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On the First Passage Time Probability Problem

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We have derived an exact solution for the first passage time probability of a stationary one-dimensional Markoffian random function from an integral equation. A recursion formula for the moments is given for the case that the conditional probability density describing the random function satisfies a Fokker-Planck equation. Various known solutions for special applications (noise, Brownian motion) are shown to be special cases of our solution. The Wiener-Rice series for the recurrence time probability density is derived from a generalization of Schrödinger's integral equation, for the case of a two-dimensional Markoffian random function.

I. INTRODUCTION

A STATIONARY Markoffian random function, $y(t)$, such as a velocity component of a colloidal particle in Brownian motion or of a star in a cluster, or the noise current in an R-L circuit as functions of the time, is defined by $P(y_0|y, t)dy$, the probability that $y \leqslant y(t) < y+dy$, if $y(0)=y_0$. For a continuous function, $y(t)$, the first passage time probability $\mathcal{P}(y_0|t, a)dt$ is defined as the probability that $y(t)$ passes the value a for the first time in the time interval $(t, t+dt)$ if $y(0)=y_0$.

The problem of determining $\mathcal{P}(y_0|t, a)$ has been solved for certain special cases by Schrödinger,[1] Smoluchowski,[2] Chandrasekhar,[3] and Wang and Uhlenbeck.[4] Schrödinger and Smoluchowski found the probability that a free particle in Brownian motion in a medium of high viscosity reaches a marker at a for the first time in $(t, t+dt)$ after starting from y_0 at $t=0$. For high viscosity the coordinate of the particle can be treated as a Markoffian random function (for times large as compared with the relaxation time),[5] and $P(y_0|y, t)$ is the well known expression determined by Einstein, which satisfies the ordinary diffusion equation. $\mathcal{P}(y_0|t, a)$ was obtained by observing that for $y_0<a$

$$\mathcal{P}(y_0|t, a) = -\frac{\partial}{\partial t}\int_{-\infty}^{a} P_a(y_0|y, t)dy, \qquad (1.1)$$

where $P_a(y_0|y, t)dy$ is the probability that the particle is in $(y, y+dy)$ at t and that at no time between 0 and t the particle has reached the marker at a. The probability $P_a(y_0|y, t)dy$ can be visualized as the probability of finding the particle in $(y, y+dy)$ at t if an absorbing barrier is present at a, and can thus be obtained as the fundamental solution of the diffusion equation with boundary conditions

$$P_a(y_0|-\infty, t) = P_a(y_0|a, t) = 0. \qquad (1.2)$$

Schrödinger pointed out that his expression for $\mathcal{P}(y_0|t, a)$ can be checked by means of an integral equation obtained by the following argument. The probability

$$\phi(y_0|t, a) = \int_{-\infty}^{a} P(y_0|y, t)dy$$

that the particle is at t in $(-\infty, a)$ is the sum of the probability $f(y_0|t, a)$ that the particle did not pass a in $(0, t)$, and the probability that it passed a for the first time at some time $\theta(0<\vartheta<t)$ but returned. Since

$$\mathcal{P}(y_0|t, a) = -\partial f(y_0|t, a)/\partial t, \qquad (1.3)$$

this yields an integral equation for $\mathcal{P}(y_0|t, a)$. Chandrasekhar estimated the rate of escape of stars from clusters by determining the probability that a star with given initial velocity will reach a chosen velocity for the first time in $(t, t+dt)$. He obtained this probability using the first method described above with the appropriate differential equation of the Fokker-Planck type instead of the ordinary diffusion equation. Uhlenbeck and Wang gave an explicit function for the probability that a velocity component of a free particle in Brownian motion (or the noise current in an L-R circuit) passes the value zero for the first time in $(t, t+dt)$.

The problem of the distribution of the absolute maximum of $y(t)$ in an interval is closely related to the first passage time problem. The probability $M(a, t)da$ that the absolute maximum of y in the interval $(0, t)$ lies between a and $a+da$ is given by

$$M(a, t) = \frac{\partial}{\partial a}\int_{-\infty}^{a} W(y_0)dy_0 f(y_0|t, a) \qquad (1.4)$$

where

$$W(y) = \lim_{t\to\infty} P(y_0|y, t). \qquad (1.5)$$

Problems of this type arise in engineering applications since circuits have been designed which register the absolute maximum of a current in a time gate, with the purpose of improving discrimination between signals and noise.

Apart from applications, the first passage time problem seemed of interest since it can be considered as the problem of finding the statistical properties of a branch of the inverse function $t(y)$ if the statistical properties of $y(t)$ are known.

[1] E. Schrödinger, Physik. Z. **16**, 289 (1915).
[2] M. v. Smoluchowski, Physik. Z. **16**, 318 (1915).
[3] S. Chandrasekhar, Astrophys. J. **97**, 263 (1943).
[4] M. C. Wang and G. E. Uhlenbeck, Revs. Modern Phys. **17**, 323 (1945).
[5] H. A. Kramers, Physica **7**, 284 (1940).

