|), \tag{65e}$$

$$G_6'=\varphi(|\mathbf{k}^{vi}-\tfrac{1}{2}\mathbf{q}''|)\overset{X,\,A.S.}{V_{NP}}(q''). \tag{65f}$$

(65) is of course identical to (36) written in terms of $\mathbf{q}''$ and $\mathbf{k}^{vi}$

The high energy proton group is obtained in the vicinity $\mathbf{q}''\approx 0$, $\mathbf{k}^v\approx\mathbf{k}$, $\mathbf{k}^{vi}\approx 0$. Since both $\varphi(\frac{3}{2}k)$ and $V(\frac{3}{2}k)$ are small compared to $\varphi(0)$ and $V(0)$, respectively, most of the contribution to the cross section comes from the terms G_4' and G_6'. We shall therefore consider only F_4' and F_6' in (63) with a corrected expression for $\psi_{k^{vi}}(\mathbf{r})$ in order to obtain the high energy proton differential cross section.

If we now select a particular proton direction and energy, there still remain two degrees of freedom; these are the two variables needed to specify the direction of $\mathbf{k}^{vi}$. In order to obtain the differential cross section as a function of the proton momentum, we must square the matrix element (62), multiply by the appropriate density of final states, and integrate over Ω^{vi}, the direction associated with $\mathbf{k}^{vi}$. The differential cross section for high energy protons will therefore be given by[25]

$$d\sigma=(M/12\pi^2\hbar^2k)(L/2\pi)^3(d\mathbf{k}^v d\mathbf{k}^{vi}/dE)|L^3\mathfrak{M}|^2. \tag{66}$$

Since the energy in the center-of-mass system is given from (3) and (6) as

$$E=(\hbar^2k^{v2}/(4/3)M)+(\hbar^2k^{vi2}/M),$$

we may use

$$(\partial E/\partial k^{vi})_{k^v}=(2\hbar^2k^{vi})/M.$$

[25] The factor $\frac{1}{2}$ due to the identity of the two neutrons has been included as before.

If we transform the momentum $\mathbf{k}^v$ to the momentum of the proton in the laboratory system

$$\mathbf{k}_p=\mathbf{k}^v+\tfrac{1}{2}\mathbf{k}=\mathbf{q}''+\tfrac{3}{2}\mathbf{k};\quad d\mathbf{k}^v=d\mathbf{k}_p, \tag{67}$$

we obtain for the differential cross section

$$d\sigma=(M^2/24\pi^2\hbar^4k)(1/2\pi)^3d\mathbf{k}_pk^{vi}\int d\Omega^{vi}|L^{9/2}\mathfrak{M}|^2, \tag{68}$$

where the matrix element

$$L^3\mathfrak{M}=V(q'')\int d\mathbf{r}\psi_{k^{vi}}^*(\mathbf{r})\psi_0(r)\times\{a_4\exp(-\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})+a_6\exp(\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})\} \tag{69}$$

is obtained from (62) and (63) by using only the terms $l=4$ and $l=6$.

As before the cross section will be the sum of the quartet and doublet cross sections. However, we must now remember that the two-particle wave functions $\psi_{k^{vi}}(\mathbf{r})$ for particles 1 and 3 will depend on the spin symmetry between these two particles (triplet or singlet). Since the spin states χ_5 and χ_7 are neither symmetric nor antisymmetric in particles 1 and 3, we shall change our complete set of spin states to χ_1, χ_2, χ_3, χ_4, χ_5', χ_6', χ_7', χ_8', where the four quartet spin states are unchanged and the doublet spins states are given by[26]

$$\chi_5'=[1/(6)^{\frac{1}{2}}](2a_1b_2a_3-a_1a_2b_3-b_1a_2a_3)=Q_{23}\chi_5=-\tfrac{1}{2}\chi_5+\tfrac{1}{2}\sqrt{3}\chi_7, \tag{70a}$$

$$\chi_7'=(1/\sqrt{2})(a_1a_2b_3-b_1a_2a_3)=Q_{23}\chi_7=\tfrac{1}{2}\sqrt{3}\chi_5+\tfrac{1}{2}\chi_7. \tag{70b}$$

We now obtain the cross section by summing over the spin states χ_1, χ_5', and χ_7' giving

$$d\sigma=\tfrac{2}{3}(d\sigma)_{\chi_1}+\tfrac{1}{3}(d\sigma)_{\chi_5'}+\tfrac{1}{3}(d\sigma)_{\chi_7'}, \tag{71}$$

where each $d\sigma$ is given by (68). We must use the triplet wave function $\psi_{k^{vi}}{}^T(\mathbf{r})$ in $(d\sigma)_{\chi_1}$ and $(d\sigma)_{\chi_5'}$ and the singlet wave function $\psi_{k^{vi}}{}^S(\mathbf{r})$ in $(d\sigma)_{\chi_7'}$. The corresponding values of a_4 and a_6, obtained from (70) and Table II, are

$$(a_4)_{\chi_5'}=-\tfrac{1}{2}(a_4)_{\chi_5}+\tfrac{1}{2}\sqrt{3}(a_4)_{\chi_7}=(1-3\eta)/8, \tag{72a}$$

$$(a_6)_{\chi_5'}=-\tfrac{1}{2}(a_6)_{\chi_5}+\tfrac{1}{2}\sqrt{3}(a_6)_{\chi_7}=-(1-3\eta)/8, \tag{72b}$$

$$(a_4)_{\chi_7'}=\tfrac{1}{2}\sqrt{3}(a_4)_{\chi_5}+\tfrac{1}{2}(a_4)_{\chi_7}=(1+\eta)\sqrt{3}/8, \tag{72c}$$

$$(a_6)_{\chi_7'}=\tfrac{1}{2}\sqrt{3}(a_6)_{\chi_5}+\tfrac{1}{2}(a_6)_{\chi_7}=(1+\eta)\sqrt{3}/8. \tag{72d}$$

We must now evaluate the integrals

$$J_1=\int d\Omega^{vi}\left|\int d\mathbf{r}\psi_{k^{vi}}^*(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})\right|^2 \tag{73a}$$

[26] χ_6' and χ_8' are obtained from χ_5' and χ_7', respectively, by interchanging a and b.

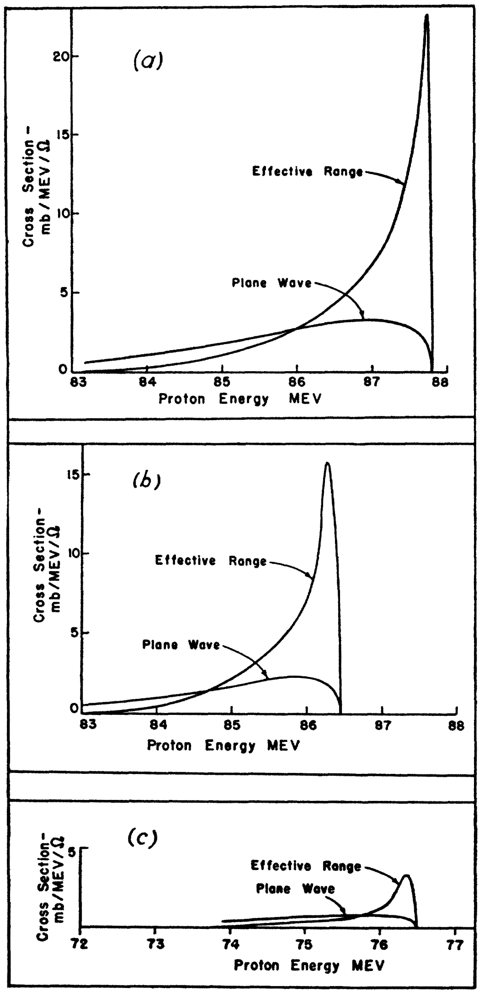

FIG. 6. Approximate energy spectrum for high energy protons in the forward direction at laboratory angles of (a) 0°, (b) 10°, and (c) 30°.

and

$$J_2=\int d\Omega^{vi}Re\left\{\int d\mathbf{r}\psi_{k^{vi}}(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})\times\int d\mathbf{r}\psi_{k^{vi}}^*(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})\right\} \quad (73b)$$

for both the triplet and singlet $\psi_{k^{vi}}(\mathbf{r})$. Since we are interested mainly in the proton cross section in the forward direction at high energies, $\mathbf{k}^{vi}$ and $\mathbf{q}''$ will be quite small. For simplicity we shall set[27] $\mathbf{q}''=0$ in J_1 and J_2. In this rough approximation only the spherically symmetric part of $\psi_{k^{vi}}(\mathbf{r})$ enters. The resulting integrals for small k^{vi} are

$$\int_0^\infty r^2dr\psi_{k^{vi}}{}^T(r)\psi_0(r) \quad\text{and}\quad \int_0^\infty r^2dr\psi_{k^{vi}}{}^S(r)\psi_0(r).$$

[27] A more accurate method of calculating J_1 and J_2 is given in Appendix G, but the present method is adequate for our purposes.

