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A Re-Evaluation of the Fundamental Atomic Constants

J. A. BEARDEN AND H. M. WATTS
The John Hopkins University, Baltimore, Maryland

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A completely new evaluation of the fundamental atomic constants by the method of least squares is presented. A number of new and highly precise experiments have been taken into account, including the measurement of: (1) the velocity of light by an exceptionally ingenious and precise method due to Hansen and Bol, (2) the absolute proton moment, (3) the ratio of magnetic moments of proton and electron, (4) the proton moment in Bohr magnetons, (5) the hyperfine structure separation of ground-state hydrogen, (6) the ratio of cyclotron to precession frequency of the proton, (7) h/mc using annihilation radiation, and (9) h/e from the x-ray high frequency limit with improved precision.

The results of the critical survey of Dorsey on the velocity of light plus the more recent measurements of c are combined to obtain a weighted-mean value: $c=299790.0\pm0.7$ km/sec. The results of the experiments listed above have been combined by several writers to compute α, e/m, and F. All of the published values, including those of certain older experiments have been corrected for the new value of c and combined in a least-squares calculation. The results are presented in a complete table of the fundamental constants and are compared with the experimental values in an isometric consistency chart. Six of the experimental results: F, N, mN, α, e/m, and h/e are found to show excellent agreement with the least-squares values. In particular the new experimental value of $h/e=(1.37928\pm0.00004)\times10^{17}$ erg sec/esu which precipitated the present work agrees well with the least-squares value, $h/e=(1.379300\pm0.000016)\times10^{-17}$ erg-sec/esu.

I. INTRODUCTION

AN effort to assess the recent work on h/e by Bearden and Schwarz[1] and that of the previous paper[2] in terms of the consistency with the present knowledge of the fundamental atomic constants led to the present rather complete analysis. The most complete recent work is the 1947 evaluation of the atomic constants by Dumond and Cohen.[3] Since then a number of very precise experiments have been reported which must also be included in any new evaluation of the constants.

The following new experimental results are of particular significance: (1) the absolute moment of the proton by Thomas, Driscoll, and Hipple,[4] (2) the ratio of proton magnetic moment to electron magnetic moment by Taub, Kusch, and Foley,[5] (3) the proton moment in Bohr magnetons by Gardner and Purcell,[6] and by R. W. Nelson, and Taub, Kusch, and Foley,[5] (4) the hyperfine structure frequency difference between certain magnetic levels of ground state hydrogen by Prodell and Kusch,[7] (5) the ratio of the cyclotron frequency to the spin precession frequency of the proton by Hipple, Sommer, and Thomas,[8] (6) the Compton wavelength of the electron by Dumond, Lind, and Watson,[9] and (7) h/e from the x-ray high frequency limit by Bearden and Schwarz[1] and Bearden, Johnson, and Watts.[2] These results have been combined by Thomas, Driscoll, and Hipple,[4] Bethe and Longmire,[10] Hipple, Sommer, and Thomas,[8] and the present writers to compute e/m, α, F, and h/m, respectively. These quantities and h/e are shown in the column *New experiments* in Table I. The quantity of interest in each case is the normalized probable error of the experiment and its weight which is inversely proportional to the square of the error. In the same table we show the same data for the group of ten experiments which were used in reference 3. On comparing the total weights of the two groups of data, it is apparent that the 1947 experimental data would have a weight of only 13 percent in a new calculation. The need for a new evaluation using the new data is inescapable.

Another factor of primary importance in a reevaluation of the atomic constants is the excellent work of Hansen and Bol[11] on the velocity of light. This work has

TABLE I. Comparison of weights of 1947 experimental results with the weights of more recent data.

Quantity measured	New experiments Error $\times10^4$	Weight	Experiments prior to 1947 Error $\times10^4$	Weight
N			0.7	2.0
F	0.31	10.4	1.0	0.9
mN			1.4	0.5
Fe/m			2.1	0.2
e/m	0.26	15.2	2.2	0.2
h/e	0.58	3.0	2.9	0.1
α	0.37	7.5	3.7	0.1
$h/(em)^{\frac{1}{2}}$			5.8	0
h/m	4.12	0.1	12.4	0
e^2/mh			11.8	0
		Total 36.2		Total 4.0

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[1] J. A. Bearden and G. Schwarz, Phys. Rev. **79**, 674 (1950).

[2] Bearden, Johnson and Watts, preceding paper.

[3] J. W. Dumond and E. R. Cohen, Revs. Modern Phys. **20**, 82 (1948); J. W. Dumond and E. R. Cohen, Revs. Modern Phys. **21**, 651 (1949).

[4] Thomas, Driscoll, and Hipple, Phys. Rev. **78**, 787 (1950); **75**, 902 (1949); **75**, 992 (1949).

[5] P. Kusch and H. M. Foley, Phys. Rev. **74**, 520 (1948); H. Taub and P. Kusch, Phys. Rev. **75**, 1481 (1949); P. Kusch and H. Taub, Phys. Rev. **75**, 1477 (1949).

[6] J. H. Gardner and E. M. Purcell, Phys. Rev. **76**, 1262 (1949), also E. M. Purcell, private communication concerning the more recent work of R. W. Nelson.

[7] A. G. Prodell and P. Kusch, Phys. Rev. **79**, 1009, (1950).

[8] Hipple, Sommer, and Thomas, Phys. Rev. **76**, 1877 (1949); J. A. Hipple, H. Sommer, and H. Thomas, Phys. Rev. **80**, 487 (1950); private communications; F. Bloch and C. D. Jeffries, Phys. Rev. **80**, 305 (1950).

[9] Dumond, Lind, and Watson, Phys. Rev. **75**, 1226 (1949).

[10] H. A. Bethe and C. Longmire, Phys. Rev. **75**, 306 (1949).

[11] W. W. Hansen and K. Bol, private communication; K. Bol, Phys. Rev. **80**, 298 (1950).

resulted in a value of C differing so greatly from the value of reference 3 that its use as an auxiliary constant would change the results of a least-squares calculation to a considerable extent. We must therefore review carefully the available information on the value of the velocity of light and its influence as an auxiliary constant on the published results of all the experiments used.

