TABLE II. R values and calculated energy displacements for ground states.

	$R_{C^{13}}$ ground	$R_{N^{13}}$ ground	γ^2 (10^{-13} erg-cm)	ΔE (one-level estimate) (kev)
$r=0.46$	−0.187	−0.212	0.57	195
$r=0.32$	−0.169	−0.178	0.96	155

at least with respect to order of magnitude, by using a one-level formula with γ^2 being estimated from the first excited state of N^{13} by $\Gamma=2k\gamma^2A^{-2}$. (Although this is not exactly applicable here, it can be used for such an order of magnitude estimation.) The results are also listed in the table. The γ^2 thus obtained is still about 0.25 of $3\hbar^2/2Ma$, which may very well be larger than the γ^2 appropriate for the ground state, so that the ΔE listed in Table II may be too large. At any rate, the displacement is in the direction to cut down the displacement of the first excited levels relative to the ground states here explained.

IV. CONCLUSIONS

It has been shown that the boundary condition postulate introduced here predicts a displacement of the first excited levels of C^{13} with respect to N^{13} in the right direction, and of sufficient magnitude to explain the experimentally known displacement. The critical dependence on the nuclear radius rules out any very definite result. It may be, of course, that there exist as yet undiscovered levels in C^{13} and N^{13} which would alter the whole analysis, and that the 3.10-Mev level of C^{13} is not really the one which corresponds to the 2.35-Mev level in N^{13}. Recent experiments on the elastic scattering of protons by C^{12} up to a proton energy of about 4 Mev (with energy resolution of order 1 kev) have failed to reveal additional N^{13} levels in the range in question here.[14]

It is a pleasure to thank Professor Wigner for suggesting this problem, and for continued advice and encouragement. I also wish to thank Professor Sherr for his interesting discussions of the experimental aspects of this problem, and Professor Breit for making available to me his tables of confluent hypergeometric functions.

[14] The author wishes to thank G. Goldhaber and R. Williamson for informing him of their results prior to publication.

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Internal Excitation and Apparent Range of Nuclear Forces in Scattering Experiments

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The effect of a speculatively considered internal excitation of nucleons on the scattering of elementary particles is examined by means of a schematic model. It is found that the relation between the integral of the square of the wave function describing the normal nucleonic state and the apparent range of force differs from that expected on the ordinary potential energy description. The effect is of the order of eight percent if one assumes the excitation to be 275 mc^2. The direction of the effect is such as to decrease the apparent range as though the meson mass were increased, somewhat as indicated by a comparison of the mass of a pi-meson with experiments on scattering. A possible necessity of corrections for the velocity dependence of nuclear forces which is considered above is briefly discussed in connection with the correlation of data on scattering with that on binding energies.

I. INTRODUCTION

IT has become customary to describe interactions between nuclear particles by means of potential energies. That such a description is of a provisory character has been well realized since the first introduction of exchange forces by Heisenberg[1] and Majorana.[2] These forces can, of course, be introduced by means of certain additive terms in the Hamiltonian without any reference to their origin. Nevertheless, Heisenberg's arguments were partly based on an expected analogy to interactions between atoms which take place in molecules. The same general feature is present in meson theories of nuclear forces.[3] According to these theories the origin of the force is not the interaction between nucleons with each other but a more primary type of process consisting in the production of mesons by nucleons. Only in special cases can the static part of the nuclear potential be split off from the relativistic part by a definite transformation. In the general Møller-Rosenfeld mixture, the potential employed in attempts to explain scattering or binding energies is only a part of a more general interaction which has not been evaluated on

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[1] W. Heisenberg, Z. Physik 77, 1 (1932).
[2] E. Majorana, Z. Physik 82, 137 (1933).
[3] H. Yukawa, Proc. Phys. Math. Soc. Japan 17, 48 (1935). C. Møller and L. Rosenfeld, Proc. Cop. 17, No. 8. G. Wentzel, *Einführung in die Quantentheorie der Wellenfelder* (Franz Deuticke, Wien, 1943).

account of internal inconsistencies of meson field theories. On the other hand the usual interpretations by means of potential energies without any other velocity dependence than that implied in the use of exchange forces lead to an apparently simple picture. The hypothesis of charge independence of nuclear forces appears to be verified rather accurately by the more recent experimental[4] and theoretical work[5] provided a long tail[6] is assumed to exist in the nuclear potential, as is expected from the meson theory of nuclear forces. Thus on the one hand one starts out with a field theoretic description of nuclear forces which is mathematically inconsistent and on the other hand success results from an application of partial results of the theory to experimental material in a rather detailed and literal manner. The situation appears to be still more paradoxical if it is remembered that the agreement of the observed mass of the pi-meson with the apparent range of nuclear force as derived from proton-proton scattering is not very good. It may be permissible, therefore, to regard the successes of the empirical approach as being due, at least in part, to an accidental inclusion in present treatments of some essential features of the correct explanation. It appears reasonable to attempt to look into other approaches to the bridging of the gap between the field theoretic formalism and the potentials used for the treatment of experiments on scattering. In view of the possibility of relating information on scattering to that on binding energies of the lighter nuclei it appears especially desirable to consider whether the features of the nuclear interaction determined by scattering are necessarily related to binding energies in the same manner as the potential energy viewpoint appears to indicate. It has been known[7] very early in the development of scattering theory that the rate of change of the phase shift with energy is essentially determined by

$$\int_0^\infty u^2 dr,$$

where u is r times the radial wave function and r is the distance between the particles. This integral determines the rate of change of the logarithmic derivative of the wave function with energy and its value is therefore one of the main data derivable from scattering experiments on the assumption of a potential energy curve explanation. It is clear, on the other hand, that such a relationship cannot hold for general velocity dependent potentials because with such general assumptions one can vary the logarithmic derivative in an arbitrary way and produce arbitrary changes in the phase shift. It appears desirable to look into this matter more quantitatively.