The first of these vanishes because the triplet wave function $\psi_{k^{vi}}{}^T(r)$ is necessarily orthogonal to the ground state $\psi_0(r)$. The remaining integral is the same as the one that occurs in the photo-magnetic disintegration of the deuteron. The integral has been evaluated by Bethe and Longmire[28] at low energy using the effective range method. The result, after proper normalization, is

$$J_s\equiv L^{\frac{1}{2}}\int_0^\infty r^2dr\psi_{k^{vi}}{}^S(r)\psi_0(r)=\frac{N\sin\delta_s}{k^{vi}}\left\{\frac{\alpha+k^{vi}\cot\delta_s}{\alpha^2+k^{vi2}}-\tfrac{1}{4}(r_{0t}+r_{0s})\right\}, \quad (74)$$

where r_{0t} and r_{0s} are the triplet and singlet effective ranges and N is the ground-state normalization factor given by

$$N^2=\alpha/[2\pi(1-\alpha r_{0t})].$$

This value of N is equivalent to that given in (44), leading to the relation between α, β and r_{0t}

$$r_{0t}=4/(\alpha+\beta)-1/\beta.$$

The singlet phase shift δ_s is given by

$$k^{vi}\cot\delta_s=-1/a_s+\tfrac{1}{2}k^{vi2}r_{0s}, \quad (75)$$

where a_s in the singlet scattering length. The integral J_s can therefore be written as

$$J_s=\left\{\frac{\alpha}{2\pi(1-\alpha r_{0t})}\right\}^{\frac{1}{2}}\{k^{vi2}+(-1/a_s+\tfrac{1}{2}k^{vi2}r_{0s})^2\}^{-\frac{1}{2}}\times\left\{\frac{\alpha-1/a_s+\frac{1}{2}k^{vi2}r_{0s}}{\alpha^2+k^{vi2}}-\tfrac{1}{4}(r_{0t}+r_{0s})\right\}. \quad (76)$$

Since the integrals J_1 and J_2 vanish in first approximation for triplet wave functions, the cross section becomes, using (68), (69), (71), and (72)

$$d\sigma=(M^2k^{vi}/48\pi^2\hbar^4k)V^2(q'')(1+\eta)^2J_s{}^2k_p{}^2dk_pd\Omega_p. \quad (77)$$

If we had used a plane wave for $\psi_{k^{vi}}(\mathbf{r})$ in the same approximation ($\mathbf{q}''=0$), we would have obtained the same cross section (77) with a different value for J_s

$$J_s{}^{P.W.}=\varphi(k^{vi})/4\pi=\left(\frac{\alpha}{2\pi(1-\alpha r_{0t})}\right)^{\frac{1}{2}}\left\{\frac{1}{\alpha^2+k^{vi2}}-\frac{1}{\beta^2+k^{vi2}}\right\}. \quad (78)$$

In Fig. 6 we have plotted the approximate spectrum for high energy protons in the forward direction for both the effective range and plane wave values of J_s at angles of 0°, 10°, and 30°. We have used the values:

$$r_{0s}=2.7\times10^{-13}\text{ cm},\quad r_{0t}=1.71\times10^{-13}\text{ cm},\quad a_s=-23.8\times10^{-13}\text{ cm}. \quad (79)$$

[28] H. A. Bethe and C. Longmire, Phys. Rev. 77, 647 (1950).

From the curves we see that the high energy protons are confined to high energies and small angles. This is what we had supposed, since these high energy protons come from neutron-proton exchange collisions which favor small momentum transfers. The angular dependence of the cross section (77) comes about primarily from the factor $V^2(q'')$. For this reason the angular distribution of the high energy protons should be very similar to the angular dependence in neutron-proton scattering.

One feature of the high energy proton spectrum which is rather surprising is that the protons seem to be crowded more toward higher energies than would be expected in the plane wave approximation. This is due to the interaction between the two neutrons, and since the energy of the neutrons is small, most of the contribution to the cross sections occurs for a singlet spin state between the two neutrons. The energy width of the proton peak is proportional to $(1/a_s)^2$, whereas the energy width for the plane wave calculation is proportional to α^2. The ratio of these two widths,

$$\Delta E_p/\Delta E_p^{P.W}=(1/a_s)^2/\alpha^2\approx 1/30, \tag{80}$$

is quite small and accounts for the highly peaked proton spectrum.

The sharp peak in the energy distribution could be used in practice in order to obtain very nearly monochromatic beams of neutrons. For this purpose, the inverse process is used; i.e., protons of given energy E are permitted to fall on a deuterium target. Then, in the forward direction, neutrons will be emitted whose energy is slightly below $E-\epsilon$ (ϵ = binding energy). According to Fig. 6a, the width of the energy distribution is only about 1 Mev, and is not appreciably increased if neutrons up to 10° angle with the proton beam are included. To get an appreciable neutron yield, probably liquid deuterium should be used as a target, but apart from this technical difficulty the neutron beam obtainable seems far superior to that from other methods.

We have only performed a rough calculation of the high energy proton distribution. More exact calculations can be performed along the lines mentioned in Appendix G. The work is considerably more involved and the major changes will occur where k^{vi} and q'' are not small, that is, for proton energies away from the maximum and for large scattering angles. However, our results should remain essentially unchanged.

The total cross section for the production of high energy protons can be obtained approximately by retaining only the a_4 and a_6 terms in (35) and (36). These are the terms which represent neutron-proton exchange collisions and which therefore are mainly responsible for high energy protons. The result is that the cross section for high energy protons is approximately 20 mb.

As mentioned earlier there is also a group of low energy protons in the laboratory system resulting from neutron-neutron and ordinary neutron-proton collisions which break up the deuteron. Since small momentum transfers are most probable, the incident neutron[29] will continue in the forward direction and will be essentially free. We may therefore use the representation of the matrix element in (33) and (34) with particle 3 free.

Since we are now asking for a final state with particle 3 only "slightly" scattered, only the terms F_1 and F_3 in (34) will contribute. If we use the approximation $\mathbf{q}=0$ in F_1 and F_3, we see that we have to evaluate the same integrals as before:

$$\int_0^\infty r^2dr\psi_{k''}{}^T(r)\psi_0(r)\quad\text{and}\quad\int_0^\infty r^2dr\psi_{k''}{}^S(r)\psi_0(r).$$

Once again, only the singlet wave function contributes and gives a value for the integral given by (74) with k^{vi} replaced by k''.

We must now remember that we are fixing the angle and energy of the *proton* in the final state. The remaining degree of freedom to be removed is therefore the angle of the vector $\mathbf{k}^{vi}$ *not* $\mathbf{k}''$. This integration will be quite complicated, but we shall make some rough approximations to allow us to carry the calculation through in order to obtain qualitative results.