II. THE VELOCITY OF LIGHT

It will be recalled that in the least squares evaluation of reference 3, certain auxiliary constants such as: the Rydberg constant, the velocity of light, the conversion from Siegbahn to absolute wavelength, etc. are used. Of the auxiliary constants involved, only the velocity of light appears to deserve a reconsideration.

In 1944 Dorsey made a very careful study[12] of all the significant experiments on the velocity of light up to that time. He reached three particularly interesting conclusions: (1) there is no evidence for a secular change in the velocity of light, (2) all of the work prior to 1934 shluld be dismissed from any calculation of C, and (3) the mean of the 5 best experimental results is 299773 ±10 km/sec. The last conclusion deserves further attention. The five experiments considered in (3) above are the following:

(a) Michelson, Pease, and Pearson reported in 1935 the results of work with a 32-face prismatic mirror using a light path the major part of which is in vacuum. There was no careful search for systematic errors. Result: 299774±10 km/sec.

(b) Karolus and Mittelstaedt reported in 1928 and 1929 on Kerr Cell measurements, the errors of which were well studied. Result: 299778±20 km/sec.

(c) Anderson in 1937 reported on Kerr cell experiments and discussed the errors in detail. Result 299771±15 km/sec.

(d) Anderson in 1941 reported further Kerr cell results: 299776 ±14 km/sec.

(e) Hüttell in 1940 reported on Kerr cell work, but failed to provide adequate data for an assessment of his results. If he is given the benefit of the doubt on several items including the correction for the index of refraction of air, his result is 299768±10 km/sec.

In the above experiments one notices the interesting point that the three experiments which were most carefully reported claim errors ranging from 14 to 20 km/sec, while the incompletely reported experiments both claim errors of only 10 km/sec. If one were to take these errors literally, the five experiments would have weights of 4, 1, 2, 2, and 4 respectively, which would give as much weight to each of the questioned experiments as to the whole set of the carefully reported experiments. Dorsey compromises by weighting the five experiments equally. The important point to be gained from Dorsey's work is that the error which should properly be assigned to the velocity of light as of 1944 is actually several times that used in the recent evaluations of the atomic constants.

TABLE II. Weighted mean evaluation of the velocity of light, in km/sec.

Worker	Value	Error, km/sec	Weight
Dorsey	299773	10.0	1.0
Essen	92.5	3.0	11.1
Aslakson	92	2.4	17.4
Bergstrand	92.7	2.0	25.0
Hansen and Bol	89.3	0.8	156.3
Weighted mean	(299790.0±0.7) km/sec		

More recently certain additional experiments have been reported:

(f) Essen[13] reported in 1947 and Essen and Gordon-Smith[13] in 1948 on measurements using a microwave cavity resonator. Result: 299793±9 km/sec. In 1950 Essen reported[14] further results of 299792.5±3.

(g) Aslakson[15] reported in 1949 the result of measurements using "Shoran" to measure the velocity of radio waves. He reports a result of 299792.3±2.4 km/sec, although there is perhaps reason to question the smallness of the claimed uncertainties in view of the variations in the velocity of propagation[16] over the long airpaths that he used.

(h) Bergstrand[17, 18] reported in 1948 and 1949 the results of Kerr cell experiments over 4 km and 9 km paths with a result of 299796±2 km/sec. However, a correction[19] in 1950 corrects that result to 299793±2 km/sec and presents the result of new measurements as 299792.7±0.25 km/sec. The error quoted represents only the random error and there is no evidence of an evaluation of the systematic errors. Until further data are available on these experiments we shall adopt as Bergstrand's value 299792.7±2 km/sec.

(i) Hansen and Bol in unpublished work[11] using a microwave cavity resonator obtain a result of 299789.3±0.8 km/sec. We shall discuss this experiment further below.

The several results are shown in Table II, utilizing Dorsey's computed result to summarize all of the earlier work.

It is to be noted that again in the period 1944 to 1950, just as in 1934 to 1941, the techniques have improved so greatly as to render data obtained in the prior period of trivial weight. Even if one were arbitrarily to double the claimed errors in all experiments since 1944, the work prior to that time would have a weight of only two percent in the weighted mean value.

The fact that the work of Hansen and Bol is as yet unpublished and that the claimed error is so small necessitated a detailed consideration of their work in the present paper. Dr. Bol has been exceptionally kind in participating in a careful and critical discussion of the experiments performed by the late Professor Hansen and himself at Stanford and permitting the use of his results prior to publication. The experiment consisted in essence of the measurement of the resonant frequency and the length of a high-Q microwave cavity. From the

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[12] N. E. Dorsey, *The Velocity of Light* (American Philosophical Society, New Series, 34, Part I, Philadelphia, 1944).

[13] L. Essen, Nature **159**, 611 (1947). L. Essen and A. C. Gordon-Smith, Proc. Roy. Soc. **A194** (No. 1038), 348 (1948).

[14] L. Essen, Nature **165**, 582 (1950).

[15] Aslakson, Nature **164**, 711 (1949).

[16] R. L. Smith-Rose, Proc. Inst. Radio Engrs. **38**, 16 (1950).

[17] L. E. Bergstrand, Sv. Vetensk. No. **20** (Stockholm) (1948).

[18] E. Bergstrand, Sv. Vetensk. No. **20** (Stockholm) (1949).

[19] E. Bergstrand, Nature **165**, 405 (1950).

two measurements and the theoretical relationship of cavity length to the free space wavelength of the radiation, the velocity of propagation of the radiation can be determined.

The unusual accuracy of the experiment can be attributed largely to several highly ingenious variations on the cavity measurement of c:

(1) The measurement of the resonant frequency in vacuum in both the TE_{012} and the TE_{021} modes in order to suppress the importance of the diameter of the cylindrical cavity. The cavity length is chosen to place the resonant frequencies of the two modes only a few megacycles, out of 3000, apart.

(2) The use of optical flats (silver plated cast iron) for the end plates of the cavity with 3-point suspension of the upper flat on spacing rods whose ends are zones of a sphere of diameter equal to the rod length. This minimizes spacing uncertainties due to skewness of the assembly.

(3) The use of a gap between the cylindrical portion and each end plate of the cavity to suppress undesired modes of oscillation, and the correction of the results for the effect of the gap.

The various sources of error will be discussed in detail in a forthcoming paper by Bol.