In Sec. II we have given an integral equation for $\mathcal{P}(y_0|t, a)$ without making the assumption that $P(y_0|y, t)$ satisfies a Fokker-Planck equation. The integral equation yields an exact expression for $\mathcal{P}(y_0|t, a)$ in terms of $P(y_0|y, t)$. Schrödinger's integral equation is shown to follow from our simpler integral equation. A simple expression for $\mathcal{P}(y_0|t, a)$ has been obtained for the case in which a is a point of symmetry of the problem. The formula given by Uhlenbeck and Wang for the Gaussian case is a special case of this result. In Sec. III we have derived the differential equation method from our result in case $P(y_0|y, t)$ satisfies a Fokker-Planck equation and have given a recursion formula for the moments of the first passage time. In Sec. IV we have discussed $\mathcal{P}(y_0|t, a)$ and $f(y_0|t, a)$ as solutions of the adjoint of the Fokker-Planck equation. In order to show the connection between the integral equation method and the approach of Wiener and Rice leading to the Wiener-Rice series[6] for the recurrence time probability of a random function in Sec. V we have derived the series from an integral equation for the case of a two-dimensional stationary Markoffian random function.[7]

II. THE INTEGRAL EQUATION

The fundamental integral equation is obtained by classifying the functions $y(t')$ for which $y(0)=y_0$ and $y\leqslant y(t)<y+dy$ according to the time $\vartheta>0$ at which they pass the value a for the first time ($y_0<a\leqslant y$). One thus obtains:

$$P(y_0|y, t)=\int_0^t \mathcal{P}(y_0|\vartheta, a)P(a|y, t-\vartheta)d\vartheta. \quad (2.1)$$

This integral equation is solved by a Laplace transformation. Using the notation f_L for the Laplace transform of a function f, we have

$$P_L(y_0|y, \lambda)=\int_0^\infty e^{-\lambda t}P(y_0|y, t)dt \quad (2.2)$$

and

$$\mathcal{P}_L(y_0|\lambda, a)=\int_0^\infty e^{-\lambda t}\mathcal{P}(y_0|t, a)dt, \quad (2.3)$$

and from (2.1)

$$P_L(y_0|y, \lambda)=\mathcal{P}_L(y_0|\lambda, a)P_L(a|y, \lambda) \quad (2.4)^8$$

and

$$\mathcal{P}(y_0|t, a)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{P_L(y_0|y, \lambda)}{P_L(a|y, \lambda)}d\lambda, \quad (2.5)$$

with $\gamma>0$ and otherwise arbitrary real, and $y_0<a\leqslant y$.

[6] S. O. Rice, Bell System Tech. J. **24**, 46, 64 (1945).

[7] The main results of our calculations have been stated in the following abstracts: A. J. F. Siegert, Phys. Rev. **70**, 449 (1946); **71**, 469 (1947); **71**, 485 (1947); **73**, 1271 (1948).

[8] Moments of $\mathcal{P}(y_0|t, a)$ can be obtained by expanding $P_L(y_0|y, \lambda)/P_L(a|y, \lambda)$ in powers of λ.

From (2.3) and (2.4) it follows that

$$\mathcal{P}(a|t, a)=0 \text{ for } t>0 \quad (2.6)$$

with

$$\int_0^\epsilon \mathcal{P}(a|t, a)=1 \text{ for any real } \epsilon>0 \quad (2.7)$$

which is in agreement with (2.1) and (2.8).

From our solution follows a property of $\mathcal{P}(y_0|ta)$ which can also be obtained directly from the argument leading to (2.1); i.e.,

$$\mathcal{P}(y_0|t, b)=\int_0^t \mathcal{P}(y_0|\vartheta, a)\mathcal{P}(a|t-\vartheta, b)d\vartheta \quad (2.8)$$

for $y_0<a<b$.

Schrödinger's integral equation[1] can be derived from (2.1) by integration over y from a to ∞. We define $f(y_0|t, a)$ as in Sec I and notice that $f(y_0|0, a)=1$ for $y_0\leqslant a$, and that

$$\mathcal{P}(y_0|t, a)=-\partial f(y_0|t, a)/\partial t. \quad (2.9)$$

Defining further

$$\phi(y_0|t, a)=\int_{-\infty}^a P(y_0|y, t)dy \quad (2.10)$$

and making use of

$$\int_{-\infty}^\infty P(y_0|y, t)dy=1, \quad (2.11)$$

we have

$$\begin{aligned}1-\phi(y_0|t, a)&=\int_0^t d\vartheta\mathcal{P}(y_0|\vartheta a)[1-\phi(a|t-\vartheta, a)]\\&=f(y_0|0, a)-f(y_0|t, a)\\&\quad-\int_0^t \mathcal{P}(y_0|\vartheta, a)\phi(a|t-\vartheta, a)d\vartheta\end{aligned}$$

or

$$f(y_0|t, a)=\phi(y_0|t, a)-\int_0^t \mathcal{P}(y_0|\vartheta, a)\phi(a|t-\vartheta, a)d\vartheta \quad (2.12)$$

or, since $\phi(a|0, a)=\frac{1}{2}$,

$$\mathcal{P}(y_0|t, a)=2\left\{-\frac{\partial}{\partial t}\phi(y_0|t, a)+\int_0^t \mathcal{P}(y_0|\vartheta, a)\frac{\partial\phi(a|t-\vartheta, a)}{\partial t}d\vartheta\right\}. \quad (2.12')$$