As we have seen in the calculation of the high energy proton distribution, the sharp peak was due to the smallness of $|1/a_s|$ compared to α. The integral J_s can be given approximately from (74) by neglecting the terms with r_{0s} and r_{0t}. We then have

$$J_s'\approx\frac{N}{[(1/a_s)^2+k''^2]^{\frac{1}{2}}}\cdot\frac{\alpha}{\alpha^2+k''^2}. \tag{81}$$

The cross section is now given by

$$d\sigma=(M^2/72\pi^2\hbar^4k)(1/2\pi)^3d\mathbf{k}_pk^{vi}$$
$$\times\int d\Omega^{vi}(4\pi)^2V^2(q)J_s'^2\frac{3}{64}[4\xi^2(1-\delta)^2+(1-\eta)^2] \tag{82}$$

obtained from (77) by replacing $4\pi k^{vi}$ by $k^{vi}\int d\Omega^{vi}$, q'' by q, J_s by J_s' in (76), and

$$\tfrac{3}{16}(1+\eta)^2=[(a_4)_{\chi_7'}+(a_6)_{\chi_7'}]^2$$

by

$$(3/64)[4\xi^2(1-\delta)^2+(1-\eta)^2]=[(a_1)_{\chi_7}+(a_3)_{\chi_7}]^2.$$

The vectors $\mathbf{q}$ and $\mathbf{k}^{vi}$ are given in terms of $\mathbf{q}''$ and $\mathbf{k}^{vi}$ as

$$\mathbf{q}=\mathbf{k}^{vi}-(\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}''); \tag{83a}$$

$$\mathbf{k}''=-\tfrac{1}{2}\mathbf{k}^{vi}-\tfrac{3}{4}(\mathbf{k}+\mathbf{q}''). \tag{83b}$$

The vector $\mathbf{q}''$ (and therefore $\mathbf{k}_p=\frac{3}{2}\mathbf{k}+\mathbf{q}''$) and the *magnitude* of $\mathbf{k}^{vi}$ are fixed during the angular integration over $d\Omega^{\mathrm{VI}}$. Both q and k'' will therefore vary over the

[29] The free neutron may also be the antisymmetrized neutron, but, since we have antisymmetrized only the initial state, we may consider only collisions in which the incident neutron (particle 3) remains free.

integration. However, the effect of the variation of q will be dwarfed by that of k'', since μ is considerably larger than both α and $|1/a_s|$; q can then be set equal to zero in $V(q)$.

If we now measure the angle of $\mathbf{k}^{vi}$ with respect to the fixed vector $\mathbf{k}+\mathbf{q}''$, we can change the angular variable Ω^{VI} to the magnitude of $\mathbf{k}''$ by using (83b). In fact,

$$\tfrac{3}{4}k^{vi}|\mathbf{k}+\mathbf{q}''|d\Omega^{vi}=2\pi dk''^2;$$
$$k^{vi}d\Omega^{vi}=(8\pi/3)(dk''^2/|\mathbf{k}+\mathbf{q}''|).$$

Since we are interested mainly in small $\mathbf{k}_p$, $\mathbf{q}''$ will be approximately $-\frac{3}{2}\mathbf{k}$, so that we have

$$k^{vi}d\Omega^{vi}\approx(16\pi/3)(dk''^2/k).$$

The cross section now becomes

$$\frac{d\sigma_p}{d\mathbf{k}_p}=K\int_{k''^2_{\min}}^{k''^2_{\max}}\frac{dk''^2}{[(1/a_s)^2+k''^2](\alpha^2+k''^2)^2},\qquad(84)$$

where K is a constant and $k''_{\min}$ and $k''_{\max}$ are given by

$$\left.\begin{aligned}k''_{\min}&=|\tfrac{1}{2}k^{vi}-\tfrac{3}{4}|\mathbf{k}+\mathbf{q}''||,\\ k''_{\max}&=\tfrac{1}{2}k^{vi}+\tfrac{3}{4}|\mathbf{k}+\mathbf{q}''|.\end{aligned}\right\}\qquad(85)$$

k^{vi} is given from conservation of energy by

$$k^{vi}=\tfrac{1}{2}\sqrt{3}(k^2-|\mathbf{k}+\mathbf{q}''|^2-(4/3)M\epsilon/\hbar^2)^{\frac{1}{2}},$$

where the binding energy ϵ now becomes important. For small $\mathbf{k}_p$ we can write

$$|\mathbf{k}+\mathbf{q}''|=|\tfrac{1}{2}\mathbf{k}-\mathbf{k}_p|\approx\tfrac{1}{2}k-(\mathbf{k}\cdot\mathbf{k}_p)/k$$

and

$$k^{vi}\approx\tfrac{1}{2}\sqrt{3}[\tfrac{3}{4}k^2+\mathbf{k}\cdot\mathbf{k}_p-(4/3)M\epsilon/\hbar^2]^{\frac{1}{2}}$$
$$\approx\tfrac{3}{4}k+\tfrac{1}{2}[(\mathbf{k}\cdot\mathbf{k}_p)/k]-\tfrac{2}{3}M\epsilon/\hbar^2k.$$

We can therefore set $k''^2_{\max}=\infty$ and

$$k''^2_{\min}\approx[\tfrac{3}{8}k+(\tfrac{1}{4}\mathbf{k}\cdot\mathbf{k}_p/k)-\tfrac{1}{3}(M\epsilon/\hbar^2k)-\tfrac{3}{8}k$$
$$+\tfrac{3}{4}(\mathbf{k}\cdot\mathbf{k}_p)/k]^2\approx\{[(\mathbf{k}\cdot\mathbf{k}_p)/k]-\tfrac{1}{3}M\epsilon/\hbar^2k\}^2.$$

The integral in (84) can be evaluated, and for $\alpha^2\gg(1/a_s)^2$ is approximately

$$d\sigma_p\approx\frac{K}{2\alpha^4}\ln\left\{\frac{\alpha^2+\{[(\mathbf{k}\cdot\mathbf{k}_p)/k]-\frac{1}{3}M\epsilon/\hbar^2k\}^2}{(1/a_s)^2+\{[(\mathbf{k}\cdot\mathbf{k}_p)/k]-\frac{1}{3}M\epsilon/\hbar^2k\}^2}\right\}\times k_p dk_p^2 d\Omega_p.\qquad(86)$$

This represents a peaked distribution of low energy protons, but *not* isotropic. The maximum cross section at a given angle, for $(1/a_s)^2\ll\alpha^2$, occurs approximately at

$$k_p\cos\theta\approx\tfrac{1}{3}(M\epsilon/\hbar^2k)+\alpha,$$

where θ is the angle of the proton momentum with respect to the incident direction in the laboratory system.

Since $\frac{1}{3}M\epsilon/\hbar^2k^2\approx\frac{3}{4}\epsilon/E_0$, where E_0 is the incident energy, $\frac{1}{3}M\epsilon/\hbar^2k$ will be considerably smaller than α (a ratio of $\sim 1/20$). The "width" of the peak will therefore vary approximately as

$$\Delta E_p\sim\alpha^2/\cos^2\theta.\qquad(87a)$$

For angles near 90°, $(\mathbf{k}\cdot\mathbf{k}_p)/k$ must be replaced by k_p^2/k so that

$$\Delta E_p(90°)\sim\alpha k.\qquad(87b)$$

The distribution of low energy protons is now seen to be approximately egg-shaped, with the peak in the spectrum occuring at increasing energies as the angle to the incident direction is increased up to 90°. The term containing the binding energy in (86) will favor the forward direction over the backward direction slightly, but the main feature of the low energy proton group is the large difference between the 0° and 90° scattering.

It was mentioned before that the low energy proton group may favor the forward direction rather than being isotropic, because of the over-all forward motion of the system. Although the binding energy term *is* in this direction, this situation should still exist for no binding energy. The reason for its disappearance is as follows:

(1) The reason that the forward direction should be favored is that the two-particle collisions can only result in an additional *forward* motion to the struck particle.

(2) We have found from (81) and (84), however, that only final states for which k'' is of order of magnitude α give much contribution. Since

$$\mathbf{q}+\tfrac{1}{2}\mathbf{k}''=-(\tfrac{3}{2}\mathbf{k}+\mathbf{q}'')=-\mathbf{k}_p,$$

q will necessarily be about the same size as k'', that is, $q\sim\alpha$.

(3) For this reason we have neglected q compared with μ, that is we have taken $V(q)=V(0)$. Since we have also set $q=0$ in the exponentials, we have effectively considered only *very* small momentum transfers which give practically no additional forward motion to the struck particle.

If we were to consider the case $q\neq 0$, we would have to evaluate both the singlet and triplet integrals in (73) and we would obtain a low energy proton distribution which would favor the forward direction. However, because of the above discussion the effect would be rather small.