The random errors from the various sources are:

(1) Contour variations	<3 in 10^7
(2) Due to error in Q	~ 1 in 10^7
(3) Frequency measurement	<3 in 10^7
(4) Coupling loop	~ 2 in 10^7
(5) Uncertainty in spacing between end-plates	<5 in 10^7.

The most significant known systematic error is the effect of the thin layer of silver sulfides, oxides, etc. on the side walls upon the resonant frequency. The true electrical diameter of the cavity is greater than the mechanical diameter by the skin depth of the walls for the particular mode of oscillation that is excited. The presence of a film of unknown thickness, with a conductivity and dielectric constant differing from that of the adjacent silver will reduce the observed resonant frequency of the cavity. Bol estimates that the value of C will be increased by approximately 0.3 km/sec when this skin effect is properly corrected for, but he prefers to consider the result provisional until experiments have been completed to measure the correction. Bol obtains a provisional value of

$$C = 299789.3 \pm (0.3 \text{ random} + 0.5 \text{ systematic}) \text{ km/sec}.$$

in which he uses a very conservative estimate of the skin effect error. Fortunately a shift in the value of C of the order of magnitude expected as a result of the correction for surface film effect will not affect the output values of the least squares calculation of the fundamental constants.

In the work which follows we shall use the weighted mean value calculated earlier using all of the published data,

$$C = (299790.0 \pm 0.7) \text{ km/sec}.$$

III. FURTHER NEW EXPERIMENTS

The several significant new experiments the results of which we shall use differ remarkably from the experiments of the 1947 work of Dumond and Cohen.[3] In this interval a whole new class of experiments of which the nuclear resonance experiments are characteristic have come to dominate the field of precision measurements. A technique basic to several of the experiments is the accurate measurement of ratios of certain characteristic resonance frequencies by making two measurements in the same homogeneous magnetic field. Thus the absolute value of the field is unimportant, providing the two samples are accurately located (in turn) at the same point in the field.

In the measurements of Hipple and his co-workers[4,8] at the Bureau of Standards, an electromagnet of unusually homogeneous field was constructed and painstakingly calibrated in absolute units. This instrument has already been used to great advantage in the absolute determination of the proton moment which enables the very accurate calculation of e/m and F.

A. Measurement of the Absolute Proton Moment

In recent papers[4] Thomas, Driscoll, and Hipple reported the results of an elaborate and highly accurate measurement of the proton moment in absolute units, using the technique of Purcell, Torrey, and Pound[20] to detect the nuclear resonance. The considerable pains taken to make an accurate absolute measurement of the magnetic field strength resulted in an accuracy an order of magnitude better than that in previous experiments. Their result when corrected for the diamagnetic effect[21] is

$$\gamma_p = (2.675305 \pm 0.00006) \cdot 10^4 \text{ sec}^{-1} \text{ gauss}^{-1}.$$

B. The Proton Moment in Bohr Magnetons and the Ratio of Magnetic Moments of Proton and Electron

There have been two recent accurate determinations of the proton moment in Bohr magnetons. Gardner and Purcell and R. W. Nelson[6] measured the precession frequency of the proton, $\omega_p = \gamma_p H_0$ and the cyclotron frequency of a free electron, $\omega_e = eH_0/mc$ in the same magnetic field. The proton resonance in mineral oil was observed by the method of Bloembergen, Purcell, and Pound[20] at 14.24 Mc/sec, and the electron resonance was observed at 9360 Mc/sec. The 657th harmonic of the proton frequency is compared with the electron frequency to obtain the result:

$$\omega_e/\omega_p = 657.4685 \pm 0.008.$$

A careful search for systematic errors was made and any resultant uncertainties are included in the above error. When the diamagnetic correction[20] of 2.68×10^{-5} the field at the proton is made, one obtains

$$\mu_p = (1.521026 \pm 0.00002) \cdot 10^{-3} \text{ Bohr magnetons}$$

in which we have used

$$\omega_e/\omega_p = (eH_0/mc)/\gamma_p H_0 = e/\gamma_p mc, \quad \gamma_p = 2\mu_p/\hbar,$$
$$\omega_e/\omega_p = e\hbar/2\mu_p mc, \quad \mu_p = (\omega_p/\omega_e)(e\hbar/2mc),$$
$$\omega_p/\omega_e = \mu_p/\mu_1.$$

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[20] Purcell, Torrey, and Pound, Phys. Rev. **69**, 37 (1946). Bloembergen, Purcell and Pound, Phys. Rev. **73**, 679 (1948).

[21] H. A. Thomas, Phys. Rev. **80**, 901 (1950); N. F. Ramsay, Phys. Rev. **78**, 699 (1950).

TABLE III. Weighted mean evaluation of μ_p.

	Value	Error $\times 10^4$	Weight
Gardner-Purcell	1.521026	0.2	25.0
Kusch-Foley-Taub	1.521057	0.7	2.0
Weighted mean	1.521027	±0.2	Total 27.0

The other significant determination of μ_p/μ_1, the proton moment in Bohr magnetons is that of Taub and Kusch,[5] in which a carefully planned series of molecular beam magnetic resonance experiments lead in a somewhat indirect manner to the desired experimental value. The experiments consist of:

(1) The measurement of g_p, the g-value of the proton in the NaOH molecule and g_J, the g-value for total electronic angular momentum of ground state caesium and indium, all in the same magnetic field. The g_p is found from the familiar flip-over frequency of the proton, while the g_J are found from the frequencies corresponding to transitions between the hyperfine structure levels of the ground states of the appropriate atoms. This measurement produced accurate values for the ratios

$$g_p/g_{J\mathrm{Cs}}=(1.51911\pm0.00003)\cdot10^{-3}$$

and

$$g_p/g_{J\mathrm{In}}=(4.56877\pm0.00007)\cdot10^{-3},$$

where $g_{J\mathrm{Cs}}$ is the g_J value for Cs^{133} and $g_{J\mathrm{In}}$ is the g_J value for In^{115}.

(2) The measurement of the ratios

$$g_{J\mathrm{Cs}}/g_{J\mathrm{Na}}=1.000134,$$

and

$$g_{J\mathrm{In}}/g_{J\mathrm{Na}}=1/3(1.00243),$$

where $g_{J\mathrm{Na}}$ is the g_J value of Na^{23}.