Since the meson mass does not agree with a literal interpretation by means of the Yukawa relation a slight modification of the view is taken here. It is supposed that there exists a state of internal excitation of one of the nucleons somewhat as in the strong coupling formulation of meson theories. In view of the many unsatisfactory features of the existing forms of strong coupling theories the internal excitation is here introduced *ad hoc* so as not to confuse the feature of such an hypothesis with details of the strong coupling theories. It appears to the writers that an internal excitation may have physical significance quite apart from the strong coupling theories and that it merits consideration also from another viewpoint. While there is practically no doubt[8] that mesons have some connection with nucleon-nucleon interactions the evidence for considering the primary process to consist in the production of virtual or real mesons is very weak. Were one to consider the meson emission as occurring at a later stage, the divergence difficulties would be reduced and possibly eliminated. For then an isolated nucleon would not have to be supposed as capable of producing mesons; this property could in fact be assigned to special nucleonic states which arise only as a consequence of nucleon-nucleon interactions. There would then be no mesic self-energy of a nucleon. In order to have such a theory in a satisfactory form, one would have to formulate it relativistically and the presence of nucleons in states of negative energy would offer a possible difficulty. The interactions responsible for the formation of internally activated states would have to be treated in such a way as not to introduce the self-energy difficulty over again through collisions with particles in negative energy states. The fact, however, that the separation of states in accordance with the sign of energy is relativistically invariant makes it possible to postulate that the property of activation is peculiar to collisions between particles having energies with the same sign. It is realized that there would be some difficulty in avoiding a divergence on account of collisions of particles in negative energy states with each other. The reduced probability of these particles occupying the same volume in phase space is of help at this point, however. Also the situation is different from that in the non-activation types of field theories because the meson field is produced only by activated nucleonic states having a smaller density than that of nucleons in normal states. No attempt is being made to develop such a theory in the present paper. That it may be possible to do so has been one of the reasons for carrying through the calculations, however.

[4] E. Melkonian, Phys. Rev. **76**, 1744 (1949). Hughes, Burgy, and Ringo, Phys. Rev. **77**, 291 (1950).

[5] J. Schwinger, Phys. Rev. **78**, 135 (1950).

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[8] Burfening, Gardner, and Lattes, Phys. Rev. **75**, 382 (1949). W. H. Barkas, Phys. Rev. **75**, 1467 (1949). S. B. Jones and R. S. White, Phys. Rev. **75**, 1468 (1949); **76**, 588 (1949). Kaplon, Peters, and Bradt, Phys. Rev. **76**, 1735 (1949). Smith, Gardner, and Bradner, Phys. Rev. **77**, 562 (1950).

The model used is highly schematic. A state of internal excitation of the composite system consisting of two nucleons is introduced. It is supposed that the chance of formation of such a state exists only when the two particles are sufficiently close to each other. No attempt is made to derive the law which determines the manner with which the transition probability to activated states depends on the internucleonic distance. For purposes of simplicity it is supposed in most of the calculations presented here that the matrix element responsible for the transition is energy independent and in most of the work this value is taken as independent of r for $r<b$ and to have the value 0 for $r>b$. It is found that the integral over the square of the radial function is no longer quite so simply related to the apparent range of force, and that there is an appreciable correction arising from this circumstance. The effect of taking the correction into account is to increase the apparent meson mass by roughly ~ 9 percent if the internal excitation energy is $\sim 275\ mc^2$ where m is the electronic mass and c is the velocity of light. Such a correction cannot be of direct quantitative significance, there being other possible modifications in the theory. For example there might be more than one state of internal excitation, the assumptions concerning the variation of the transition probability with distance may be in error etc. The fact that the meson mass corresponding to observation for proton-proton scattering ($\sim 320\ m$) agrees somewhat better with the corrected rather than the uncorrected apparent range of force has presumably only qualitative significance. The existence of the correction and its approximate magnitude may be, however, more pertinent to the physical situation. The finite extension of the region within which the interaction matrix element has a nonvanishing value is not in itself in contradiction with relativity since it would be possible to assign the law of formation for the activated state arbitrarily in the rest frame and to derive from it the behavior of the system in other frames. It is admittedly objectionable, however, to introduce this law in the arbitrary manner in which this is done below.

The general nature of the range correction considered can be seen from the fact that if the interaction could be treated by second-order perturbation theory, a velocity dependence would be brought in through the variation of the energy differences occurring in the denominators of usual formulas. The estimates of these effects made on p. 1062 of the paper by Breit, Thaxton, and Eisenbud[7] indicate that a progressive increase in depth of roughly 1.6 percent per 2.4 mc^2 change in the energy of relative motion is approximately equivalent to a shortening of the range by the factor $(21.6/16)^{\frac{1}{2}}=1.16$, which corresponds to a 16 percent effect. A nine percent effect on range should correspond, therefore, roughly to a progressive increase in depth of 0.9 percent in 2.4 mc^2. An energy difference of 2.4 mc^2 is 0.87 percent of the meson mass energy, 275 mc^2, in agreement with estimates of velocity dependence effects on range made below. The effect is thus approximately such as though it were caused by variations in the energy denominators. It appeared worth while nevertheless to make somewhat more systematic and independent calculations because the accuracy of the second order perturbation approximation is somewhat uncertain in this case.