It should be mentioned that the *free* particle calculation with $q=0$ predicts an isotropic low energy proton component varying as $\varphi^2(k_p)$ and therefore having an energy width $\Delta E_p\sim\alpha^2$, agreeing qualitatively with the more correct width given by (87a) near the incident direction. As can be seen from (86), however, the spectrum predicted does not agree.

VII. CONCLUSIONS

(1) The total cross section (inelastic plus elastic) can be calculated using plane waves for the final state of

the three particles involved. No such simple method is available for the inelastic scattering alone, or for the differential cross section.

(2) The total cross section is given within about 20 percent by elementary interference theory, in which the amplitudes of the waves scattered by a neutron and a proton at rest are added, taking into account spin and symmetry. A better approximation is obtained by considering the motion of neutron and proton in the initial state of the deuteron. (See Sec. II.)

(3) From the total cross section one cannot deduce a unique value for the neutron-neutron cross section. Different assumptions on the exchange properties of the neutron-neutron force lead to values of σ_{NN} at 90 Mev from 20 mb for ordinary forces to 62 mb for pure exchange forces (Table III). The interference between the scattering from neutron and proton may give either a positive (for ordinary N–N forces) or negative (exchange forces) contribution. (See Sec. III.)

(4) The elastic scattering permits a decision between the various types of neutron-neutron forces, being greatest for an ordinary force (Table IV). The experimental value of about 50 mb is closest to the result for a Serber type N–N force (Sec. IV).

(5) The cross section for the production of low energy protons does not agree with the N–N cross section (Sec. V and Table IV) but is generally smaller. The observed cross section is again compatible with a Serber force.

(6) The energy distribution of the protons in the forward direction show a very sharp peak near the incident neutron energy (Fig. 6a, b) which is closely related to the cross section for the photo-magnetic disintegration of the deuteron. The reverse reaction, i.e., the bombardment of deuterons with high energy protons, should yield very nearly monochromatic neutrons.

(7) In Table IV, we have summarized the most important results not contained in Table III.

APPENDIX A.

EVALUATION OF INTEGRAL I

The integral I given in (38a) can be evaluated completely once the radial dependence of the potential and the deuteron ground state are decided upon. We shall use the Yukawa potential and Hulthén wave function as given in (43) and (44).

The variable Ω'' in (38a) represents the angle of $\mathbf{k}''$ and can be taken with respect to $\mathbf{q}$ if k'' and q are kept constant. Let us now change from the variable Ω'' to the variable $z=|\mathbf{k}''+\frac{1}{2}\mathbf{q}|$ just as we did in (14) when setting up the integral over the final states.

$$k''d\Omega''=4\pi z dz/q.$$

The z integration must now be performed first between the limits $|\frac{1}{2}q-k''|$ and $\frac{1}{2}q+k''$, so that I is given by

$$I=2\pi\int_0^{2k} dq V^2(q)\int_0^Q 2k''dk''\int_{\frac{1}{2}q-k''}^{\frac{1}{2}q+k''} z\varphi^2(z)dz, \qquad \text{(A1)}$$

where $Q^2=\frac{3}{4}q(2k-q)$ and where $\frac{1}{2}q-k''$ has been written without an absolute value sign, since the integrand is an odd function of z. We would now like to interchange the order of the k'' and z integrations and can do so with proper regard to the limits. The integral then becomes

$$I=2\pi\int_0^{2k} dq V^2(q)\int_{\frac{1}{2}q-Q}^{\frac{1}{2}q+Q} zdz\,\varphi^2(z)\int_{|z-\frac{1}{2}q|}^{Q} 2k''dk''. \qquad \text{(A2)}$$

If the k'' integration is now performed, we obtain

$$I=2\pi\int_0^{2k} dq V^2(q)\int_{\frac{1}{2}q-Q}^{\frac{1}{2}q+Q} zdz\,\varphi^2(z)\{zq-(z^2+q^2-\tfrac{3}{2}kq)\}. \qquad \text{(A3)}$$

The region of integration in the qz plane is shown in Fig. 5.

We now have to determine $V(q)$ and $\varphi(z)$. Using (43) and (44) we readily obtain

$$V(q)=\left(\frac{4\pi V_0}{\mu}\right)\left(\frac{1}{\mu^2+q^2}\right), \qquad \text{(A4)}$$

$$\varphi(z)=4\pi N\left(\frac{1}{\alpha^2+z^2}-\frac{1}{\beta^2+z^2}\right). \qquad \text{(A5)}$$

If we now interchange the order of the q and z integrations, we obtain

$$I=2\pi\left(\frac{16\pi^2NV_0}{\mu}\right)^2\int_{-\frac{1}{2}k}^{\frac{3}{2}k} zdz\left(\frac{1}{\alpha^2+z^2}-\frac{1}{\beta^2+z^2}\right)^2 \times\int_{z-}^{z+} dq\left\{\frac{q(z+\frac{3}{2}k)-z^2-q^2}{(\mu^2+q^2)^2}\right\},$$

where

$$z\pm=\tfrac{3}{4}k+\tfrac{1}{2}z\pm\{(\tfrac{3}{4}k+\tfrac{3}{2}z)(\tfrac{3}{4}k-\tfrac{1}{2}z)\}^{\frac{1}{2}}.$$

If we now perform the q integration, we obtain

$$I=2\pi\left(\frac{16\pi^2NV_0}{\mu}\right)^2\frac{1}{2\mu^3}\int_{-\frac{1}{2}k}^{\frac{3}{2}k} zdz\left(\frac{1}{\alpha^2+z^2}-\frac{1}{\beta^2+z^2}\right)^2 \times(z^2+\mu^2)(w-\tan^{-1}w), \qquad \text{(A6)}$$

where

$$w=\{2\mu/(\mu^2+z^2)\}\{(\tfrac{3}{4}k+\tfrac{3}{2}z)(\tfrac{3}{4}k-\tfrac{1}{2}z)\}^{\frac{1}{2}}. \qquad \text{(A7)}$$

The remaining integration can fortunately be performed by contour integration if we notice that w is real for $-\frac{1}{2}k\leq z\leq\frac{3}{2}k$ and imaginary for other values of z along the real axis. Therefore, $\tan^{-1}w$ will be real for $-\frac{1}{2}k\leq z\leq\frac{3}{2}k$; for other real values of z, since

$$\tan^{-1}i|w|=\frac{1}{2i}\ln\left(\frac{1-|w|}{1+|w|}\right),$$

$\tan^{-1}w$ will be imaginary as long as $|w|<1$. If we write

$$w=\{2i\mu/(\mu^2+z^2)\}\{z^2-(\tfrac{1}{2}z+\tfrac{3}{4}k)^2\}^{\frac{1}{2}},$$

we see that for $z<-\frac{1}{2}k$ and for $z>\frac{3}{2}k$ we may write

$$|w|\leq 2\mu z/(\mu^2+z^2)\leq 1.$$

We can therefore extend the integral along the entire real axis and take the real part in order to obtain the integral between the limits $-\frac{1}{2}k$ and $\frac{3}{2}k$. If we now close the contour in the upper half-plane, we find that we must avoid the four branch points $z=-\frac{1}{2}k$, $z=\frac{3}{2}k$, and $z=z_1$, $z=z_2$, the two branch points for which $w=\mp i$. The contour is shown in Fig. 7. The integrals from z_1 to ∞ and z_2 to ∞ can be readily evaluated, since $\tan^{-1}w$ changes by $\pm 2\pi i$ on going around the branch points z_1 and z_2. z_1 and z_2 are given by

$$z_1=\tfrac{1}{2}i\mu+\{\tfrac{3}{4}\mu^2+\tfrac{3}{2}i\mu k\}^{\frac{1}{2}},\ w=-i, \qquad \text{(A8a)}$$

$$z_2=-\tfrac{1}{2}i\mu+\{\tfrac{3}{4}\mu^2-\tfrac{3}{2}i\mu k\}^{\frac{1}{2}},\ w=+i, \qquad \text{(A8b)}$$

TABLE IV. Calculated cross sections in millibarns for different exchange character of the N–N interaction.