(3) The measurement of the g_J values of Li^6, Li^7, Na^{23}, and K^{39} with the result that they are equal to within one part in 40,000. Kusch and Taub conclude that "within the precision of these measurements, the g_J of these atoms is equal to the spin g, g_e. For if perturbations were to affect the value of g_J, they would presumably affect the different atoms by different amounts."

The results of the three experiments of Kusch, Taub, and Foley are then properly combined as follows:

$$(g_p/g_{J\mathrm{Na}})_{\mathrm{Cs}}=(g_p/g_{J\mathrm{Cs}})(g_{J\mathrm{Cs}}/g_{J\mathrm{Na}})=1.51931\times10^{-3}$$

and

$$(g_p/g_{J\mathrm{Na}})_{\mathrm{In}}=(g_p/g_{J\mathrm{In}})(g_{J\mathrm{In}}/g_{J\mathrm{Na}})=1.51923\times10^{-3},$$

from which

$$(g_p/g_{\mathrm{Na}})_{\mathrm{mean}}=(1.51927\pm0.00010)\cdot10^{-3},$$

where it is to be noted that the first two values are entirely independent in that the transition field which is the essence of the apparatus is different in the two experiments and the field strengths used differed widely.

Next, the experimental result that the g_J of Na is equal to g_e, the spin g of the electron, leads to the result

$$g_p/g_e=(1.51927\pm0.00010)\cdot10^{-3}.$$

This value, together with the good agreement of the experimental value of g_e in reference 5 with Schwinger's value as corrected by Karplus and Kroll,[22]

$$g_e=2(1.001145),$$

leads to

$$g_p=(3.04202\pm0.00015)\cdot10^{-3}.$$

The proton magnetic moment in Bohr magnetons is then computed from g_p and a diamagnetic correction of 4.7×10^{-5} is added to obtain

$$\mu_p=(1.521057\pm0.0007)\cdot10^{-3}\text{ Bohr magnetons.}$$

The Gardner-Purcell value and the Kusch-Foley-Taub value of μ_p are combined in Table III to obtain a weighted mean value.

C. Hyperfine Structure of Hydrogen

Prodell and Kusch have reported[7] on a precise measurement of the hyperfine structure separation of H and D, of which only the former is of interest here. The atomic-beam magnetic resonance method was used to observe the frequency necessary to induce transitions among certain magnetic levels of the hyperfine structure multiplets of ground-state hydrogen in a weak field. The measurement is absolute in that it depends on a fundamental time standard and nothing else. The result is

$$\nu_H=1420.4051\pm0.006\text{ Mc/sec.}$$

D. Ratio of Proton Cyclotron Frequency to Proton Precession Frequency

Hipple, Sommer, and Thomas[8] have reported the precise measurement of the ratio of the cyclotron frequency of the proton to its nuclear precession frequency. The primary features in this experiment are the measurement of both frequencies in the highly homogeneous magnetic field described in reference 4, plus the development of a new device called the omegatron for the measurement of the cyclotron resonance frequency of the proton. Their result $\omega_p/\omega_c=2.792685\pm0.00006$ is combined with the result of Bloch and Jeffries,[8] $\omega_p/\omega_c=2.79245\pm0.00020$ to obtain an uncorrected weighted mean value of $\omega_p/\omega_c=2.792746\pm0.000056$.

We apply a diamagnetic correction of 7.8×10^{-5} after Thomas[21] to obtain

$$\omega_p/\omega_c=2.792746\pm0.000056.$$

E. Compton Wavelength of the Electron

Dumond, Lind, and Watson have reported[9] the result of a precision measurement of the wavelength of

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[22] J. Schwinger, Phys. Rev. **73**, 416 (1948); R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950). The value given is our result using $1/\alpha=137.043$ in the formula of Karplus and Kroll.

annihilation radiation from Cu^{64} using the two-meter focusing curved crystal spectrometer. A careful consideration of all of the instrumental and theoretical effects leads to the published result:

$$h/mc=(0.024271\pm 0.000010)\cdot 10^{-8}\text{ cm}.$$

It is to be noted that the effect of the initial velocity of the annihilating positron or electron leads to no appreciable average shift of the photon frequency. Instead, the first-order effect is a line-broadening similar to the Doppler effect.

IV. INPUT DATA DERIVED FROM THE NEW EXPERIMENTS

A. Calculation of e/m

The proton precession frequency, ω_p, and the electron cyclotron frequency, ω_e, are given theoretically by

$$\omega_p=\gamma_p H_0,\quad \omega_e=eH_0/m,$$

where γ_p is the gyromagnetic ratio of the proton, and H_0 is the dc magnetic field strength. Therefore

$$e/m=\gamma_p\omega_e/\omega_p$$

and after Thomas, Driscoll, and Hipple[4, 8] we have calculated

$$e/m=(1.758912\pm 0.00005)\cdot 10^7\text{ emu/g}$$

by using their value of γ_p and the frequency ratio of Gardner and Purcell.[6]

B. Calculation of α

Bethe and Longmire[10] point out that the Fermi hyperfine-structure formula for hydrogen can be modified to include: (1) Breit's relativistic correction, (2) the reduced mass effect, (3) the electromagnetic correction to the electron magnetic moment, (4) the correction for the distributed nature of the proton magnetic moment, and (5) the correction for the distributed nature of the electron magnetic moment. The result is

$$\nu_H=\alpha^2\left[\frac{16}{3}\left(\frac{\mu_p}{\mu_1}\right)CR_\infty\left(1+\frac{2\alpha^2}{3}\right)\left(1+\frac{m}{M}\right)^{-3}\right.$$
$$\left.\times\left(1+\frac{\alpha}{2\pi}-\frac{2.97\alpha^2}{\pi^2}\right)\left(1-\frac{(g_p-g_D)\alpha m}{g_p\mu}\right)\left(1-\frac{\alpha^2}{2\pi}\right)\right],$$

where μ_1 is the Bohr magneton, μ is the meson mass, g_D is the deuteron g-factor, and the other quantities have been previously defined.

We use the results quoted in III c, and b, plus E. R. Cohen's* value of R_∞, and ignore the last two factors in obtaining a result

$$1/\alpha=137.0432\pm 0.0009.$$

C. Calculation of F

From the relation

$$F=\gamma_p M_p\omega_c/\omega_p$$

and the values of γ_p and ω_c/ω_p quoted in III a, and d, plus the isotopic weight of the proton,[3]

$$M_p=1.007580\pm 0.000003,$$

Hipple, Sommer, and Thomas calculate

$$F=9652.03\pm 0.3\text{ emu/g}$$

on the physical scale.