II. GENERAL RELATIONS

In order to estimate the possible effect of the internal excitation of nucleons the following schematic treatment will be used. The state of the system will be described by

$$\Psi=\psi_0(\mathbf{r})u_0+\psi_1(\mathbf{r})u_1+\cdots, \tag{1}$$

where $u_1, u_2, \cdots$ are functions of internal coordinates while $\mathbf{r}$ is the relative displacement vector of the nucleons. The internal coordinates refer to states of mesons. These will be denoted schematically by x_{In}. The Hamiltonian contains operators acting on the $\mathbf{r}$ and x_{In}. Phenomenologically one is sure that in the part of configuration space corresponding to large distances,

$$r=|\mathbf{r}|,$$

there are well defined functions $u_j(x_{In})$ which represent internal states of excitation of the nucleons and that the corresponding ψ_j are then solutions of the free particle equations for two bodies. In regions of configuration space within which the nucleons interact the functions u_j will be taken to be the same as for the larger r. The Hamiltonian will be supposed to have the form

$$H=-\frac{\hbar^2}{2\mu}\Delta_r+H_{In}+H', \tag{1.1}$$

where $\mu=$reduced mass$=M/2$, $M=$mass of nucleon, $H_{In}=$Hamiltonian describing the formation of mesons around individual nucleons, and $H'=$remainder of the Hamiltonian which is, therefore, responsible for the specific nuclear interaction.

The operator H_{In} is such that

$$H_{In}u_j=E_ju_j, \tag{1.2}$$

where E_j is the internal excitation energy of nucleons when they are well separated. The functions u_j form a complete orthonormal set for the x_{In}. Substitution of Eq. (1.1) into Eq. (1) gives

$$\Sigma_j u_j\left[-\frac{\hbar^2}{2\mu}\Delta_r-E+E_j\right]\psi_j+\Sigma_j H'\psi_j u_j=0,$$

so that

$$-\frac{\hbar^2}{2\mu}\Delta_r\psi_i+E_i\psi_i+\Sigma_j(u_i, H'\psi_j u_j)_\tau=E\psi_i. \tag{1.3}$$

The scalar products in the last term do not contain the internal coordinates of the meson system and are functions of $\mathbf{r}$ only. The symbol τ is used in the subscript because π-mesons are presumably connected or possibly

responsible for the interaction. For a fixed $\mathbf{r}$

$$(u_i(x_{In}), H'(\mathbf{r}, x_{In})\psi_j(\mathbf{r})u_j(x_{In}))_x$$
$$=\psi_j(\mathbf{r})(u_i(x_{In}), H'(\mathbf{r}, x_{In})u_j(x_{In}))_x=H_{ij}'(\mathbf{r})\psi_j(r).$$

The last scalar product is a function of $\mathbf{r}$ and of i, j only. One has, therefore,

$$-\frac{\hbar^2}{2\mu}\Delta\psi_i+E_i\psi_i+\Sigma_jH_{ij}'(\mathbf{r})\psi_j(\mathbf{r})=E\psi_i, \quad (1.4)$$

a coupled system of equations involving interaction matrix elements H_{ij}' which themselves depend on $\mathbf{r}$ and functions $\psi_j(\mathbf{r})$ which have to be determined by solving Eq. (1.4). The subscript r in Δ will be dropped from now on.

The diagonal elements of $\|H_{ij}'\|$ contribute terms of the potential energy type. If only such terms were present one would have independent solutions for each i. This situation is analogous to the behavior of a diatomic molecule when its motion of vibration is describable by a potential energy curve. The presence of nondiagonal terms produces coupling between states of motion along different potential energy curves.

In order to simplify the treatment only two potential energy curves will be considered. The system is then described by

$$\left[-\frac{\hbar^2}{2\mu}\Delta+H'_{00}(\mathbf{r})\right]\psi_0(\mathbf{r})+H'_{01}(\mathbf{r})\psi_1(\mathbf{r})=E\psi_0(\mathbf{r}),$$
$$\left[-\frac{\hbar^2}{2\mu}\Delta+H'_{11}(\mathbf{r})\right]\psi_1(\mathbf{r})+H'_{10}(\mathbf{r})\psi_0(\mathbf{r})=(E-E_1)\psi_1(\mathbf{r}). \quad (2)$$

The internal energy is here standardized by $E_0=0$.

By employing different assumptions about H_{00}', H_{01}', H_{11}' one can obtain a variety of conditions intermediate between the extreme situation of the strict potential energy curve description and that of the interaction being caused mainly by H_{01}'. The latter condition is farthest from the usual assumptions and will, therefore, be considered first. The notation

$$H_{01}'(\mathbf{r})=H'D(\mathbf{r})$$

will be used. Here H' is a constant which is in general complex while $D(\mathbf{r})$ is a real function. Since ψ_0, ψ_1 must satisfy standard regularity conditions at $r=0$ and $r=\infty$, one may solve the second line of Eq. (2) for ψ_1 by the formula

$$\psi_1(\mathbf{r})=-(MH'^*/4\pi\hbar^2)\int D(\mathbf{r}')|\mathbf{r}-\mathbf{r}'|^{-1}$$
$$\times[\exp(-\kappa|\mathbf{r}-\mathbf{r}'|)]\psi_0(\mathbf{r}')d\mathbf{r}' \quad (2.1)$$

and the problem becomes describable by an integro-differential equation in one variable:

$$-(\hbar^2/M)\Delta\psi_0(\mathbf{r})-D(\mathbf{r})(M|H'|^2/\hbar^2)$$
$$\times\int G(\mathbf{r}, \mathbf{r}')D(\mathbf{r}')\psi_0(\mathbf{r}')d\mathbf{r}'=E\psi_0(\mathbf{r}), \quad (2.2)$$

where the Green's function

$$G(\mathbf{r}, \mathbf{r}')=\{\exp[-\kappa|\mathbf{r}-\mathbf{r}'|]\}/[4\pi|\mathbf{r}-\mathbf{r}'|] \quad (2.3)$$

with

$$\kappa=[M(E_1-E)/\hbar^2]^{\frac{1}{2}} \quad (2.4)$$

is such that

$$[\Delta-\kappa^2+\delta(\mathbf{r}-\mathbf{r}')]G(\mathbf{r}, \mathbf{r}')=0. \quad (2.5)$$

It is assumed here that κ is real which implies that $E<E_1$.