	1	0	−1
N–N force	1	$\frac{1}{2}(1+P)$	P
Total elastic	80	60	30
Elastic scattering giving low energy deuterons	65	50	25
Low energy protons	1	19	50
N–N cross section	20	30	62

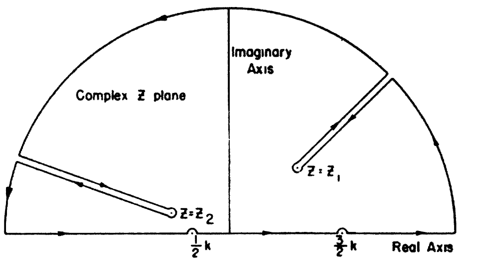

FIG. 7. Contour integration for I in the complex z plane.

where the "angle" of the square roots is taken between 0 and π. The only poles of the integrand inside the contour are at $z=i\alpha$ and $z=i\beta$, and residues must be taken at these points.

If we now evaluate the residues, subtract the integrals along the branch cuts, and take the real part we obtain

$$I=\left(\frac{16\pi^2NV_0}{\mu}\right)^2\frac{\pi^2}{\mu^3}\left[\frac{2\mu^2-\alpha^2-\beta^2}{\beta^2-\alpha^2}\mathrm{Im}\{W(i\alpha)-W(i\beta)\}\right.$$
$$+\frac{\mu^2-\alpha^2}{2\alpha}\mathrm{Re}\{W'(i\alpha)\}+\frac{\mu^2-\beta^2}{2\beta}\mathrm{Re}\{W'(i\beta)\}$$
$$+\frac{\mu^2-\alpha^2}{2}\mathrm{Re}\left\{\frac{1}{\alpha^2+z_1^2}-\frac{1}{\alpha^2+z_2^2}\right\}+\frac{\mu^2-\beta^2}{2}\mathrm{Re}\left\{\frac{1}{\beta^2+z_1^2}-\frac{1}{\beta^2+z_2^2}\right\}$$
$$\left.+\frac{2\mu^2-\alpha^2-\beta^2}{2(\beta^2-\alpha^2)}\ln\left|\frac{(\alpha^2+z_1^2)(\beta^2+z_2^2)}{(\alpha^2+z_2^2)(\beta^2+z_1^2)}\right|\right], \quad \text{(A9)}$$

where

$$W(z)=w-\tan^{-1}w;\quad W'(z)=\frac{w^2}{1+w^2}\frac{dw}{dz},$$

and w is given by (A7). Using the numerical values

$$\left.\begin{aligned}\mu^{-1}&=1.18\times10^{-13}\text{ cm}\\ \alpha^{-1}&=4.31\times10^{-13}\text{ cm}\\ \beta^{-1}&=0.616\times10^{-13}\text{ cm}\\ k^{-1}&=0.720\times10^{-13}\text{ cm}\end{aligned}\right\} \quad \text{(A10)}$$

which have previously been given, we obtain

$$I=\{(2\pi)^3/2\mu^2\}\{4\pi V_0/\mu\}^2(0.739). \quad \text{(A11)}$$

From (45) I_0 can be evaluated to give

$$I_0=\{(2\pi)^3/2\mu^2\}(4\pi V_0/\mu)^2(0.858). \quad \text{(A12)}$$

We therefore have

$$I=0.86I_0. \quad \text{(A13)}$$

The discussion of this result follows (45) in Sec. III.

APPENDIX B.

EVALUATION OF INTEGRALS II AND II_0

We shall first show that at high energy II given in (46) goes over into II_0 given in (47). In this limit we may replace the elliptical region in the r_1q plane by the strip $0\leq q\leq\frac{3}{2}k$ and $-\infty<r_1<\infty$ as before (Fig. 5). The upper limit on the r_2 integration can also be made infinite. Let us now express $\varphi(r_1)$ in terms of $\psi_0(r)$ by integrating (21) over angles

$$\varphi(r_1)=(4\pi/r_1)\int_0^\infty xdx\psi_0(x)\sin r_1x. \quad \text{(B1)}$$

The integral II then goes over into

$$II_0=32\pi^3\int_0^{\frac{3}{2}k}dqV^2(q)\int_{-\infty}^\infty dr_1\int_0^\infty xdx\psi_0(x)\sin r_1x$$
$$\times\int_{q-r_1}^\infty dr_2\int_0^\infty ydy\psi_0(y)\sin r_2y. \quad \text{(B2)}$$

If we interchange the order of integration for r_2 and y, we can perform the r_2 integration giving[30]

$$II_0=32\pi^3\int_0^{\frac{3}{2}k}dqV^2(q)\int_{-\infty}^\infty dr_1\int_0^\infty xdx\psi_0(x)\sin r_1x\int_0^\infty dy\psi_0(y)$$
$$\times[\cos qy\cos r_1y+\sin qy\sin r_1y]. \quad \text{(B3)}$$

Only that part of the integrand which is even in r_1 contributes, so that the cosine term vanishes. If we now perform the r_1 integration, we obtain a delta-function; in fact,

$$\int_{-\infty}^\infty\sin xr_1\sin yr_1dr_1=\pi\delta(x-y)\quad\left\{\begin{matrix}x\geq0\\ y\geq0\end{matrix}\right.,$$

so that II_0 becomes

$$II_0=(2\pi)^3\int_0^{\frac{3}{2}k}qdqV^2(q)\int_0^\infty\psi_0^2(x)4\pi(\sin qx/qx)x^2dx. \quad \text{(B4)}$$

This is clearly the same as

$$II_0=(2\pi)^3\int_0^{\frac{3}{2}k}qdqV^2(q)\int\psi_0^2(r)\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r} \quad \text{(B5)}$$

after performing the angular integration on $\mathbf{r}$ in (B5).

We can carry out the x integration in (B4) by using the expression (44) for $\psi_0(x)$. This gives

$$II_0=(2\pi)^34\pi N^2\int_0^{\frac{3}{2}k}dqV^2(q)\{\tan^{-1}[q/2\alpha]$$
$$+\tan^{-1}[q/2\beta]-2\tan^{-1}[q/(\alpha+\beta)]\}. \quad \text{(B6)}$$

This integral can be readily performed by numerical methods and gives

$$II_0=0.49I_0. \quad \text{(B7)}$$

In order to evaluate II in (38b) we shall change the variables of integration from k'' and Ω'' to $r_1=|\mathbf{k}''+\frac{1}{2}\mathbf{q}|$ and $r_2=|\mathbf{k}''-\frac{1}{2}\mathbf{q}|$.[31] The volume element becomes

$$k''^2dk''d\Omega''=2\pi r_1dr_1r_2dr_2/q.$$

We must now determine the limits on the r_1 and r_2 integrations to give as the region of integration the sphere of radius Q shown in Fig. 8. For a fixed value of r_1, the limits on r_2 will depend on whether Q is greater than or less than $\frac{1}{2}q$ and whether r_1 is greater or less than $|Q-\frac{1}{2}q|$. Specifically,

$$Q\geq\tfrac{1}{2}q\left\{\begin{matrix}0\leq r_1\leq Q-\frac{1}{2}q, & |q-r_1|\leq r_2\leq q+r_1 & \text{(B8a)}\\ Q-\frac{1}{2}q\leq r_1\leq Q+\frac{1}{2}q, & |q-r_1|\leq r_2\leq(3kq-r_1^2-q^2)^{\frac{1}{2}} & \text{(B8b)}\end{matrix}\right.$$

$$Q\leq\tfrac{1}{2}q:\ \tfrac{1}{2}q-Q\leq r_1\leq\tfrac{1}{2}q+Q,\quad |q-r_1|\leq r_2\leq(3kq-r_1^2-q^2)^{\frac{1}{2}}. \quad \text{(B8c)}$$

Let us now consider the integral over the regions

$$Q\geq\tfrac{1}{2}q\left\{\begin{matrix}\frac{1}{2}q-Q\leq r_1\leq0, & |q-r_1|\leq r_2\leq(3kq-r_1^2-q^2)^{\frac{1}{2}}\\ 0\leq r_1\leq Q-\frac{1}{2}q, & |q-r_1|\leq r_2\leq(3kq-r_1^2-q^2)^{\frac{1}{2}}\end{matrix}\right\}. \quad \text{(B9)}$$

Since the integrands are odd functions of r_1 and r_2, if we change the sign of r_1 in the first of the two regions, we must then subtract the two regions of integration from one another. This means that the region

$$0\leq r_1\leq Q-\tfrac{1}{2}q,\quad q+r_1\leq r_2\leq(3kq-r_1^2-q^2)^{\frac{1}{2}}$$

must be subtracted from the region,

$$0\leq r_1\leq Q-\tfrac{1}{2}q,\quad |q-r_1|\leq r_2\leq(3kq-r_1^2-q^2)^{\frac{1}{2}},$$

leaving the region

$$Q\geq\tfrac{1}{2}q:\quad 0\leq r_1\leq Q-\tfrac{1}{2}q,\quad |q-r_1|\leq r_2\leq q+r_1.$$

The region given in (B8a) is therefore equivalent to the regions in (B9). If these regions are now combined with the region in (B8b), we obtain the same region as in (B8c), so that the integral

$$II=\int_0^{2k}qdqV^2(q)\int_0^Qk''^2dk''\int d\Omega''\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|)$$

may finally be written as the single integral

$$II=2\pi\int_0^{2k}dqV^2(q)\int_{\frac{1}{2}q-Q}^{\frac{1}{2}q+Q}r_1dr_1\varphi(r_1)\int_{q-r_1}^{(3kq-r_1^2-q^2)^{\frac{1}{2}}}r_2dr_2\varphi(r_2).$$

[30] The upper limit gives no contribution since an exponential convergence factor must be used.