The integral equation (2.12) can also be solved by finding the Laplace transform. We obtain

$$f_L(y_0|\lambda, a)=\phi_L(y_0|\lambda a)-\mathcal{P}_L(y_0|\lambda a)\phi_L(a|\lambda, a) \quad (2.13)$$

or since

$$\mathcal{P}_L(y_0|\lambda, a)=1-\lambda f_L(y_0|\lambda, a) \quad (2.13')$$

we have

$$\mathcal{P}_L(y_0|\lambda, a)=[1-\lambda\phi_L(y_0|\lambda, a)]/[1-\lambda\phi_L(a|\lambda, a)] \quad (2.14)$$

and

$$f_L(y_0|\lambda, a)=\frac{\phi_L(y_0|\lambda, a)-\phi_L(a|\lambda, a)}{1-\lambda\phi_L(a|\lambda, a)}. \quad (2.15)$$

If specially the problem has a point of symmetry s defined by

$$P(s|y, t)=P(s|2s-y, t), \quad (2.16)$$

one has immediately an explicit solution for the probability density of the first passage time through s, since then

$$\phi(s|t, s)=\tfrac{1}{2} \text{ for all } t \quad (2.17)$$

and, therefore,

$$\mathcal{P}(y_0|t, s)=-2(\partial/\partial t)\int_{-\infty}^{s} P(y_0|y, t)dy. \quad (2.18)$$

The result of Wang and Uhlenbeck [reference 4, Eq. (82)] is a special case of (2.18) obtained by taking for $P(y_0|y, t)$ the Gaussian distribution defined by Eq. (36) of reference 4, for which $s=0$.

It is to be expected that the existence of the first passage time probability density $\mathcal{P}(y_0|t, a)$ imposes certain limitations on the choice of $P(y_0|y, t)$. We note that the expression $P_L(y_0|y, \lambda)/P_L(a|y, \lambda)$ must be independent of y for $y_0<a<y$.

III. THE SOLUTION WHEN $P(y_0|y, t)$ SATISFIES A FOKKER-PLANCK EQUATION. RECURSION FORMULA FOR THE MOMENTS OF $\mathcal{P}(y_0|t, a)$

If the limits of the first and second moments

$$A(z)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty} dy(y-z)P(z|y, \Delta t)$$

and

$$B(z)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty} dy(y-z)^2P(z|y, \Delta t)$$

exist and all higher moments tend to zero faster than Δt for $\Delta t\to 0$ $P(y_0|y, t)$ satisfies the two differential equations[9]

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial y}[A(y)P]+\frac{1}{2}\frac{\partial^2}{\partial y^2}[B(y)P]\equiv LP \quad (3.1)$$

and

$$\frac{\partial P}{\partial t}=A(y_0)\frac{\partial P}{\partial y_0}+\tfrac{1}{2}B(y_0)\frac{\partial^2 P}{\partial y_0^2}\equiv L_0^+P \quad (3.1')$$

with initial condition $P(y_0|y, 0)=\delta(y-y_0)$ and boundary conditions $P(y_0|\pm\infty, t)=0$ for finite y_0, and $P(\pm\infty|y, t)=0$ for finite y. The Laplace transform $P_L(y_0|y, \lambda)$, therefore, satisfies the differential equations

$$(L-\lambda)P_L=-\delta(y-y_0) \quad (3.2)$$

and

$$(L_0^+-\lambda)P_L=-\delta(y-y_0). \quad (3.2')$$

To derive the differential equation method of references 1 and 2 we construct a function $P_a(y_0|y, t)$ which satisfies Eq. (3.1) for $y<a$ with initial condition $P_a(y_0|y, 0)=\delta(y-y_0)$ and boundary conditions

$$P_a(y_0|-\infty, t)=P_a(y_0|a, t)=0,$$

by writing its Laplace transform as

$$P_{aL}(y_0|y, \lambda)=P_L(y_0|y, \lambda) - P_L(y_0|a, \lambda)P_L(a|y, \lambda)/P_L(a|a, \lambda). \quad (3.3)$$

Using Eq. (2.4) and integrating we get

$$\int_{-\infty}^{a} P_a(y_0|y, \lambda)dy=\phi_L(y_0|\lambda a)-\mathcal{P}_L(y_0|\lambda, a)\phi_L(a|\lambda, a) =f(y_0|\lambda, a) \quad (3.4)$$

according to Eq. (2.13) and further

$$-\frac{\partial}{\partial t}\int_{-\infty}^{a} P_a(y_0|y, t)dy=-\frac{\partial f(y_0|t, a)}{\partial t}=\mathcal{P}(y_0|t, a).$$

The differential equation method of references 1 and 2 thus leads to the same result as our method, if $P(y_0|y, t)$ satisfies the Fokker-Planck Eq. (3.1).