By means of the Fourier representation

$$e^{-\kappa r}/r=(1/2\pi^2)\int[e^{i\mathbf{k}\mathbf{r}}/(\kappa^2+k^2)]d\mathbf{k}$$

and the expansion

$$e^{i\mathbf{k}\mathbf{r}}=\Sigma_Li^L(2L+1)P_L(\mathbf{k}\mathbf{r}/kr)F_L(kr)/(kr),$$

where the P_L are Legendre functions and the F_L are standard expressions in Bessel functions of half-integral order, one obtains

$$G(\mathbf{r}, \mathbf{r}')=(1/2\pi^2)\Sigma_L(-)^L(2L+1)P_L(\mathbf{r}\mathbf{r}'/rr')$$
$$\cdot\int_0^\infty[F_L(kr)F_L(kr')/rr'](k^2+\kappa^2)^{-1}dk \quad (2.6)$$

which gives the expansion of G in terms of Legendre functions of the cosine of the angle between $\mathbf{r}$ and $\mathbf{r}'$. For $L=0$ the sum on the right of Eq. (2.6) contributes the spherically symmetric part

$$\langle G(\mathbf{r}, \mathbf{r}')\rangle=(e^{-\kappa r}/4\pi\kappa rr')\sinh(\kappa r'), \quad (r>r'), \quad (2.7)$$

the latter expression being readily obtainable directly from the definition of G.

Assuming for simplicity that ψ_0 is spherically symmetric and introducing

$$F=r\psi_0$$

one has

$$d^2F/dr^2+k^2F-\int_0^\infty v(r, r')F(r')dr'=0, \quad (3)$$

$$k^2=(M/\hbar^2)E, \quad (3.1)$$

with the convention that F stands for $F(r)$ unless otherwise specified. The quantity v is given by

$$v(r, r')=-A^2D(r)D(r')[\exp(-\kappa|r-r'|)$$
$$-\exp(-\kappa|r+r'|)]/(2\kappa), \quad (3.2)$$

where

$$A^2=(M/\hbar^2)^2|H'|^2. \quad (3.3)$$

For scattering at $E<E_1$ the function F has to be continued from $r=0$ to $r=\infty$. According to Eq. (2.1) the corresponding $\psi_1(\mathbf{r})$ is spherically symmetric and vanishes rapidly at $r=\infty$. With the simplifying assumptions made here the solution of the scattering problem reduces, therefore, to the determination of the logarithmic derivative of F at a value of r, $r=b$ beyond which the last term in Eq. (3) is negligible. One finds

by a repetition of the usual[9] Green's theorem proof type of calculation that

$$[\partial(\partial F/F\partial r)/\partial(k^2)]_{r=b}=-F^{-2}(b)\int_0^b F^2dr$$

$$+F^{-2}(b)\int_0^b\int_0^b F(r)[\partial v(r,r')/\partial(k^2)]F(r')drdr'. \quad (3.4)$$

The presence of the last term is characteristic of the velocity dependence of v which is caused by the dependence of κ on the energy of the system.

In order to apply this result to the scattering of protons by protons it suffices to introduce

$$u=C_0\mathfrak{F}/\sin K_0, \quad u^s=C_0\mathfrak{F}^s/\sin K_0^s,$$

where K_0 is the phase shift and the notation is as in Breit and Hatcher.[10] The function $\mathfrak{F}$ now replaces F of the charge free case while $\mathfrak{F}^s$, K_0^s are the modifications of $\mathfrak{F}$ and K_0 for the standard reference potential which is eventually made to have zero range. In terms of the function

$$f=(C_0^2/\eta)\cot K_0+q_0/\eta-2\ln\eta \quad (4)$$

of Breit, Condon, and Present[7] in the notation of Yost, Wheeler, and Breit[11] a combination of the argument which led to Eq. (3.4) with the steps contained between Breit and Hatcher's Eqs. (2.7), (5) yields

$$[\partial(f-f^s)/\partial(k^2)]_b=(\hbar^2/\mu e^2)\Big\{\int_0^\infty[(u^s)^2-u^2]dr$$

$$+\int_0^\infty\int_0^\infty u(r)[\partial v(r,r')/\partial(k^2)]u(r')drdr'\Big\}, \quad (4.1)$$

where the right side is to be evaluated for $E=0$. Since $\partial f^s/\partial(k^2)=0$ the right side of Eq. (4.1) gives the slope of the f curve plotted against E.

III. EFFECT ON APPARENT RANGE

Since $\partial f/\partial(k^2)$ is approximately proportional to the range parameter and since without velocity dependence the first (single) integral alone gives the whole effect the ratio of the double to the single integral in Eq. (4.1) gives an effect characteristic of the velocity dependence. This ratio will now be estimated. To do so the equation determining $F(r)$ which is also satisfied by $u(r)$ will first be changed into a more transparent form. This is accomplished by noting that the second of the two terms in Eq. (3.2) can be obtained from the first by changing the sign of the first term and changing the sign of one of the two quantities r or r'. This suggests extending the problem to negative r and defining

$$D(-r)=D(r), \quad F(-r)=-F(r). \quad (5)$$

[9] G. Breit and E. Wigner, Phys. Rev. **53**, 998 (1938).
[10] G. Breit and R. D. Hatcher, Phys. Rev. **78**, 110 (1950).
[11] Yost, Wheeler, and Breit, Phys. Rev. **49**, 174 (1936).