[31] r_1 and r_2 are directly related to spheroidal coordinates. See, for example, J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 56.

The absolute value sign has been dropped from $q-r_1$, since $r_2\varphi(r_2)$ is an odd function of r_2. If we now put in the expressions (A4) and (A5) for $\varphi(z)$ and $V(q)$, we obtain

$$II=(16\pi^2V_0N/\mu)^2 2\pi\{II_{\alpha\alpha}-II_{\alpha\beta}-II_{\beta\alpha}+II_{\beta\beta}\},$$

where

$$II_{\alpha\beta}=\int_0^{2k}\frac{dq}{(\mu^2+q^2)^2}\int_{\frac{1}{2}q-Q}^{\frac{1}{2}q+Q}\frac{r_1dr_1}{\alpha^2+r_1^2}\int_{q-r_1}^{(3kq-r_1^2-q^2)^{\frac{1}{2}}}\frac{r_2dr_2}{\beta^2+r_2^2}$$

and $II_{\alpha\alpha}$, $II_{\beta\alpha}$, and $II_{\beta\beta}$ are similarly defined. We can now perform the r_2 integration giving

$$II_{\alpha\beta}=\frac{1}{2}\int_0^{2k}\frac{dq}{(\mu^2+q^2)^2}\int_{\frac{1}{2}q-Q}^{\frac{1}{2}q+Q}\frac{r_1dr_1}{\alpha^2+r_1^2}\times\{\ln(\beta^2+3qk-r_1^2-q^2)-\ln(\beta^2+(q-r_1)^2)\}.$$

Let us make the change of variable

$$x^2=3qk-r_1^2-q^2;\quad xdx=-r_1dr_1;\quad \alpha^2+r_1^2=\alpha^2+3qk-q^2-x^2$$

in the first term in the bracket. The limits on x become

$$x_{\substack{\min\\ \max}}=\{3qk-q^2-(\tfrac{1}{2}q\pm Q)^2\}^{\frac{1}{2}}=\tfrac{1}{2}q\mp Q.$$

If we make the change of variable

$$x=q-r_1;\quad dx=-dr_1;\quad \alpha^2+r_1^2\equiv\alpha^2+(q-x)^2$$

in the second term in the bracket, the limits on x also become

$$x_{\substack{\min\\ \max}}=q-(\tfrac{1}{2}q\pm Q)=\tfrac{1}{2}q\mp Q.$$

The integral $II_{\alpha\beta}$ may therefore be written as

$$II_{\alpha\beta}=\frac{1}{2}\int_0^{2k}\frac{dq}{(\mu^2+q^2)^2}\int_{\frac{1}{2}q-Q}^{\frac{1}{2}q+Q}dx\,\ln(\beta^2+x^2)\times\left\{\frac{x}{\alpha^2+3qk-q^2-x^2}-\frac{q-x}{\alpha^2+(q-x)^2}\right\}$$

with similar expressions for $II_{\alpha\alpha}$, $II_{\beta\alpha}$, and $II_{\beta\beta}$. If we now interchange the order of integration as we did for I, we can perform the q integration by partial fractions. The x integration remaining is quite complicated and can only be done numerically. The result of the numerical integration[32] is

$$II=0.81II_0;\quad II=0.46I.$$

APPENDIX C.

EVALUATION OF INTEGRALS *III*, *IV*, AND *V*

The integral

$$III=\int_0^{2k}qdqV(q)\int_0^Q k''^2dk''\int d\Omega''V(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}-\mathbf{k}''|)\varphi^2(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)$$

can be evaluated accurately but only after an extremely laborious numerical integration. We have previously mentioned that the value of this integral need not be calculated accurately, since it can be obtained indirectly from the two-particle cross sections. We shall therefore evaluate III approximately by assuming that the major contributions to the integral come near $|\mathbf{k}''+\frac{1}{2}\mathbf{q}|=0$. The integral then becomes

$$III\approx\int_0^{2k}qdqV(q)V(|\tfrac{3}{2}\mathbf{k}+\mathbf{q}|)\int_0^Q k''^2dk''\int d\Omega''\varphi^2(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|).$$

The k'' integration is now the same as the one encountered in the evaluation of I in (38a). We shall therefore make use of our experience with I in Appendix A: the integral is 14 percent less than the $Q=\infty$ value (I_0), and the limit of $2k$ on q is changed to the more accurate value $\frac{3}{2}k$. We therefore have

$$III\approx(2\pi)^3(0.86)\int_0^{\frac{3}{2}k}qdqV(q)V(|\tfrac{3}{2}\mathbf{k}+\mathbf{q}|).$$

Near $\mathbf{k}''=-\frac{1}{2}\mathbf{q}$ we have

$$|\tfrac{3}{2}\mathbf{k}+\mathbf{q}|^2=(9/4)k^2+q^2+3\mathbf{k}\cdot\mathbf{q}=(9/4)k^2-\tfrac{1}{2}q^2-2k''^2\approx(9/4)k^2-q^2.$$

So that the integral becomes

$$III\approx(2\pi)^3(0.86)(4\pi V_0/\mu)^2\int_0^{\frac{3}{2}k}\frac{qdq}{(\mu^2+q^2)(\mu^2+9k^2/4-q^2)}$$
$$\approx(2\pi)^3(0.86)(4\pi V_0/\mu)^2(1/2\mu^2)\times\{1+(9k^2/8\mu^2)\}^{-1}\ln\{1+(9k^2/4\mu^2)\}.$$

Using (43) and (A10) we therefore obtain

$$III\approx0.56I.$$

We have made a rough calculation of the error in III due to the approximations made and have found it to be quite large. This error cannot be easily calculated; but, as we have mentioned, and as we shall show, the value of III need not be known accurately. We shall include the rough estimate of the error and use the value $III\approx0.43I$.

The integral

$$IV=\int_0^{2k}qdqV(q)\int_0^Q k''^2dk''\int d\Omega''V(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}-\mathbf{k}''|)\times\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|)$$

is quite difficult to evaluate, but can be done approximately with little difficulty. We shall again assume that the deuteron is large, so that the main contribution to the integral occurs near $|\mathbf{k}''+\frac{1}{2}\mathbf{q}|=0$ and $|\mathbf{k}''-\frac{1}{2}\mathbf{q}|=0$, that is, near $k''=0$, $q=0$. We shall therefore set $k''=q=0$ compared to $\frac{3}{2}k$ in the second V term. This then gives

$$IV\approx V(\tfrac{3}{2}k)\int_0^{2k}qdqV(q)\int_0^Q k''^2dk''\int d\Omega''\times\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|).$$

The similarity between this integration and the one in (38b) for the interference term II, is now evident and we shall once again make use of our experience with II. The calculation of II in Appendix B gave the result that II is 19 percent less than the value calculated for $Q=\infty\,(II_0)$ and that the upper limit for the q integration changes to $\frac{3}{2}k$. We therefore have for IV

$$IV\approx(2\pi)^3(0.81)V(\tfrac{3}{2}k)\int_0^{\frac{3}{2}k}qdqV(q)\int\psi_0^2(r)\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}$$
$$\approx(2\pi)^3(0.81)V(\tfrac{3}{2}k)4\pi N^2\int_0^{\frac{3}{2}k}dqV(q)\times\{\tan^{-1}(q/2\alpha)+\tan^{-1}(q/2\beta)-2\tan^{-1}[q(\alpha+\beta)]\}$$

as in (B6). This integral can be readily performed by numerical methods and gives $IV\approx0.11I$.