The function $P_a(y_0|y, t)dy$ is to be interpreted for $y\leqslant a$ as the probability that the random function $y(t')$, having started as $y(0)=y_0$, reaches a value between y and $y+dy$ at t without having assumed the value a at any time $t'(\geqslant 0)$ before t. The validity of this interpretation is shown by Laplace transformation of Eq. (3.3) which results in

$$P_a(y_0|y, t)=P(y_0|y, t) -\int_0^t \mathcal{P}(y_0|\vartheta, a)P(a|y, t-\vartheta)d\vartheta. \quad (3.5)$$

If $P(y_0|y, t)$ satisfies the adjoint Fokker-Planck equation (3.1′) an integral recursion formula for the moments is obtained as follows. Equation (3.1′) may be written in the form

$$\frac{\partial P(y_0|y, t)}{\partial t}=\frac{1}{2W(y_0)}\frac{\partial}{\partial y_0}\left\{B(y_0)W(y_0)\frac{\partial P(y_0|y, t)}{\partial y_0}\right\}, \quad (3.6)$$

where $W(y)$ is the stationary distribution

$$W(y)=\frac{\text{Const}}{B}\exp\left[\int^{y} 2(A/B)dy\right]. \quad (3.7)$$

[9] Equation (3.1) is derived in reference 4, Eq. (3.1′) by an obvious variant of this derivation. Both equations are special cases of Kolmogoroff's equations [A. Kolmogoroff, Math. Ann. 104, 415 (1931)].

If $B(y_0)W(y_0)\partial P(y_0|y,t)/\partial y_0 \to 0$ for $y_0\to -\infty$ we have

$$B(y_0)W(y_0)\frac{\partial P(y_0|y,t)}{\partial y_0}=\int_{-\infty}^{y_0}2W(x)dx\frac{\partial P(x|y,t)}{\partial t}. \quad (3.8)$$

We further obtain

$$P(y_0|y,t)-P(a|y,t)=\int_a^{y_0}\frac{2dz}{B(z)W(z)}\int_{-\infty}^{z}W(x)\frac{\partial P(x|y,t)}{\partial t}dx. \quad (3.9)$$

Since, for $\lambda\neq 0$

$$\int_0^\infty e^{-\lambda t}\frac{\partial P(x|y,t)}{\partial t}dt=-\delta(y-x)+\lambda P_L(x|y,\lambda) \quad (3.10)$$

we have for $y>y_0$

$$P_L(y_0|y,\lambda)-P_L(a|y,\lambda)=\lambda\int_a^{y_0}\frac{2dz}{B(z)W(z)}\int_{-\infty}^{z}W(x)P_L(x|y,\lambda)dx. \quad (3.11)$$

With the moments denoted by $t_n(y_0|a)$ and

$$\mathcal{P}_L(y_0|\lambda,a)=P_L(y_0|y,\lambda)/P_L(a|y,\lambda)=\sum_0^\infty(-\lambda)^n t_n(y_0|a)/n!, \quad (3.12)$$

one obtains

$$\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int_{y_0}^{a}\frac{2dz}{B(z)W(z)}\int_{-\infty}^{z}W(x)t_n(x|a)dx=\frac{1}{\lambda}[1-P_L(y_0|y,\lambda)/P_L(a|y,\lambda)]=-\frac{1}{\lambda}\sum_1^\infty\frac{(-\lambda)^n}{n!}t_n(y_0|a), \quad (3.13)$$

since $t_0(y_0|a)=1$. We thus have the recursion formula

$$t_n(y_0|a)=n\int_{y_0}^{a}\frac{2dz}{B(z)W(z)}\int_{-\infty}^{z}W(x)t_{n-1}(x|a)dx. \quad (3.14)$$

Specially for the average first passage time we have

$$t_1(y_0|a)=\int_{y_0}^{a}\frac{2dz}{B(z)W(z)}\int_{-\infty}^{z}W(x)dx. \quad (3.15)$$

Since $(BW)'=2AW$, this can be written as

$$t_1(y_0|a)=\int_{y_0}^{a}dz/\bar{A}(z),$$

where $\bar{A}(z)$ is defined as an average drift velocity:

$$\bar{A}(z)=\int_{-\infty}^{z}A(x)W(x)dx\Big/\int_{-\infty}^{z}W(x)dx. \quad (3.16)$$

IV. DISCUSSION OF $f(y_0|t,a)$ AS SOLUTION OF THE ADJOINT FOKKER-PLANCK EQUATION

In the preceding section we have derived the differential equation method of references 1 and 2, which uses the ordinary Fokker-Planck equation, from our integral equation. We now show that $f(y_0|t,a)$ and $\mathcal{P}(y_0|t,a)$ are solutions of the adjoint Fokker-Planck equation and, by means of a heuristic derivation, give an interpretation of this fact. We shall also show how $f(y_0|t,a)$ and $\mathcal{P}(y_0|t,a)$ are obtained in terms of a solution $v_\lambda(y_0)$ of the homogeneous equation

$$(L_0{}^+-\lambda)v_\lambda=0, \quad (4.1)$$

where $v_\lambda(y_0)$ is that solution which is regular for $y_0\to-\infty$. The solution for the Gaussian case is given as an example.