Instead of Eqs. (3), (3.2) one has then the equivalent

$$d^2F/dr^2+k^2F-\int_{-\infty}^{+\infty}v_1(r,r')F(r')dr'=0, \quad (5.1)$$

$$v_1(r,r')=-A^2D(r)D(r')[\exp(-\kappa|r-r'|)]/(2\kappa). \quad (5.2)$$

The advantage of Eq. (5.1) is that for sufficiently large κ the contributions to the integral are more obviously localized near r. Accordingly the integro-differential equation can be approximated by a differential equation. Expansion of $D(r')F(r')$ in a Taylor series around r and integration term by term give

$$d^2F/dr^2+k^2F+A^2D(r)[1/\kappa^2+d^2/\kappa^4dr^2$$
$$+d^4/\kappa^6dr^4+\cdots]D(r)F(r)=0, \quad (5.3)$$

which may be also written symbolically as

$$d^2F/dr^2+k^2F+(A^2D/\kappa^2)(1-\kappa^{-2}d^2/dr^2)^{-1}DF=0. \quad (5.4)$$

To a first approximation the interaction is such as though the potential energy were

$$-[A^2D^2(r)/\kappa^2](\hbar^2/M). \quad (5.5)$$

The correction term to the equivalent range is obtainable from

$$\int_0^\infty\int_0^\infty u(r)v(r,r';\kappa)u(r')drdr'$$
$$=-(A^2/4\kappa)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}D(r)D(r')$$
$$\times e^{-\kappa|r-r'|}u(r)u(r')drdr' \quad (5.6)$$

so that taking into account the relation $\partial/\partial(k^2)=-\partial/(2\kappa\partial\kappa)$ one has

$$\int_0^\infty\int_0^\infty u(r)[\partial v(r,r';\kappa)/\partial(k^2)]u(r')drdr'$$
$$=(A^2/8\kappa)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}D(r)D(r')$$
$$\times\{\partial[\kappa^{-1}\exp(-\kappa|r-r'|)]/\partial\kappa\}u(r)u(r')drdr'. \quad (5.7)$$

On expanding $D(r')u(r')$ in a Taylor's series and integrating over r' this expression becomes

$$-(A^2/2)\int_{-\infty}^{+\infty}D(r)u(r)[1/\kappa^4+2d^2/\kappa^6dr^2$$
$$+3d^4/\kappa^8dr^4+\cdots]D(r)u(r)dr, \quad (5.8)$$

and hence for large κ

$$\int_0^\infty\int^\infty u(r)[\partial v(r,r';\kappa)/\partial(k^2)]u(r')drdr'$$
$$\cong-(A^2/2\kappa^4)\int_{-\infty}^{+\infty}[D(r)u(r)]^2dr. \quad (5.9)$$

On the other hand, it follows from Eq. (5.3) that

$$(A^2/\kappa^2)[D(r)]^2u(r)\cong -d^2u(r)/dr^2-k^2u(r) \quad (6)$$

and hence, according to Eq. (5.9),

$$\int_0^\infty\int_0^\infty u(r)[\partial v(r,r';\kappa)/\partial(k^2)]u(r')drdr' \cong(1/2\kappa^2)\int_{-\infty}^{+\infty} u[d^2u/dr^2+k^2u]dr. \quad (6.1)$$

The two terms on the right side of Eq. (4.1) can be thus expressed in terms of single integrals involving $u(r)$ and a factor $1/\kappa^2$. Except for this factor, the quantity κ^2 enters implicitly in the present approximation. It will be noted that for $E=0$, the term k^2u on the right side of Eq. (6.1) vanishes.

An approximate evaluation of the two terms can be carried through readily for a square well in the absence of Coulomb interaction *i.e.* for proton-neutron scattering. For zero energy the wavelength inside the well will be denoted by $2\pi/K$, the radius of the well will be referred to by b and the notation $z=Kb$ will be used. For $r>b$ the function $u(r)$ varies linearly with r and can be expressed as

$$u(r)=1+r/a_1, \quad (k=0), \quad (7)$$

since the absolute value of $u(r)$ does not matter in the comparison of the two terms. The homogeneous logarithmic derivative at $r=b$ has the value

$$Y=(rdF/Fdr)_{r=b}=z\cot z \quad (7.1)$$

and one finds that

$$a_1/b=(1-Y)/Y \quad (7.2)$$

and

$$u^{-2}\int_0^b (u^s)^2dr=(a_1/3)[1+b/a_1-(1+b/a_1)^{-2}] =(b/3Y)[1-(1-Y)^3]. \quad (7.3)$$

The other two integrals on the right side of Eq. (4.1) are

$$u^{-2}\int_0^b u^2dr=(z-\sin z\cos z)/(2K\sin^2 z), \quad (7.4)$$

$$u^{-2}\int_0^\infty\int_0^\infty u(r)[\partial v(r,r';\kappa)/\partial(k^2)]u(r')drdr' \cong -K(z-\sin z\cos z)/(2\kappa^2\sin^2 z). \quad (7.5)$$

Substitution of numbers corresponding to a square well depth of the ordinary potential theory $D=11.33$ Mev, an internal excitation $E_1=275\ mc^2$, a square well width $b=e^2/mc^2$ gives $z=1.471$, $Y=0.1472$

$$\int_0^b (u^s)^2dr\Big/\int_0^b u^2dr=1.83$$

and

$$\int_0^\infty\int_0^\infty u(r)[\partial v/\partial(k^2)]u(r')drdr'\Big/\int_0^\infty[(u^s)^2-u^2]dr \cong -0.098$$

and the effect may be expected to vary approximately inversely with the internal excitation energy.

The estimate made above is subject to various objections some of which are concerned with the approximations made in Eq. (6.1). Without further calculation it is especially unsatisfactory not to have available some estimate of the effect of the change in the shape of the function $u(r)$, produced by the coupling of the function without excitation, ψ_0, to that with excitation, ψ_1. The expansion in powers of $1/\kappa^2$ is not a good one for $|b-r|/b\ll 1$ because of the discontinuity in $D(r)$ at $r=b$. For these reasons more accurate calculations have also been made by means of two independent methods.

