

Boundary Conditions for the Description of Nuclear Reactions

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The present paper deals with a phenomenological description of nuclear reactions by means of boundary conditions. The several stages of the reaction (initial particles, compound nucleus, etc.) are described in their respective configuration spaces; and the state representing the whole system is given by a wave function in Fock space. The wave equations and the boundary conditions are deduced with the help of an analogy to vibration problems. The interaction leading to the nuclear reaction appears as a boundary condition between the different components of the Fock type wave function. A discussion is given of elastic scattering and of two-particle nuclear reactions, showing that the present procedure gives the same dependence of the cross sections on energy as the Breit-Wigner resonance formula and its generalizations. A sketch is given of the extension of the present ideas to multiple particle reactions. The whole treatment is nonrelativistic.

I. INTRODUCTION

OVER a wide range of energies of the incident particles, it is useful to follow the progress of a nuclear reaction in the following three stages:

- (1) the presence of the initial nucleus and the incident particle,
- (2) the formation of a single particle (compound nucleus) after the collision,
- (3) the appearance of two or more particles as a result of the disintegration of the compound system.

The possibility of contemplating these three stages of the reaction stems from the relatively long lifetime of the compound nucleus, for a large range of collision energies, which makes it fruitful to consider it as a separate entity.¹

The usual description of the nuclear reaction processes represents all three stages of the reaction in the configuration space of the nucleons which constitute the incident particle and the initial nucleus.² The complexity of the problem, particularly in the absence of a definite theory of nuclear forces, has made it useful to consider the coordinates of the center of mass and the relative position of the two initial particles as six of the coordinates of the configuration space of the nucleons, and to include the rest in the internal coordinates of the particles. By an appropriate division of this configuration space, Wigner and Eisenbud³ and others developed a method for the description of two-particle reactions which takes into account the general features of the phenomenon.

In the present paper we will develop a description of nuclear reactions which will not use the configuration space of all the nucleons present, but the configuration space of the reacting particles and of the compound particle, all of which will be considered as elementary particles.

¹ H. A. Bethe, *Elementary Nuclear Theory* (John Wiley and Sons, Inc., New York, 1947), p. 109. Also L. Eisenbud, Princeton dissertation, 1946, and E. P. Wigner, *Am. J. Phys.* **17**, 99 (1949).
² See reference 1 and H. A. Bethe, *Revs. Modern Phys.* **8**, 107 (1936).

³ E. P. Wigner and L. Eisenbud, *Phys. Rev.* **72**, 29 (1947) and E. P. Wigner, *Phys. Rev.* **70**, 15 (1946). This article also contains references to earlier literature.

The three stages of the nuclear reaction will not be represented in the same configuration space. The first stage, corresponding to the presence of the two initial particles, will be described in the 6-dimensional configuration space corresponding to them by the wave function $\psi_1(\mathbf{r}_1, \mathbf{r}_2, t)$. The second stage, in which a single particle (the compound nucleus) is present, will be represented by the wave function $\psi_2(\mathbf{r}_1, t)$ in 3-dimensional configuration space. Finally, the third stage, in which we have, let us say, n -particles, which are produced in the disintegration of the compound system, will be described in their $3n$ -dimensional configuration space by the wave function $\psi_3(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n, t)$. The reaction products will be described in a different configuration space from that of the initial particles, even if $n=2$, except when we have only elastic scattering. In this case, only the first two spaces are needed to describe the reaction.

As the state of the system that is undergoing a nuclear reaction may be in any of the three stages mentioned above, it is clear that its complete description can be achieved only by a wave function of the type introduced by Fock,⁴ in connection with states in which the number of particles is not a constant of the motion. For our problem, the appropriate wave function, which will be referred to from now on as the Fock wave function of the state, is given by the vector:

$$\Psi = \begin{pmatrix} \psi_1(\mathbf{r}_1, \mathbf{r}_2, t) \\ \psi_2(\mathbf{r}_1, t) \\ \psi_3(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n, t) \end{pmatrix}. \quad (1)$$

The scalar product of two Fock wave functions is defined by:

$$\begin{aligned} (\Psi, \Psi') = & \iint \psi_1^* \psi_1' d\mathbf{r}_1 d\mathbf{r}_2 + \int \psi_2^* \psi_2' d\mathbf{r}_1 \\ & + \int \dots \int \psi_3^* \psi_3' d\mathbf{r}_1 \dots d\mathbf{r}_n \quad (2) \end{aligned}$$

⁴ V. Fock, *Z. Physik* **75**, 622 (1932). Also L. Landau and R. Peierls, *Z. Physik* **62**, 188 (1930).

and the Fock wave function is normalized when $(\Psi, \Psi) = 1$.