From (3.2′) and (2.10) it follows that

$$(L_0{}^+-\lambda)\phi(y_0/\lambda,a)=-1 \text{ for } y_0<a, \quad (4.2)$$

and, therefore, using (2.15)

$$(L_0{}^+-\lambda)f_L(y_0|\lambda,a)=-1 \quad (4.2')$$

so that

$$L_0{}^+f(y_0|t,a)=\partial f(y_0|t,a)/\partial t \quad (4.3)$$

with initial condition

$$f(y_0|0,a)=1, \quad y_0<a. \quad (4.4)$$

If we demand the validity of (2.1) for $y_0\leqslant a<y$ we must extend the initial condition to include $y_0=a$. From (2.15) follows the boundary condition

$$f(a|t,a)=0, \quad t>0$$

and since

$$\phi_L(y_0|\lambda,a)\to 1/\lambda \text{ for } y_0\to-\infty,$$
$$f_L(y_0|\lambda,a)\to 1/\lambda \text{ for } y_0\to-\infty$$

or

$$f(y_0|t,a)\to 1 \text{ for } y_0\to-\infty. \quad (4.5)$$

These boundary and initial condition are easily understood on the basis of the meaning of $f(y_0|t,a)$ as the probability that $y(t')\leqslant a$ for all t' in the interval $0\leqslant t'<t$ if $y(0)=y_0$.

Conversely, a function $f(y_0|t,a)$ which is a solution of (4.3) with the above boundary conditions is also a solution of (2.12). To show this we consider the problem of finding $\phi(y_0|t,a)$ as a solution of the Fokker-Planck equation with boundary condition

$$\phi(y_0|t,a)<\infty \text{ for } y_0\to-\infty,$$
$$\phi(y_0|t,a)=\phi(a|t,a) \text{ for } y_0=a \quad (4.6)$$

and initial condition

$$\phi(y_0|0,a)=1 \text{ for } y_0<a,$$

where $\phi(a|t, a)$ is considered given. Using

$$\mathcal{P}(y_0|t, a)\equiv -\partial f(y_0|t, a)/\partial t$$

as source function we can construct $\phi(y_0|t, a)$ as

$$\phi(y_0|t, a)=f(y_0|t, a)+\int_0^t \mathcal{P}(y_0|\vartheta, a)\phi(a|t-\vartheta, a)d\vartheta \tag{4.7}$$

which is identical with (2.12).

To understand the reason why $f(y_0|t, a)$ satisfies the adjoint to the Fokker-Planck equation we give a heuristic derivation of this property, by subdivision of the time interval $(0, t)$. Consider the function $f_n(y_0|t, a)$ defined as the conditional probability that if $y(0)=y_0\leqslant a$—also $y(kt/n)\leqslant a$, where $k=1, 2\cdots n$ and $t>0$. The function $f_n(y_0|t, a)$ is given by

$$f_n(y_0|t, a)=\int_{-\infty}^a P(y_0|y_1, t/n)dy_1 \times\int_{-\infty}^a P(y_1|y_2, t/n)dy_2\cdots \times\int_{-\infty}^a P(y_{n-1}|y_n, t/n)dy_n$$

$$=[\Lambda(t/n)]^n\chi(y_0) \tag{4.8}$$

where $\chi(y_n)=1$ for $y_0\leqslant a$ and Λ is the operator defined by

$$\Lambda(t/n)g(y_0)=\int_{-\infty}^a P(y_0|y, t/n)g(y)dy.$$

Replacing n by $n+1$ and t by $(1+1/n)t$ in (4.8), we have

$$f_{n+1}(y_0|(1+1/n)t, a) =[\Lambda(t/n)]^{n+1}\chi(y_0)=\Lambda(t/n)f_n(y_0|t, a). \tag{4.9}$$

Expanding we have

$$f_{n+1}(y_0|t, a)+\frac{t}{n}\frac{\partial f(y_0|t, a)}{\partial t}+\cdots =\Lambda(t/n)f_n(y_0|t, a). \tag{4.10}$$

If the sequence f_n converges we have

$$\partial f(y_0|t, a)/\partial t =\lim_{n\to\infty}\left\{\frac{n}{t}[\Lambda(t/n)-1]f(y_0|t,a)\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{n}{t}\left[\int_{-\infty}^{\infty} P(y_0|y, t/n)dyf(y|t, a)-f(y_0|t, a)\right]\right\} +\lim_{n\to\infty}\left\{\frac{n}{t}\int_a^{\infty} P(y_0|y, t/n)f(y|t, a)\right\}. \tag{4.11}$$