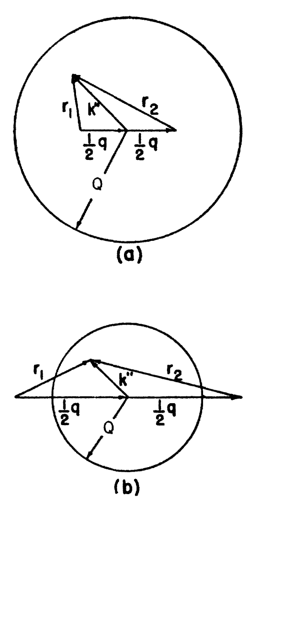

FIG. 8. Diagrams to determine limits on r_1 and r_2 after change of variables in Appendix B.

[32] All the calculations were originally performed with the incorrect value of $5\frac{1}{2}$ for β/α. The calculations for I, I_0, and II_0 were repeated for $\beta/\alpha=7$ and were found to give values that differed from the old by approximately 0 to 2 percent. Since the work involved in calculating II was quite complicated, the value of II for $\beta/\alpha=7$ was inferred from the way in which the others changed. The value given for II (and for III, IV, and V) should be good to within $\pm0.01\,I$.

The integral

$$V=\int_0^{2k} qdq\int_0^Q k''^2dk''\int d\Omega'' V(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}+\mathbf{k}''|)\times V(|\tfrac{3}{2}\mathbf{k}+\tfrac{1}{2}\mathbf{q}-\mathbf{k}''|)\varphi(|\mathbf{k}''+\tfrac{1}{2}\mathbf{q}|)\varphi(|\mathbf{k}''-\tfrac{1}{2}\mathbf{q}|)$$

can be treated in an analogous way and gives

$$V\approx(2\pi)^3(0.81)V^2(\tfrac{3}{2}k)4\pi N^2\int_0^{\frac{1}{2}k} dq\{\tan^{-1}[q/2\alpha]+\tan^{-1}[q/2\beta]-2\tan^{-1}[q/(\alpha+\beta)]\}.$$

The integral can be performed directly and gives

$$V\approx 0.03I.$$

We shall not try to estimate the errors made in the approximations used in evaluating IV and V.

We shall now demonstrate that our results do not depend critically on the numerical value of III. In order to do so, we shall repeat the calculations for Table III with the uncorrected value

$$III=0.56I.$$

The new numerical values given in Table V are quite close to those in Table III and show clearly that the exact value of the ratio III/I, is unimportant. As previously mentioned, this is because the total cross section was expressed as the sum of the three terms σ_{NP}, σ_{INTERF}, and σ_{NN}, and the experimentally observed value of σ_{NP} was used.

TABLE V. Table III recalculated with the uncorrected value of the integral $III=0.56I$ instead of the corrected value $III=0.43I$ used in Table IV.

δ	1	0	−1
N-N force	1	$\frac{1}{2}(1+P)$	P
$\xi=V_{NN}/V_{NP}{}^T$	0.46	0.79	0.83
σ_{NP}	72 mb	72 mb	72 mb
σ_{INTERF}	26 mb	15 mb	−16 mb
σ_{NN}	19 mb	30 mb	61 mb
σ_{TOT}	117 mb	117 mb	117 mb

APPENDIX D.

EFFECT OF INCLUDING THE BINDING ENERGY IN THE CONSERVATION OF ENERGY

If the binding energy had been included, the conservation of energy (7), would have been

$$k'^2+(4/3)k''^2=k^2-(4/3)(M\epsilon/\hbar^2).$$

The arguments given for replacing $\psi_{k''}(\mathbf{r})$ by a plane wave in the half-ellipse of Fig. 2 are still valid; but the ellipse is now given by

$$(q-k)^2+(4/3)k''^2=k^2(1-\delta)^2.$$

where

$$1-\delta=\{1-(4/3)M\epsilon/\hbar^2k^2\}^{\frac{1}{2}},\quad \delta\approx\tfrac{2}{3}M\epsilon/\hbar^2k^2=\tfrac{2}{3}(\alpha/k)^2.$$

In order to determine the effect of δ, we need only consider the I_0 term, which now becomes

$$I_0'=(2\pi)^3\int_{\delta k}^{(\frac{1}{2}-\delta)k} qdqV^2(q),$$

where the upper limit is determined by where the new ellipse in Fig. 5 crosses the q axis. Clearly, at the lower limit, the factor q in the integrand will make the change small; and at the upper limit the change will also be small because of the size of $V^2(\frac{3}{2}k)$. In fact, the fractional change can be given approximately as

$$\frac{I_0-I_0'}{I_0}=(0.86)\frac{(\delta k)^2/2\mu^4+(\delta k)(\frac{3}{2}k)/(\mu^2+9k^2/4)^2}{2\mu^2}=\frac{0.86\delta^2k^2}{\mu^2}+\frac{0.64\delta k^2\mu^2}{(\mu^2+9k^2/4)^2}\approx 0.002,$$

which can certainly be neglected.

APPENDIX E.

VALIDITY OF THE ASSUMPTION OF A PLANE WAVE FOR $\psi_{k''}(r)$ IN THE SHADED REGION OF FIG. 2

In the justification of the use of plane waves in the elliptical region in Fig. 2 it was assumed that $\psi_{k''}(\mathbf{r})$ could be replaced by a plane wave in the shaded region which occurs mostly for large k''. The most questionable region for this assumption is in the lower left corner of the shaded region where k'' *can* be small. The lower right corner will give a small contribution because of the factor $V^2(q)\approx V^2(2k)$.

In order to obtain a more accurate expression for $\psi_{k''}(\mathbf{r})$ in the region mentioned we shall use the first Born approximation in a deuteron potential, so that $\psi_{k''}(\mathbf{r})$ will be given by

$$\psi_{k''}(\mathbf{r})\approx L^{-\frac{3}{2}}\exp(i\mathbf{k}''\cdot\mathbf{r})+L^{-\frac{3}{2}}V_0f(\mathbf{k}'',\mathbf{r}),\tag{E1}$$

where $f(\mathbf{k}'',\mathbf{r})$ is a function depending on the details of the Born approximation and V_0 is either the triplet or singlet Yukawa well depth. We now wish to evaluate the integral in (18):

$$\int\psi_{k''}{}^*(\mathbf{r})\psi_0(r)\exp(\pm\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}.\tag{E2}$$

Since the region being considered contains only small values of q (much smaller than k'', since the boundary ellipse is given by $q\approx\frac{2}{3}k''^2/k$ near the origin), we shall set the exponential equal to unity. The remaining integral will be just the orthogonality integral and will necessarily vanish for a triplet wave function; that is,

$$\int\psi^*{}_{k''}{}^T(\mathbf{r})\psi_0(r)d\mathbf{r}=0.\tag{E3}$$

From (E1) we therefore must have

$$V_0{}^T\int f(\mathbf{k}'',\mathbf{r})\psi_0(r)d\mathbf{r}\approx-\int\exp(-i\mathbf{k}''\cdot\mathbf{r})\psi_0(r)d\mathbf{r}=-\varphi(k'').\tag{E4}$$

In the case of a singlet potential we have

$$\int\psi^*{}_{k''}{}^S(\mathbf{r})\psi_0(r)d\mathbf{r}\approx L^{-\frac{3}{2}}\varphi(k'')+L^{-\frac{3}{2}}V_0{}^S\int f(\mathbf{k}'',\mathbf{r})\psi_0(r)d\mathbf{r}.\tag{E5}$$

Using (E4), this becomes

$$\int\psi^*{}_{k''}(\mathbf{r})\psi_0(r)d\mathbf{r}\approx L^{-\frac{3}{2}}\varphi(k'')(1-V_0{}^S/V_0{}^T).$$

We therefore reach the following conclusions.