The first method amounts to an exact solution for the special case of $D(r)$ having the value 0 when $r>b$ and a constant nonvanishing value for $r<b$. The constant value of $AD(r)$ in $0<r<b$ will be referred to as V so that

$$V=AD(r), \quad (r<b). \quad (8)$$

The coupled system of equations already dealt with under Eq. (2) assumes the form

$$\begin{aligned}(d^2/dr^2+k^2)F(r)+VG(r)&=0,\\ VF(r)+(d^2/dr^2-\kappa^2)G(r)&=0,\end{aligned} \quad (8.1)$$

where $F(r)$, $G(r)$ replace ψ_0, ψ_1. The general solution of these equations subject to $F=G=0$ at $r=0$ is

$$\begin{aligned}F&=C_1\sin(\omega_1 r)+C_2\sinh(|\omega_2|r),\\ G&=[C_1(\omega_1^2-k^2)\sin(\omega_1 r) \quad (r<b) \quad (8.2)\\ &\quad +C_2(\omega_2^2-k^2)\sinh(|\omega_2|r)]/V,\end{aligned}$$

where

$$\begin{aligned}\omega_1^2-k^2&=-\kappa_0^2/2+[(\kappa_0^2/2)^2+V^2]^{\frac{1}{2}},\\ \omega_2^2-k^2&=-\kappa_0^2/2-[(\kappa_0^2/2)^2+V^2]^{\frac{1}{2}},\end{aligned} \quad (8.3)$$

and where only the case

$$\omega_2^2<0 \quad (8.4)$$

is being considered. The quantity

$$\kappa_0^2=k^2+\kappa^2=ME_1/\hbar^2 \quad (8.5)$$

is independent of the energy E. For $r>b$ the function G is of the form const $\exp(-\kappa r)$ and the continuity of the logarithmic derivative of G at $r=b$ gives the condition

$$-\kappa=[C_1(\omega_1^2-k^2)\omega_1 c_1+C_2(\omega_2^2-k^2)|\omega_2|ch_2]/[C_1(\omega_1^2-k^2)s_1+C_2(\omega_2^2-k^2)sh_2], \quad (8.6)$$

where

$$\begin{aligned}c_1&=\cos(\omega_1 b), \quad s_1=\sin(\omega_1 b),\\ sh_2&=\sinh(|\omega_2|b), \quad ch_2=\cosh(|\omega_2|b).\end{aligned} \quad (8.7)$$

The condition just mentioned gives the ratio C_1/C_2

and determines the solution within a constant factor. A reasonably lengthy calculation yields

$$\int_0^\infty\int_0^\infty F(r)[\partial v/\partial(k^2)]F(r')drdr'$$
$$=(V^2/4\kappa)\Big\{-\frac{2\kappa C_1^2}{(\kappa^2+\omega_1^2)^2}\Big(b-\frac{\sin(2z_1)}{2\omega_1}\Big)$$
$$-\frac{4\kappa C_1C_2}{\omega_1^2+|\omega_2|^2}\Big[\frac{1}{(\kappa^2+\omega_1^2)^2}+\frac{1}{(\kappa^2+\omega_2^2)^2}\Big]$$
$$\times[|\omega_2|s_1ch_2-\omega_1c_1sh_2]$$
$$-\frac{2\kappa C_2^2}{(\kappa^2+\omega_2^2)^2}\Big(-b+\frac{\sinh(2z_2)}{2|\omega_2|}\Big)+\Xi\Big\}, \quad (8.8)$$

in the notation

$$z_1=\omega_1 b, \quad z_2=|\omega_2|b \quad (8.81)$$

and with

$$\Xi=-2\Big[C_1\frac{(\kappa-\omega_1^2\kappa^{-1})s_1+2\omega_1c_1}{(\kappa^2+\omega_1^2)^2}$$
$$+C_2\frac{(\kappa-\omega_2^2\kappa^{-1})sh_2+2|\omega_2|ch_2}{(\kappa^2+\omega_2^2)^2}\Big]$$
$$\times\Big[C_1\frac{\omega_1c_1}{\kappa^2+\omega_1^2}+C_2\frac{|\omega_2|ch_2}{\kappa^2+\omega_2^2}\Big]. \quad (8.82)$$

Similarly

$$\int_0^b F^2dr=\tfrac{1}{2}C_1^2\Big(b-\frac{\sin(2z_1)}{2\omega_1}\Big)$$
$$+2C_1C_2\frac{|\omega_2|s_1ch_2-\omega_1c_1sh_2}{\omega_1^2+|\omega_2|^2}$$
$$+\tfrac{1}{2}C_2^2\Big(-b+\frac{sh(2z_2)}{2|\omega_2|}\Big), \quad (8.9)$$

$$\int_0^b F_s^2dr=(C_1s_1+C_2sh_2)^2[(b^2/3)+a_1b(a_1+b)]/(a_1+b)^2. \quad (8.91)$$

In applying these formulas a simplification results from the fact that z_2 is a number of the order of 5 so that one may set $\tanh z_2=1$. In this approximation it is convenient to introduce the quantity

$$q=(C_2/C_1)\sinh z_2, \quad (9)$$

which has the significance of the ratio of the contribution to the wave function F at $r=b$ arising from the term in $\sinh(|\omega_2|b)$ to the amplitude of the sinusoidal part of F. One has some further simplifications arising in the case $k^2=0$ on account of the relation

$$\omega_2^2+\omega_1^2+\kappa^2=k^2=0 \quad (9.1)$$

which enable a simplification to be made in the form of the denominators in Eqs. (8.8), (8.82). It is thus found for $k^2=0$, in the limit of $\exp(-z_2)\ll 1$ that

$$\int_0^\infty\int_0^\infty F(r)[\partial v/\partial(k^2)]F(r')drdr'$$
$$\cong C_1^2b[(Vb^2)^2/(4\zeta)][-2\zeta z_2^{-4}(1-s_1c_1/z_1)$$
$$-4\zeta q(z_1^{-4}+z_2^{-4})(z_2s_1-z_1c_1)/(z_1^2+z_2^2)$$
$$-2\zeta q^2/(z_1^4z_2)+\Xi/C_1^2b^4], \quad (9.2)$$