If the moment conditions for the derivation of the Fokker-Planck equation are fulfilled, the first term on the right-hand side becomes

$$A(y_0)\partial f/\partial y_0+B(y_0)\partial^2 f/\partial y_0^2.$$

For any fixed $y_0<a$ the second term vanishes, since

$$\int_a^\infty P(y_0|y, t/n)f(y|t, a)dy\leqslant\int_a^\infty P(y_0|y, t/n)dy \leqslant\int_a^\infty [(y-y_0)/(a-y_0)]^4P(y_0|y, t/n)dy, \leqslant\int_{-\infty}^\infty [(y-y_0)/(a-y_0)]^4P(y_0|y, t/n)dy \tag{4.12}$$

and

$$\lim_{n\to\infty}\frac{n}{t}\int_{-\infty}^\infty (y-y_0)^4P(y_0|y, t/n)dy=0. \tag{4.13}$$

For any value of $y_0<a$ the function $f(y_0|t, a)$ thus satisfies the adjoint Fokker-Planck Eq. (3.2). For $y_0=a$, $t>0$ we have $f(a|t, a)=0$; for $t=0$, $y_0<a$ we have $f(y_0|0, a)=1$ directly from (4.8). The inequality $f(y_0|t, a)\leqslant 1$ also follows from this equation.

If a solution $v_\lambda(y_0)$, regular for $y_0\to-\infty$, of the homogeneous equation

$$(L_0^{\prime +}-\lambda)v_\lambda=0 \tag{4.14}$$

is known, one obtains $f_L(y_0|\lambda, a)$ in the form

$$f_L(y_0|\lambda, a)=\lambda^{-1}\{1-v_\lambda(y_0)/v_\lambda(a)\}, \tag{4.15}$$

since this expression satisfied (4.2′) and vanishes for $y_0=a$. Using (2.13′) one then obtains

$$\mathcal{P}_L(y_0|\lambda, a)=v_\lambda(y_0)/v_\lambda(a). \tag{4.16}$$

For the one-dimensional Gauss-Markoff function[10] ($A(y_0)=-y_0$; $B/2=1$) one has, for example,

$$v_\lambda(y_0)=\exp(y_0^2/4)D_{-\lambda}(-y_0) \tag{4.17}$$

and thus

$$\mathcal{P}_L(y_0|\lambda, a)=\exp[\tfrac{1}{4}(y_0^2-a^2)] D_{-\lambda}(-y_0)/D_{-\lambda}(-a), \tag{4.18}$$

where $D_\nu(z)$ is the solution of Weber's equation defined by Whittaker and Watson [*Modern Analysis*, p. 347 ff]. This result can also be obtained from Eq. (2.4) using

$$P_L(y_0|y, \lambda)=(2\pi)^{-\frac{1}{2}}\Gamma(\lambda)\exp[\tfrac{1}{4}(y_0^2-y^2)] \cdot D_{-\lambda}(y)D_{-\lambda}(-y_0) \tag{4.19}$$

(valid for $y_0<y$). This expression follows from (3.2) and (3.2′) and can be checked by means of the series expansion for $P(y_0|y, t)$ given by Uhlenbeck and Ornstein.[11]

[10] The recurrence time problem for the corresponding random process (Ehrenfest model) has been solved by R. Bellman and T. E. Harris, Ann. Math. Statistics (in print).

[11] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).

For $y_0 \ll a \ll -1$ one expects the first passage time to approach the value $\ln(y_0/a)$. Actually one obtains $\mathcal{P}(y_0|t, a) \sim \delta(t - \ln y_0/a)$, by using the asymptotic form of the Weber function

$$D_{-\lambda}(-y_0) \sim (-y_0)^{-\lambda} \exp(-y_0^2/4) \text{ (for } y_0 \to -\infty). \quad (4.20)$$

For a and t both of the order of unity, or greater, the smallest root λ_0 of $D_{-\lambda}(-a)$ dominates the behavior of $\mathcal{P}(y_0|t, a)$, and λ_0 can be estimated by analogy with the quantum-mechanical problem of an oscillator with a reflecting wall at a.