D. Calculation of h/e

When the new value of the velocity of light is used in the formula

$$h/e=(V\lambda/C^2)\cdot 10^8,$$

V being in absolute volts and λ in centimeters, the weighted-mean value of Bearden and Schwarz and of Bearden, Johnson and Watts as computed in reference 2 gives

$$h/e=(1.37928\pm 0.00004)\times 10^{-17}\text{ erg sec/esu}.$$

E. Compton Wavelength of the Electron, h/mc

We use the result of III e and multiply by C to obtain

$$h/m=7.2762\pm 0.0030\text{ erg sec/g}.$$

V. CORRECTIONS ON OLD INPUT DATA

A careful study of the original references[23–34] was made on the remaining experimental input data of reference 3 which are not being replaced by new input data. The point of this survey was to determine which of the

TABLE IV. Revised values of 1947 data.

Quantity	Value	c-dependence	Corrected value
N	6.02338×10^{23}	none	——
mN	5.48541×10^{-4}	none	——
Fe/m	1.69824×10^{11}	none	$^*1.69778\times 10^{11}$
$h/(me)^{\frac{1}{2}}$	1.00084×10^{-8}	$c^{-\frac{1}{2}}$	1.000818×10^{-8}
e^2/mh	3.8197×10^{34}	c^{+1}	3.81987×10^{34}

* We have included the correction pointed out in reference 10 to Fe/m to account for the effect of scattering by the atomic nuclei.

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* *Note.*—We are indebted to E. R. Cohen for communicating to us in advance of publication the result of a recent study in which he has obtained $R_\infty=109737.323\pm 0.024\text{ cm}^{-1}$.

[23] J. A. Bearden, J. Appl. Phys. **12**, 395 (1941).

[24] J. A. Bearden, Phys. Rev. **54**, 698 (1938).

[25] F. G. Dunnington, Rev. Modern Phys. **11**, 68 (1939).

[26] C. E. Robinson, Phil. Mag. **22**, 1129 (1936).

[27] Robinson and Clews, Proc. Roy. Soc. London **A176**, 28 (1940).

[28] Robinson and Mayo, Proc. Roy. Soc. London **A173**, 192 (1939).

[29] Drinkwater, Richardson, and Williams, Proc. Roy. Soc. London **A174**, 164 (1940).

[30] W. V. Houston, Phys. Rev. **51**, 446 (1937); Phys. Rev. **30**, 608 (1927).

[31] D. Y. Chu, Phys. Rev. **55**, 175 (1939).

[32] R. C. Williams, Phys. Rev. **54**, 568 (1938).

[33] C. E. Robinson, Phys. Rev. **55**, 423A (1939).

[34] C. D. Shane and F. H. Spedding, Phys. Rev. **47**, 33 (1935).

TABLE V. Equations of condition and associated normalized errors.

Quantity measured	n	a_n	b_n	c_n	ω_n	ϵ_n
N	1	0	1	0	−5.62	0.710
F	2	1	0	0	2.84	0.304
mN	3	−4	5	3	5.11	1.37
Fe/m	4	6	−5	−3	−1.28	2.06
e/m	5	5	−5	−3	−0.909	0.255
h/e	6	−1	1	1	6.79	0.290
α	7	2	−2	−1	2.832	0.065
$h/(me)^{\frac{1}{2}}$	8	1.5	−1.5	−0.5	13.77	5.80
h/m	9	4	−4	−2	1.32	4.12
e^2/mh	10	6	−6	−4	0.41	11.78

Cross Terms:

$\epsilon_{25}{}^2=\epsilon_{52}{}^2=+0.0484,$

$\epsilon_{57}{}^2=\epsilon_{75}{}^2=-0.00845$

subject experiments used the velocity of light in computing the published values from the experimental data. The results are given in Table IV.

VI. LEAST SQUARES METHOD

The method of least squares has been discussed adequately in numerous texts and journals, of which one of the most readable is *The Calculus of Observations* by Whittaker and Robinson.[35] It is easily shown[3] that all of the experimental quantities for which measured values are given in III and IV can be expressed as products of various powers of F, N, m, and h, and certain auxiliary constants such as c which are known to a high precision. It has been shown that these equations can be reduced through a Taylor expansion to linear equations in terms of deviations from origin values of the variables F, N, m, and h. Further, by virtue of the exceptional precision with which R_∞ is known, the equation for R_∞ can be used to eliminate one of the variables. We follow Dumond and Cohen[3] in eliminating m, so that we will express each experimentally observed quantity in terms of F, N, and h.

As an example, consider the fine-structure constant,

$$\alpha=2\pi e^2/ch,$$

but $F=Ne$, so

$$\alpha=2\pi F^2/chN^2,$$

The linearized equation is given by

$$(\partial\alpha/\partial F)_0(F-F_0)+(\partial\alpha/\partial N)_0(N-N_0)+(\partial\alpha/\partial H)_0(H-H_0)=(V-V_0),$$

where h has been changed to H for convenience of notation, the subscript 0 denotes an origin value chosen close enough to the expected "true" value to render second order terms unimportant, V_0 is the corresponding origin value of α, and v without subscript is the observed value of α. On performing the indicated differentiations and evaluating the derivatives with the origin values, we obtain

$$2V_0\frac{(F-F_0)}{F_0}-2V_0\frac{(N-N_0)}{N_0}-V_0\frac{(H-H_0)}{H_0}=(V-V_0).$$

Divide through by M_0 and use the definitions

$$x=(F-F_0)/F_0,\quad y=(N-N_0)/N_0,\quad z=(H-H_0)/H_0,$$
$$v=(V-V_0)/V_0,$$

to obtain

$$2x-2y-z-v=0,$$

the desired linear equation of condition for the observable, α.