where

$$\Xi/(C_1^2b^4)\cong-2\{[(\zeta-z_1^2\zeta^{-1})s_1+2z_1c_1]/z_2^4$$
$$+q(\zeta+z_2)^2/(\zeta z_1^4)\}[z_1c_1/z_2^2-qz_2/z_1^2], \quad (9.3)$$

and

$$\zeta=\kappa b. \quad (9.31)$$

The remaining integrals simplify as follows,

$$[1/(C_1^2b)]\int_0^b F^2dr\cong\tfrac{1}{2}(1-s_1c_1/z_1)$$
$$+2q(z_2s_1-z_1c_1)/(z_1^2+z_2^2)+q^2/(2z_2), \quad (9.4)$$

$$[1/(C_1^2b)]\int_0^b F_s^2dr$$
$$\cong(s_1+q)^2[(b^2/3)+a_1(a_1+b)]/(a_1+b)^2. \quad (9.5)$$

In order to apply these relations for preassigned Y, ζ and z_1, it is necessary to be able to determine z_2 and q from ζ and z_1. This may be done by means of

$$z_2=(z_1^2+\zeta^2)^{\frac{1}{2}}, \quad (10)$$

$$q\cong(z_1^2/z_2^2)(\zeta s_1+z_1c_1)/(\zeta+z_2), \quad (10.1)$$

$$Y\cong(z_1c_1+qz_2)/(s_1+q) \quad (10.2)$$

and

$$(Vb^2)^2=z_1^2z_2^2. \quad (10.3)$$

Equations (10), (10.1) supply all the quantities on the right side of Eq. (10.2) as functions of z_1. The left side of Eq. (10.2) is known from experiment and hence z_1 can be determined by adjusting it so that Eq. (10.2) is satisfied. An alternative procedure is to introduce

$$Y_1=z_1\cot z_1. \quad (10.4)$$

It follows from Eqs. (10), (10.1), (10.2) that

$$Y_1\cong[Y+z_1^2\zeta(Yz_2^{-2}-z_2^{-1})/(\zeta+z_2)]/$$
$$[1+z_1^2(z_2^{-1}-Yz_2^{-2})/(\zeta+z_2)] \quad (10.5)$$

which is suitable for the determination of z_1 by successive approximations. A trial value of z_1 on the right side gives a value of Y_1 which determines an improved z_1 which may again be used on the right side of Eq. (10.4).

Corresponding to $D=11.33$ Mev, $b=e^2/mc^2$, $E_1=275$ mc^2, $Y=0.1472$, is the value $\zeta=5.17$. Equation (10.5),

for the above values, is satisfied for $z_1=1.62$ and $Y_1=-0.079$. Substitution of these numbers in the above equations gives

$$\int_0^b (u^s)^2dr \Big/ \int_0^b u^2dr=1.76,$$

and

$$\int_0^\infty\int_0^\infty u(r)[\partial v/\partial(k^2)]u(r')drdr' \Big/ \int_0^\infty [(u^s)^2-u^2]dr=-0.088$$

corresponding to an 8.8 percent decrease in $f^{(1)}$ and therefore approximately to the same fractional amount in the value of the range parameter.

In the second of the two methods supplementing Eq. (6.1) the potential is adjusted so as to represent by means of u the wave function for a square well of the ordinary, potential energy curve, theory. The physical model is thus different from that just considered. A comparison of numerical results obtained by the two methods will be seen to show that the value of the correction for energy dependence is relatively insensitive to the details of the model. The representation of the square well wave function is carried out only approximately. At each stage in the process the function u is approximated by a function u_a consisting of a number of straight line segments joining each other continuously at a set of preassigned values r denoted here as r_i, $(i=1, 2, \cdots, n)$. The function u_a is continuous everywhere but the derivative du_a/dr is discontinuous at the points r_i. The possibility of employing straight line segments in the representation of u owes its origin to the fact that it suffices to make the evaluations for $E=0$. The discontinuities in du_a/dr are treated as the limiting forms of conditions approached by functions having very high curvature in the vicinity of the values $r=r_i$. Such a condition can be reproduced by making $D(r)$ have the value zero except in the vicinity of the values $r=r_i$, where $D(r)$ is made to have very large values. The limiting form of $D(r)$ leading to the desired behavior of u_a is

$$D(r)=\Sigma_i a_i\delta(r-r_i), \tag{11}$$

where δ is Dirac's δ function and the a_n are a set of adjustable parameters. In accordance with Eq. (3.2)

$$v(r, r')=-A^2\Sigma_{i,j}a_ia_j\delta(r-r_i)\delta(r'-r_j)U(r, r'), \tag{11.1}$$

where

$$U(r, r')=[\exp(-\kappa|r-r'|)-\exp(-\kappa|r+r'|)]/(2\kappa). \tag{11.2}$$

Equation (3) assumes the form

$$[d^2/dr^2+k^2]u_a(r) + A^2\Sigma_{i,j}a_ia_j\delta(r-r_i)U(r, r_j)u_a(r_j)=0. \tag{11.3}$$

According to this equation u_a is a sine function with wavelength $2\pi/k$ except at the values $r=r_i$, where

$$du_a/dr\Big|_{r_i-0}^{r_i+0}=-A^2a_i\Sigma_jU(r_i, r_j)a_ju_a(r_j). \tag{11.4}$$