V. DERIVATION OF THE WIENER-RICE SERIES FROM AN INTEGRAL EQUATION

Let $y(t)$ be Markoffian in y and $\dot{y}$, and be stationary and characterized by the conditional probability

$$P_2(y_0, \dot{y}_0 | y_1, \dot{y}_1; t_1 - t_0) dy_1 d\dot{y}_1 = \text{cond. prob.} \{\text{if } y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0, \text{ then } y_1 < y(t_1) < y_1 + dy_1, \dot{y}_1 < \dot{y}(t_1) < \dot{y}_1 + d\dot{y}_1\}. \quad (5.1)$$

Let further $\mathcal{P}_k(t_0, \dot{y}_0|t, \dot{y}) dt dy$ be the conditional probability that, if $y(t_0) = 0$ and $\dot{y}(t_0) = \dot{y}_0$, then y will go through zero with slope between $\dot{y}$ and $\dot{y} + d\dot{y}$ in the interval $(t, t+dt)$ and will have k other roots between t_0 and t. Because y is assumed to be two-dimensional Markoffian, there is the following recursion formula for $\mathcal{P}_k$:

$$\mathcal{P}_{k+1}(t_0, \dot{y}_0|t, \dot{y}) = \int_{t_0}^{t} d\vartheta \int_{-\infty}^{\infty} d\dot{\eta} \mathcal{P}_0(t_0, \dot{y}_0|\vartheta, \dot{\eta}) \mathcal{P}_k(\vartheta, \dot{\eta}|t, \dot{y}). \quad (5.2)$$

If we define the function $\Omega F(t_0, \dot{y}_0|t, \dot{y})$ by

$$\Omega F(t_0, \dot{y}_0|t, \dot{y}) = \int_{t_0}^{t} d\vartheta \int_{-\infty}^{\infty} d\dot{\eta} \mathcal{P}_0(t_0, \dot{y}_0|\vartheta, \dot{\eta}) F(\vartheta, \dot{\eta}|t, \dot{y}) \quad (5.3)$$

we have

$$\mathcal{P}_k(t_0, \dot{y}_0|t, \dot{y}) = \Omega^k \mathcal{P}_0(t_0, \dot{y}_0|t, \dot{y}). \quad (5.4)$$

We now define $p(t_0, \dot{y}_0|t, \dot{y}) dt d\dot{y}$ as the conditional probability that y goes through zero between t and dt, with a slope between $\dot{y}$ and $\dot{y}+d\dot{y}$ if $y(t_0)=0$, $\dot{y}(t_0) = \dot{y}_0$, without regard to other roots. We then have

$$p(t_0, \dot{y}_0|t, \dot{y}) = \sum_{k=0}^{\infty} \mathcal{P}_k(t_0, \dot{y}_0|t, \dot{y}) = \sum_{k=0}^{\infty} \Omega^k \mathcal{P}_0(t_0, \dot{y}_0|t, \dot{y}) = (1-\Omega)^{-1} \mathcal{P}_0(t_0, \dot{y}_0|t, \dot{y}) \quad (5.5)$$

or

$$\mathcal{P}_0(t_0, \dot{y}_0|t, \dot{y}) = p(t_0, \dot{y}_0|t, \dot{y}) - \int_{t_0}^{t} d\vartheta \int_{-\infty}^{\infty} d\dot{\eta} \mathcal{P}_0(t_0, \dot{y}_0|\vartheta, \dot{\eta}) p(\vartheta, \dot{\eta}|t, \dot{y}). \quad (5.6)$$

This can now be considered as an integral equation for $\mathcal{P}_0$. It expresses the fact that to obtain $\mathcal{P}_0$ one must *deduct* from $p(t_0, \dot{y}_0|t, \dot{y})$ the probability that y has gone through zero for the first time after t_0 at some time $\vartheta < t$ with any slope $\dot{\eta}$, and passes again through zero (after an arbitrary number of passages) in the time interval $(t, t+dt)$ with slope $(\dot{y}, \dot{y}+d\dot{y})$. This integral equation is thus obtainable by a direct generalization of a method applied first by Schrödinger to the first passage time problem for particles in Brownian motion. The function $p(t_0, \dot{y}_0|t, \dot{y})$ can be expressed in terms of $P_2(y_0, \dot{y}_0|y_1, \dot{y}_1; t_1 - t_0)$ since

$$p(t_0, \dot{y}_0|t, \dot{y}) dt d\dot{y} = P_2(0, \dot{y}_0|0, \dot{y}, t-t_0)|\dot{y}| dt d\dot{y}. \quad (5.7)$$

To derive the Wiener-Rice series we define

$$\Lambda F(t_0, \dot{y}_0|t, \dot{y})$$

by

$$\Lambda F(t_0, \dot{y}_0|t, \dot{y}) = \int_{t_0}^{t} d\vartheta \int_{-\infty}^{\infty} d\dot{\eta} F(t_0, \dot{y}_0|\vartheta, \dot{\eta}) p(\vartheta, \dot{\eta}|t, \dot{y})$$

and write (5.6) in the form

$$(1+\Lambda)\mathcal{P}_0(t_0, \dot{y}_0|t, \dot{y}) = p(t_0, \dot{y}_0|t, \dot{y}) \quad (5.8)$$

or

$$\mathcal{P}_0(t_0, \dot{y}_0|t, \dot{y}) = \sum_{k=0}^{\infty} (-)^k \Lambda^k p(t_0, \dot{y}_0|t, \dot{y}), \quad (5.9)$$

where

$$\Lambda^k p(t_0, \dot{y}_0|t, \dot{y}) = \int_{t_0}^{t} dt_k \int_{-\infty}^{\infty} d\dot{y}_k \int_{t_0}^{t_k} dt_{k-1} \int_{-\infty}^{\infty} d\dot{y}_{k-1} \cdots \int_{t_0}^{t_2} dt_1 \int_{-\infty}^{\infty} d\dot{y}_1 p(t_0, \dot{y}_0|t_1, \dot{y}_1) \cdots p(t_k, \dot{y}_k|t, \dot{y}). \quad (5.10)$$