We will modify the equations of condition to the form

$$a_nX+b_nY+c_nZ+\omega_n=0$$

wherein the three new variables are defined by

$$X=10^4x,\quad Y=10^4y,\quad Z=10^4z.$$

Also we define

$$\omega_n=-10^4v_n,\quad \epsilon_n=10^4e_n,$$

where e_n is the normalized probable error in the experimental determination of the quantity V_n (V_n was simply V in our example). The coefficients of the equations of condition for the ten experimental results discussed in IV and V are given in Table V, using the auxiliary constants listed later in Table VII. All masses are expressed according to the chemical scale of atomic weights.

The values listed for ω_n and ϵ_n are carried beyond the number of significant figures as one effort to minimize accumulation of error due to successive roundings-off.

Before the equations of condition of Table V can be used to obtain the normal equations, the individual equations must be divided by their respective $\epsilon_n{}^2$'s to obtain the weighted equations of condition. In order to obtain the normal equation in the variable X we multiply each weighted equation through by its a_n and add the ten equations so obtained. By a similar process the normal equations in Y and Z are obtained. However, in the present case F and e/m are correlated through their common use of γ_p (see Sec. IV) and α and e/m are correlated through the quantity μ_p/μ_1. The normal equations must therefore be obtained by a more general method due to E. R. Cohen[36] which includes the cross terms listed in Table V in a 10 × 10 error matrix. The resultant normal equations were then solved for X, Y, and Z, and the latter were used to compute F, N, and h.

Since the method of handling correlated variables is complicated and particularly subject to computational errors we sought a different means of solving the problem as a check. A new method was evolved[37] which allowed expressing all of the pertinent observations in

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[35] E. T. Whittaker and G. Robinson, *The Calculus of Observations* (Blackie and Son, Ltd., London, 1948).

[36] E. R. Cohen, Phys. Rev. **81**, 162 (1951). We are deeply indebted to J. W. Dumond for pointing out to us the error in principle involved in a solution which ignores this correlation.

[37] J. A. Bearden and H. M. Watts. Phys. Rev. **81**, 160 (1951).

a set of uncorrelated equations of condition. The results of the uncorrelated method were in excellent agreement with those of the correlated method. Owing to the considerable difference in complexity of calculating and checking both the values of the atomic constants and the associated errors, the simpler uncorrelated method was adopted in the remainder of the work.

The least-squares values of F, N, and h, obtained by both methods are shown in Table VI.

TABLE VI. Least-squares values of the base variables.

F	9649.54 emu equiv.$^{-1}$ (Chem.)
N	6.02402×10^{23} (Chem.)
h	6.62363×10^{-27} (erg sec)

All of the other fundamental constants can be computed in terms of these plus the auxiliary constants.

The errors can be computed in the usual way[35] for a

TABLE VII. Least squares adjusted values of the fundamental atomic constants.

F	Faraday	(9649.54 ± 0.18)emu equiv^{-1} (Chem.) (9652.17 ± 0.17) emu equiv^{-1} (Phys.)
N	Avogadro's number	$(6.02402\pm0.00017)\times10^{23}$ (Chem.) $(6.02566\pm0.00016)\times10^{23}$ (Phys.)
h	Planck's constant	$(6.62363\pm0.00016)\times10^{-27}$ erg sec.
m	Electron mass	$(9.10710\pm0.00022)\times10^{-28}$ g
e	Electronic charge	$(4.80217\pm0.00006)\times10^{-10}$ esu. $(1.601844\pm0.000021)\times10^{-20}$ emu
h/e		$(1.379300\pm0.000016)\times10^{-17}$ erg sec esu^{-1}
e/m	Specific electronic charge	$(1.758896\pm0.000028)\times10^{7}$ emu g^{-1}
h/m		(7.27304 ± 0.00007) cm^2 sec^{-1}
α	Fine structure constant	$(7.296953\pm0.000037)\times10^{-3}$
α^2		$(5.324552\pm0.000054)\times10^{-5}$
$1/\alpha$		137.0435 ± 0.0007
h/mc	Compton wavelength of electron	$(2.426045\pm0.000025)\times10^{-10}$ cm
$e/4\pi mc$	Zeeman displacement per gauss	$(4.668885\pm0.00008)\times10^{-5}$ cm^{-1} gauss^{-1}
a_0	First Bohr radius	$(0.5291483\pm0.0000024)\times10^{-8}$ cm
a_0'	Separation of electron and proton in the ground state of H^1	$(0.5291342\pm0.0000024)\times10^{-8}$ cm
a_0''	Radius of the electron orbit referred to center of mass for normal H^1	$(0.5294224\pm0.0000024)\times10^{-8}$ cm
$\frac{1}{16}R_H\alpha^2$	Doublet separation in hydrogen	(0.3649900 ± 0.0000037) cm^{-1}
σ	Stefan-Boltzmann constant	$(5.6694\pm0.0011)\times10^{-5}$ erg cm^{-2} deg^{-4} sec^{-1}
k	Boltzmann's constant	$(1.38020\pm0.00007)\times10^{-16}$ erg deg^{-1}
c_1	First radiation constant	$(4.99060\pm0.00012)\times10^{-15}$ erg cm
c_2	Second radiation constant	(1.43870 ± 0.00007) cm deg
$\lambda_{max}T$	Wien displ. law constant	(0.289762 ± 0.000014) cm deg
μ_1	Bohr magneton	$(9.27100\pm0.00017)\times10^{-20}$ erg gauss^{-1}
$(3K/N)^{\frac{1}{2}}$	Multiplier of (Curie const.)$^{\frac{1}{2}}$ to give magnetic moment per molecule	$(2.62173\pm0.00009)\times10^{-20}$ (erg mole deg$^{-1\frac{1}{2}}$)
h/k	Atomic specific heat constant	$(4.79903\pm0.00023)\times10^{-11}$ sec deg
μ	Reduced mass of electron in H^1	$(9.10214\pm0.00022)\times10^{-28}$ g
$2m/\hbar^2$	Schrödinger constant for fixed nucleus	$(1.638995\pm0.000045)\times10^{27}$ erg^{-1}
$2\mu/\hbar^2$	Schrödinger constant for the H^1 atom	$(1.638103\pm0.000045)\times10^{27}$ erg^{-1}
mN	Atomic weight of the electron	$(5.486137\pm0.00009)\times10^{-4}$ (Chem.) $(5.487629\pm0.00009)\times10^{-4}$ (Phys.)
λ_0	Wavelength associated with 1 ev	$(12396.32\pm0.16)\times10^{-8}$ cm
M/m	Ratio proton mass to electron mass	(1836.093 ± 0.044)
ν_0	Frequency associated with 1 ev	$(2.418378\pm0.000029)\times10^{14}$ sec^{-1}
E_0	Energy associated with 1 ev	$(1.601844\pm0.000021)\times10^{-12}$ erg
h/e	Potential associated with unit frequency	$(1.379300\pm0.000016)\times10^{-17}$ esu
E_1	Energy associated with unit wave number	$(1.985698\pm0.000048)\times10^{-16}$ erg
v_0	Speed of 1-ev electron	$(5.931098\pm0.000046)\times10^{7}$ cm sec^{-1}
	Conversion factor atomic mass units to Mev 1 amu $=931.128\pm0.017$ Mev	
mC^2	Energy equivalent of electron mass	(0.510969 ± 0.000009) Mev
$(R_0/F)\times10^{-8}$	Energy associated with 1°K	$(8.61632\pm0.00042)\times10^{-5}$ ev
T_0	"Temperature" associated with 1 ev	(11605.9 ± 0.6) deg K
n_0	Loschmidt's number	$(2.68754\pm0.00010)\times10^{19}$ cm^{-3}
	Auxiliary Constants Used	
c	Velocity of light	(299790.0 ± 0.7) km sec^{-1}
R_∞	Rydberg constant for infinite mass	109737.323 ± 0.024 cm^{-1}
R_H	Rydberg constant for H^1 atom	(109677.581 ± 0.008)cm^{-1}
ν_H	Hyperfine structure splitting in ground state H^1	$(1420.4051\pm0.0003)\times10^{6}$ sec.$^{-1}$
	Atomic masses (Physical scale)*	H$^1=1.0081283\pm0.0000028$ $M_p=1.007580\pm0.000003$
R_0	Gas constant per mole	$(8.31436\pm0.00038)\times10^{7}$ erg mole^{-1} deg^{-1}
V_0	Standard volume of perfect gas	(22414.6 ± 0.6) cm^3 mole^{-1}
	Conversion factors NBS Int. units to absolute electrical units 1 NBS ohm $-p$ absolute ohm $p=1.000495$ 1 NBS v $=pq$ absolute v $pq=1.000330$ 1 NBS amp $=q$ absolute amp $q=0.999835$	
λ_g/λ_s	Ratio of grating in Siegbahn scales of wavelength	1.002030 ± 0.000020
d_{20}	Grating space of calcite at 20°C	$(3.03567\pm0.00005)\times10^{-8}$ cm
ρ	Density of calcite at 20°C	(2.71030 ± 0.00003) g cm^{-3}
r	Ratio physical to chemical scale of atomic weights	(1.000272 ± 0.000005)