(1) In the case of a triplet interaction we have subtracted too great an amount in the region near the origin. A calculation of the integral using a plane wave in this region shows that I should be increased by about 4 percent.

(2) In the case of the singlet interaction we have again subtracted too much in the region near the origin, but this time by an amount

$$4\text{ percent}\{1-(1-V_0{}^S/V_0{}^T)^2\},$$

since the integral (E2) appears squared in (18). Using the value of $V_0{}^S/V_0{}^T=0.686$, this becomes $3\frac{1}{2}$ percent, or approximately the same as in the triplet case.

This adjustment is also necessary in the other integrals, II–V, and will be approximately the same. For this reason the ratios (48), (49), (50), and (51) will *not* change appreciably and all the numerical results in Table III should be valid.

The use of a plane wave for the $\psi_{k''}(\mathbf{r})$ should be better at higher k''. The failure of the plane wave to be orthogonal to the ground state for the triplet case makes little difference, since the factor $\exp(\pm\frac{1}{2}i\mathbf{q}\cdot\mathbf{r})$ is now a rapidly oscillating function. The reason is that the most important values of q are large because of the factor $V(q)$.

We have investigated the validity of using the plane wave in the entire shaded region in some detail and found that the correction is quite appreciable—in fact, it is about one-half of the 14 percent discrepancy between the values of I and I_0. However, this correction must also be applied to the integrals II–V. As mentioned before, this will leave the ratios (48), (49), (50), and

(51) only slightly changed and will not appreciably affect the numerical results in Table III.

APPENDIX F.

EXAMPLE OF EVALUATION OF COEFFICIENTS a_l IN TABLE I

As an example of the manner in which the a_l's in (28) and in Table I are obtained, we shall evaluate the doublet χ_5 coefficients in (28). This means that both the initial and final spin states will be χ_5. If we then insert (26) into (25), we obtain for the matrix element

$$\mathfrak{M}=(\Psi_f\chi_5,\ \{1-P_{13}Q_{13}\}V_{NN}(r_{13})\{w_n+m_nP_{13}\}\Psi_i\chi_5) +(\Psi_f\chi_5,\ \{1-P_{13}Q_{13}\}V_{NP}(r_{23}) \times\{w_p+m_pP_{23}+b_pQ_{23}+h_pP_{23}Q_{23}\}\Psi_i\chi_5). \quad \text{(F1)}$$

From (30) we find that

$$(\chi_5,\chi_5)=1,\quad (\chi_5,Q_{13}\chi_5)=-\tfrac{1}{2},$$
$$(\chi_5,Q_{23}\chi_5)=-\tfrac{1}{2},\quad (\chi_5,Q_{13}Q_{23}\chi_5)=-\tfrac{1}{2},$$

so that (F1) becomes

$$\mathfrak{M}=(\Psi_f,\ \{(w_n+\tfrac{1}{2}m_n)+(\tfrac{1}{2}w_n+m_n)P_{13}\}V_{NN}(r_{13})\Psi_i) +(\Psi_f,\ \{(w_p-\tfrac{1}{2}b_p)+(m_p-\tfrac{1}{2}h_p)P_{23}+(\tfrac{1}{2}w_p+\tfrac{1}{2}b_p)P_{13} +(\tfrac{1}{2}m_p+\tfrac{1}{2}h_p)P_{13}P_{23}\}V_{NP}(r_{23})\Psi_i). \quad \text{(F2)}$$

If we now use (29) and the initial state

$$\Psi_i=L^{-\frac{3}{2}}\exp(-i\mathbf{k}\cdot\mathbf{x})\psi_0(r).$$

(F2) goes directly into (28) with the a_l's as given in the doublet χ_5 row of Table I.

APPENDIX G.

A MORE ACCURATE CALCULATION OF J_1 AND J_2 IN EQ. (73)

We were faced with the evaluation of

$$J_1=\int d\Omega^{vi}\left|\int d\mathbf{r}\psi_{k^{vi}}{}^*(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})\right|^2 \quad \text{(G1a)}$$

and

$$J_2=\int d\Omega^{vi}Re\left\{\int d\mathbf{r}\psi_{k^{vi}}(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r}) \times\int d\mathbf{r}\psi_{k^{vi}}{}^*(\mathbf{r})\psi_0(r)\exp(\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})\right\} \quad \text{(G1b)}$$

for both the triplet and singlet $\psi_{k^{vi}}(\mathbf{r})$. Since we are interested in these integrals only for small values of k^{vi}, we must find an approximate expression for $\psi_{k^{vi}}(\mathbf{r})$ at low energies. We shall therefore expand $\psi_{k^{vi}}(\mathbf{r})$ into spherical harmonics with the intention of using only the lowest order terms. We shall set

$$\psi_{k^{vi}}(\mathbf{r})=\Sigma_l A_l(k^{vi}r)Y_{l0}(\mathbf{k}^{vi},\mathbf{r}), \quad \text{(G2)}$$

where the $Y_l{}^0(\mathbf{k}^{vi},\mathbf{r})$ are spherical harmonics in terms of the angle of $\mathbf{k}^{vi}$ with respect to the polar axis in the direction $\mathbf{r}$, and the $A_l(k^{vi}r)$ are the corresponding coefficient in the expansion. If we also expand the exponential

$$\exp(\tfrac{1}{2}i\mathbf{q}''\cdot\mathbf{r})=\Sigma_{l''}B_{l''}{}^*(\tfrac{1}{2}q''r)Y_{l''}{}^0(\mathbf{r},\mathbf{q}'') \quad \text{(G3)}$$

and use the addition theorem

$$Y_l{}^0(\mathbf{k}^{vi},\mathbf{r})=[4\pi/(2l+1)]^{\frac{1}{2}}\Sigma_m Y^*{}_l{}^m(\mathbf{k}^{vi},\mathbf{q}'')Y_l{}^m(\mathbf{r},\mathbf{q}''), \quad \text{(G4)}$$

the integral J_1 in (G1a) becomes

$$J_1=\int d\Omega^{vi}\int d\mathbf{r}\int d\mathbf{r}'\sum_{\substack{ll'l''l'''\\ mm'}}\left\{\frac{(4\pi)^2}{(2l+1)(2l'+1)}\right\}^{\frac{1}{2}}A_l{}^*(k^{vi}r) \times A_{l'}(k^{vi}r')Y^*{}_l{}^m(\mathbf{k}^{vi},\mathbf{q}'')Y_l{}^m(\mathbf{r},\mathbf{q}'')Y_{l'}{}^{m'}(\mathbf{k}^{vi},\mathbf{q}'') \times Y_{l'}{}^{m'}(\mathbf{q}'',\mathbf{r}')B_{l''}{}^*(\tfrac{1}{2}q''r)B_{l'''}(\tfrac{1}{2}q''r') \times Y^*{}_{l''}{}^0(\mathbf{r},\mathbf{q}'')Y_{l'''}{}^0(\mathbf{r}',\mathbf{q}''). \quad \text{(G5)}$$

Since

$$\int d\Omega_x Y^*{}_l{}^m(\mathbf{x},\mathbf{a})Y_{l'}{}^{m'}(\mathbf{x},\mathbf{a})=\delta_{mm'}\delta_{ll'},$$

the integral reduces to

$$\Sigma_l\left|\int A_l(k^{vi}r)B_l(\tfrac{1}{2}q''r)\psi_0(r)r^2dr\right|^2\{4\pi/(2l+1)\}. \quad \text{(G6)}$$

From this result we see that the various angular momentum states in the sum (G2) do not interfere with one another in the cross section. We can therefore evaluate the cross section for different l's and add the individual results. The same will be true for the second integral J_2 in (G1b).

The integral for $l=0$ at low energy can be treated quite satisfactorily by the effective range method, but for higher l's we shall use the corresponding components of the plane wave approximation. The result is that the integrals J_1 and J_2 can be written as

$$J=J_{l=0}+J_{PW}-J_{PW\ l=0},$$

where $J_{l=0}$ is the $l=0$ term evaluated by the effective range method, J_{PW} is the integral evaluated using a plane wave for $\psi_{k^{vi}}(\mathbf{r})$, and $J_{PW\ l=0}$ is just the $l=0$ term of the plane wave approximation.