We now define, with $t_0 < t_j' < t$

$$w_k(t_0, \dot{y}_0|t_1', t_2', \cdots t_k', t) = \int_{-\infty}^{\infty} d\dot{y} \int_{-\infty}^{\infty} d\dot{y}_k \cdots \int_{-\infty}^{\infty} d\dot{y}_1 p(t_0, \dot{y}_0|t_1, \dot{y}_1) \times p(t_1, \dot{y}_1|t_2, \dot{y}_2) \cdots p(t_k, \dot{y}_k|t, \dot{y}), \quad (5.11)$$

where $t_1, t_2, t_3 \cdots t_k$ is the same set of values as t_1', $t_2' \cdots t_k'$, only rearranged, such that

$$t_1 < t_2 < t_3 \cdots < t_k.$$

w_k is thus symmetric in the variables t_j', because whatever the order of the t_j' may be, they appear in order of size on the right hand side. We note that

$$w_k(t_0, \dot{y}_0|t_1', \cdots, t_k', t) dt_1' \cdots dt_k' dt$$

is the conditional joint probability that, if $y(t_0) = 0$ and

$\dot{y}(t_0)=\dot{y}_0$, then the function $y(t')$ has roots in the intervals $(t_1', t_1'+dt_1')\cdots(t_k', t_k'+dt_k')$, $(t, t+dt)$, regardless of any other roots in the interval (t_0, t). Because of the symmetry of w_k, we have

$$\int_{t_0}^{t} dt_1' \int_{t_0}^{t} dt_2' \cdots \int_{t_0}^{t} dt_k' w_k(t_0, \dot{y}_0 | t_1', t_2', \cdots t_k', t)$$

$$= k! \int_{t_0}^{t} dt_k' \int_{t_0}^{t_k'} dt_{k-1}' \cdots \int_{t_0}^{t_2'} dt_1' w_k(t_0, \dot{y}_0 | t_1', \cdots t_k', t)$$

$$= k! \int_{-\infty}^{\infty} d\dot{y} \Lambda^k p(t_0, \dot{y}_0 | t, \dot{y}). \quad (5.12)$$

Integration of (5.9) over $\dot{y}$ and use of (5.12) yields

$$\int_{-\infty}^{\infty} d\dot{y} \mathcal{P}_c(t_0, \dot{y}_0 | t, \dot{y}) = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_k w_k(t_0, \dot{y}_0 | t_1, t_2 \cdots t_k, t). \quad (5.13)$$

We now define $W_0(t_0|t)$ by the relation

$$W_0(t_0|t) = \int_{-\infty}^{\infty} d\dot{y}_0 W_2(0, \dot{y}_0) \int_{-\infty}^{\infty} d\dot{y} \mathcal{P}(t_0, \dot{y}_0 | t, \dot{y}) \quad (5.14)$$

and $\hat{w}_k(t_0 t_1, \cdots, t_k, t)$ by the relation

$$\hat{w}_k(t_0 | t_1, \cdots, t_k, t) = \int_{-\infty}^{\infty} d\dot{y}_0 W_2(0, \dot{y}_0) w_k(t_0, \dot{y}_0 | t_1, t_2, \cdots, t_k, t). \quad (5.15)$$

Then $W_0(t_0|t)dt$ is the conditional probability that, if $y(t_0)=0$, then there is a root of y in the interval $(t, t+dt)$ and no root between t_0 and t, while $\hat{w}_k(t_0|t_1, \cdots, t_k, t)dt_1\cdots dt_k dt$ is the conditional probability that, if $y(t_0)=0$, then y has roots in the intervals (t_j, t_j+dt_j) [$j=1, 2, \cdots, k$] and in $(t, t+dt)$, regardless of other roots. With these definitions we obtain from (5.13)

$$W_0(t_0|t) = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_k \times \hat{w}_k(t_0 | t_1, \cdots, t_k, t), \quad (5.16)$$

which is the Wiener-Rice series for our case.

These considerations can obviously be generalized immediately to the case of a function y which is Markoffian in $y, \dot{y}, \ddot{y}, \cdots, y^{(n)}$ by replacing $\dot{y}$ by $\{\dot{y}, \ddot{y}, \cdots, y^{(n)}\}$ and $d\dot{y}$ by the volume element in the space $\{\dot{y}, \ddot{y}, \cdots, y^{(n)}\}$, whereby, however, $|\dot{y}|$ retains its original meaning in Eq. (5.7).

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