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* K. T. Bainbridge, "Isotopic weights of the fundamental isotopes," Preliminary Report No. 1, National Research Council (June, 1948).

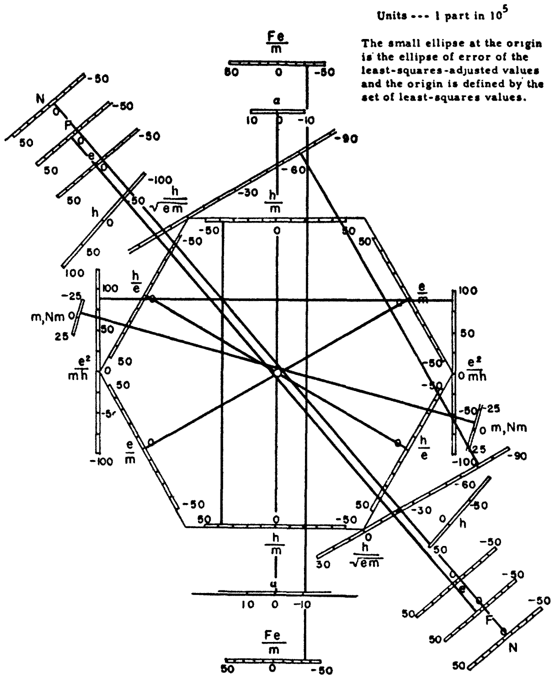

FIG. 1. Isometric consistency chart showing deviation of the input data from the least-squares-adjusted values.

function F_n, (we use F_n to denote an output value to distinguish it from ω_n, an input value for the same quantity).

$$F_n(X, Y, Z)=a_nX+b_nY+c_nZ$$

through use of the relation

$$\sigma^2(F_n)=a_n^2\sigma_{XX}^2+b_n^2\sigma_{YY}^2+c_n^2\sigma_{ZZ}^2+2a_nb_n\sigma_{XY}^2+2a_nc_n\sigma_{XZ}^2+2b_nc_n\sigma_{YZ}^2.$$

The normalized errors, σ, are obtained from the characteristic determinant of the normal equations as follows: Let A be the 3×3 characteristic determinant with elements A_{ij}, then

$$\sigma_{ij}^2=(-)^{i+j}A_{ij}^{-1},$$

where A_{ij}^{-1} is the minor of A_{ij} divided by det$A$. If the coefficients of the normal equations are arranged in the determinant in the order a_n, b_n, c_n in each row, and the rows represent the normal equations in X, Y, Z in that order, then for i or j in σ_{ij}^2 read 1 for X, 2 for Y, and 3 for Z.

The errors computed thus far are mean errors for an observation of unit weight. In order to convert to probable error for the weight appropriate to a given set of weighted equations of condition, one must multiply the above results by the factor

$$W=0.6745[(\sum w_id_i^2)/(s-f)]^{\frac{1}{2}}.$$

The factor 0.6745 arises from the conversion from mean to probable error, s is the number of equations of condition (10 in this case), f is the number of variables (3 in this case), and $\sum w_id_i^2$ is the sum of the squares of the residuals of the 10 weighted equations of condition obtained by substituting in them the computed least-squares values of the three variables.

On applying these and earlier relationships we obtain finally the set of least-squares adjusted values of the fundamental atomic constants and their errors as shown in Table VII.

The input data of the ten experiments used in the calculations are shown on an isometric consistency chart[38] in Fig. 1. The origin of the chart is the set of least-squares-adjusted values of Table VI, and the normalized deviations from its origin are shown on the appropriate scales in units of 1 part in 10^5. It will be noted that the input values group rather evenly about the origin, but that some deviations of input from output values are in the order of several parts in 10^4. The small ellipse about the origin is the ellipse of error of the output values.