For $E=0$ this equation can be used to determine the quantities a_i in such a way as to represent a preassigned set of values of the $u_a(r_i)$. In fact the $u_a(r_i)$ determine the left side of Eq. (11.4) and hence one is dealing with n simultaneous quadratic equations in the n unknowns a_i. Denoting temporarily by Σ the sum occurring in Eq. (11.4), one may regard these equations as n linear equations with Σ entering as a parameter. The solution of these equations gives the a_i as expressions proportional to $1/\Sigma$ and their substitution into the formula giving Σ as a linear expression in the a_i furnishes a value of Σ^2. The a_i are, therefore, determined to within a sign common to all of them. The correction term for velocity dependence which enters Eq. (4.1) is not affected by a change of sign of all of the a_i, however, and is, therefore, uniquely determined by the choice of $u_a(r)$. The value of A is seen to be immaterial in this procedure because the whole matter can be put entirely in terms of the products Aa_i. Explicitly

$$\int_0^\infty\int_0^\infty u(r)[\partial v(r, r'; \kappa)/\partial(k^2)]u(r')drdr' = -A^2\Sigma_{i,j}a_ia_jU(r_i, r_j)u(r_i)u(r_j) \times[1+\kappa r_>-\kappa r_<\coth\kappa r_<]/(2\kappa^2), \tag{11.5}$$

where $r_<$, $r_>$ are respectively the smaller and greater of the pair of values r_i, r_j. For $n=1$ the function u_a is uniquely determined by the value of du/dr for $r>b$ and by the requirement

$$\int_0^\infty (u_a^2-u^2)dr=0.$$

For $n>1$ these requirements alone are not sufficient to determine u_a. However u_a is chosen so as to approximate u as closely as possible for a preassigned n.

For $n=1$ calculation shows a decrease in range of about five percent. For $n=2$ one obtains a decrease in range of between about five percent and seven percent depending on the r_i chosen. These figures are of the same order of magnitude as those obtained above but are smaller.

It may be shown however that this difference is not unexpected. It is seen from Eq. (5.7) that

$$\partial[\exp(-\kappa|r-r'|)/\kappa]/\partial\kappa$$

is the desired quantity. Differentiation yields

$$-(1+\kappa[r-r'|)\exp(-\kappa|r-r'|)/\kappa^2.$$

In the applications of the method just presented, which have been carried out, the term $\kappa|r-r'|$ is not given

a full chance to appear because for the relatively small number of segments chosen the importance of contributions due to $r=r'$ is exaggerated as a consequence of the rapid decay of the exponential function containing κ. In fact for $n=1$ the term is totally lacking. Since

$$\int_0^\infty [\exp(-\kappa|r-r'|)]d|r-r'| = \int_0^\infty (\kappa|r-r'|)\exp(-\kappa|r-r'|)d|r-r'| = \kappa^{-1},$$

one would expect this term to give an effect of close to a factor of 2. The fact that the limits go from 0 only to b modifies this figure somewhat.

A rough evaluation of this effect for $n=1$ can be made making use of the fact that if the term in $\kappa|r-r'|$ were neglected the contributions to the integral representing the correction for range would have relative weights $\sin^2 z dr$ where z is the phase of the internal function. A numerical estimate shows that about half of $\int_0^b \sin^2 z dr$ is caused by contributions arising in $0<r<3b/4$. Through most of this range of values of r the value of $\exp\{-\kappa|\pm b-r|\}$ is small compared to unity and the inclusion of $\kappa|r-r'|$ in addition to 1 in the factor multiplying the exponential may be expected practically to double this contribution, since the limits of integration for r' may be replaced by $\pm\infty$. In the range $3b/4<r<b$ the contributions arising from $-b<r'<r$ may again be treated approximately as though the lower limit of integration were $-\infty$ giving rise to a doubling of a quarter of the whole effect. Contributions from $r<r'<b$ are harder to refine by this consideration and will be left out of account. The expected correction factor is thus $2(1/2)+2(1/4)=1.5$ which would lead to an effect on $f^{(1)}$ and approximately on the range parameter of $5(1.5)=7.5$ percent in reasonably good agreement with other estimates. These considerations indicate that the effect is relatively insensitive to the choice of $D(r)$, this choice having been made in one of the calculations so as to have $D(r)$ constant in $0<r<b$ and in another so as to reproduce the square well wave function.

IV. DISCUSSION

The general viewpoint of the present paper could perhaps be objected to on the grounds that in some forms of meson theories it is possible to eliminate a set of variables by a contact transformation and that the static interaction occurs then in the equations without velocity dependence. To take into account the energy of meson formation would appear to be superfluous, therefore. The method of removing field variables is not capable of dealing with the whole interaction, however, and is applicable only in special cases. This objection to the consideration of velocity dependence would appear to be especially inapplicable if the formation of mesons involves the excitation of a nucleon as a preliminary step.

It appears to be very desirable to emphasize that the calculations made in the present paper cannot be interpreted as indicating in a definite way an improved agreement between the range of nuclear force derived from proton-proton scattering and the mass of the pi-meson through the Yukawa formula. In the tentative form of theory attempted above the connection of the meson field with the shape of the nuclear potential is replaced by the introduction of the function $D(r)$ which might itself be explained perhaps as in the strong coupling forms of meson theory. Without a more complete view regarding the origin of $D(r)$ a quantitative comparison is not meaningful. It appears reasonable to claim on the other hand that if one adjusts $D(r)$ so as to reproduce the Fermi intercept expected for the Yukawa potential by means of the function called F in the present paper then the velocity dependence of the force gives rise to an apparent shortening of the range of force when the latter is inferred from scattering experiments.

The equations considered above are not equivalent to the ordinary way of describing nucleons. There is present an additional component of the wave functions for each spin direction and in terms of the component describing relative motion in the unactivated state the differential equation is not an ordinary differential equation of the second order as is clear from Eqs. (5.3), (5.4). Scattering experiments and observations on binding energies will be connected, therefore, in a somewhat different way from that expected for standard types of forces. Exact adjustments of potential well parameters to fit both sets of phenomena would not be adequate for obtaining a true picture and such phenomenologic parameters would not even be expected to be obtainable quite consistently from different data.

In view of the qualitative rather than quantitative interest of the present problem the values of the parameters employed for the nuclear potential well were only approximate and the effects for proton-proton scattering were inferred from the somewhat similar situation for the proton-neutron case.