The same situation is again presented in Fig. 2 on a scale ten times as large as that of Fig. 1. It is noted that only six of the ten experimental results agree well enough with the least-squares value to appear on this highly expanded consistency chart. These six, however, agree an order of magnitude better with the output values than did the results of 1947. This means, of course, that the highly refined recent experiments are converging nicely on what we may hope are the true values. Further, the excellent consistency of the results points to an improved recognition, or elimination, of the

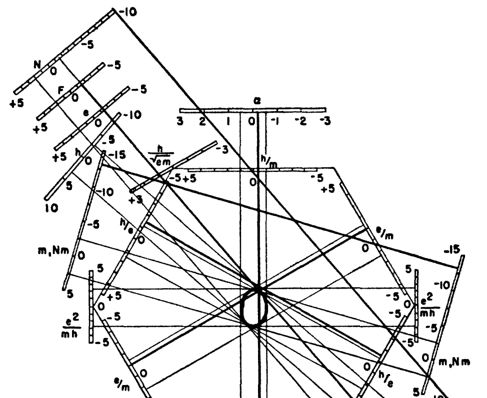

FIG. 2. Isometric chart comparing deviation of experimental values from least-squares-adjusted values with probable errors of latter. (Light lines—probable errors; heavy lines—experimental values; units—1 part in 10^6.) 1. α: Gardner and Purcell; 2. e/m: Hipple, Sommer, and Thomas; 3. mN: Drinkwater, Richardson, and Williams; 4. N: Bearden; 5. h/e: Bearden and Schwarz and Bearden, Johnson, and Watts; 6. F: Hipple, Thomas, and Driscoll.

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[38] J. W. Dumond, Phys. Rev. 58, 457 (1940).

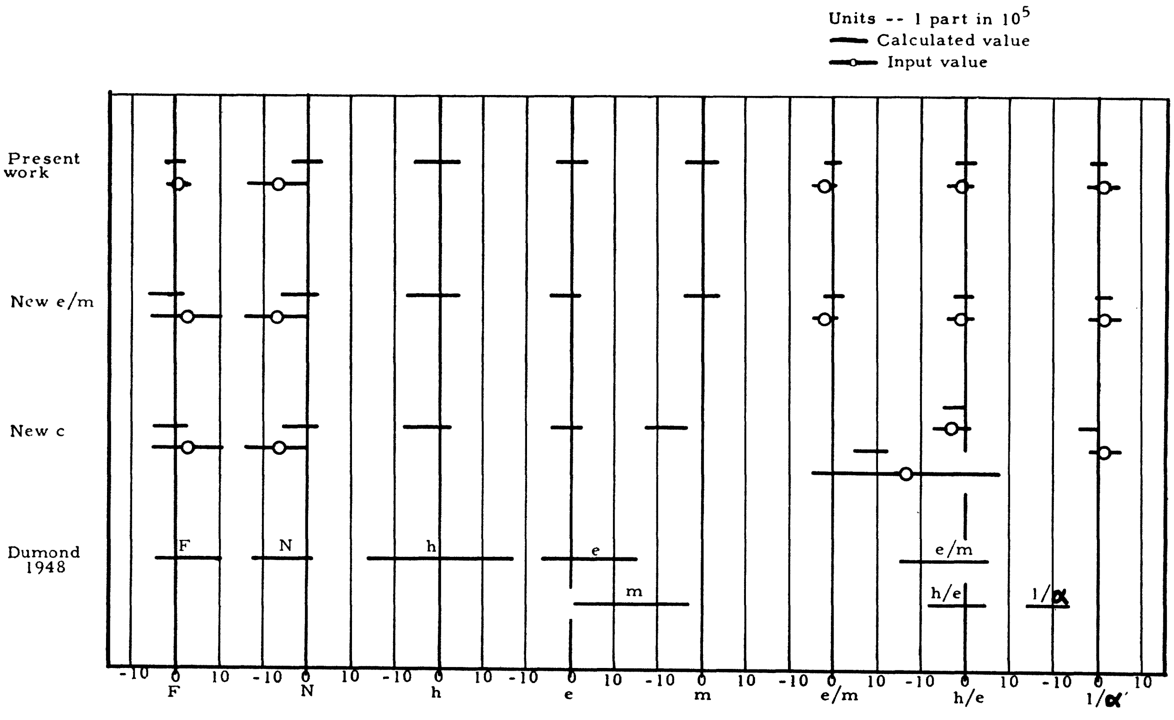

FIG. 3. Comparison of values and errors of several atomic constants.

systematic errors whose persistent presence has necessitated the least-squares approach. The results can be considered to confirm the Hansen and Bol value of c, since a minor change in c distorts Fig. 2 radically.

Figure 3 illustrates the trend in the values and the probable errors of several selected atomic constants since Dumond's calculation in 1948. The present least-squares-adjusted values were arbitrarily used as origins in Fig. 3. Early in 1950 we made a least-squares calculation using the new value of c and other new data available at that time. These results are shown under the heading "New c." It is noted that the probable errors in these results have dropped to about half their prior values and the overlap of input with output values is good. Next the new results[4] on e/m represented such a significant improvement on the error in the input value of e/m that a new calculation was made and the results are shown under the heading "New e/m." Finally, the new experimental value of F obtained by Hipple and his co-workers (see Sec. IV c) necessitated a third calculation, the results of which are given under the heading "Present Work." One sees that the errors have shrunk further and the consistency of input with output values is excellent.**

We acknowledge the invaluable assistance of Drs. Kees Bol, D. E. Kerr, and D. D. King in examining and assessing the experiments of Hansen and Bol. In addition, we thank Messrs. D. Barbieri and H. T. David for advice concerning least squares, and Mr. F. T. Johnson for aid in various parts of the work. We are greatly indebted to Professor T. H. Berlin for critically reviewing the manuscript, and to the Bureau of Ordnance U. S. N. for financial assistance which made this work possible.

** *Note added in proof.*—The original monograph of the present work released last summer has elicited numerous private communications which together with recently published data modify the input values noticeably. In order to prevent obsolescence of the present work we have incorporated these new data. Accordingly, the table of values represents our knowledge of the atomic constants as of December 1, 1950.