The interpretation given to the components of (1) is the following: $\psi_1^*(\mathbf{r}_1, \mathbf{r}_2, t)\psi_1(\mathbf{r}_1, \mathbf{r}_2, t)d\mathbf{r}_1d\mathbf{r}_2$ represents the probability of finding, at time t , the state in the form of the two initial particles in the neighborhood of $\mathbf{r}_1, \mathbf{r}_2$, and similarly for the other components.

Our problem is now to determine the wave equations satisfied by the Fock wave function, as well as to describe the flux of probability which takes place in the course of time, from the configuration space of the initial particles to that of the particles produced in the reaction.

To guide us in this problem we consider the analogous problem of the vibration of continuous media.

II. ANALOGY WITH VIBRATION PROBLEMS

The usefulness of the analogy between vibration problems in continuous media and the above description of nuclear reactions rests on the following considerations:

- (a) the equations obeyed by the displacements in the vibration problem are similar to those (Schroedinger or Klein-Gordon) which the wave functions in configuration space satisfy.
- (b) The probability density in configuration space, which is the quadratic form $\psi\psi^*$, can be compared to the energy density in the vibration problem, which is also a quadratic form of the derivatives of the displacement. In both cases we have conservation theorems. In the vibration problem, the total energy of the vibrating medium is constant, while in configuration space, the total probability equals unity for normalized wave functions, and is independent of time.
- (c) For a state represented by a Fock wave function, such as the nuclear reaction above, there will be a flux of probability between the configuration spaces of different dimensions. A similar problem appears when we consider vibrations of continuous media of different dimensions which are in contact, and where we observe a transfer of energy between the media.

A simple example of vibrating media of different dimensions in contact is given by the two semi-infinite strings **I** and **III** of linear densities ρ_1, ρ_3 , and tension T , which are coupled at $x=0$ by the mass point **II** of mass M_2 (Fig. 1).

We have in this case one-dimensional continuous media represented by the two strings, in contact with the zero-dimensional mass point **II**. When the system is set into vibration, there is a continuous exchange of energy between the strings and the mass point.

If a disturbance is produced in **I**, all the energy of the system will be concentrated there; but, in the course of time, some of the energy is communicated to the mass point **II**, which, in turn, passes it to **III** and gives some back to **I** in the form of a reflected wave.

In the Fock space picture, something similar happens

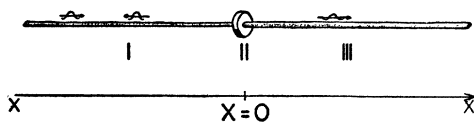


FIG. 1. One-dimensional elastic system composed of two strings (**I** and **III**) coupled by a mass (**II**).

in the description of a nuclear reaction. At the beginning, all the probability is concentrated in the configuration space of the two initial particles. Later on, there is a flow into the configuration space of the compound system, and finally, into the configuration space of the particles produced in the reaction.

A spontaneous disintegration of a particle, such as the α -decay, would correspond in the vibration picture to the energy of the system being concentrated initially in the mass point **II**. This energy would be communicated to the strings, and it can be seen easily that, because of this radiation of energy, the amplitude of the vibrations of **II** suffers an exponential decrease with time.

We designate by $u_1(x, t)$ the lateral displacement of the string **I** and by $u_3(x, t)$ the lateral displacement of string **III**, the displacement of the mass point **II** is given by $u_2(t)$. The "state" of the vibrating system is given by the vector:

$$U = \begin{bmatrix} u_1(x, t) \\ u_2(t) \\ u_3(x, t) \end{bmatrix}. \tag{3}$$

The equations of motion for the strings when $x \neq 0$ will be the ordinary wave equations:

$$\begin{aligned} \rho_1(\partial^2 u_1 / \partial t^2) - T(\partial^2 u_1 / \partial x^2) &= 0 \\ \text{and} \\ \rho_3(\partial^2 u_3 / \partial t^2) - T(\partial^2 u_3 / \partial x^2) &= 0. \end{aligned} \tag{4}$$

The equation of motion of the particle **II** will no longer be that of a free particle, and a simple way to find it is through the principle of conservation of energy. The total energy of the vibrating system is given by

$$\begin{aligned} H = \frac{1}{2} \int_{-\infty}^0 [\rho_1(\partial u_1 / \partial t)^2 + T(\partial u_1 / \partial x)^2] dx + \frac{1}{2} M_2 (du_2 / dt)^2 \\ + \frac{1}{2} \int_0^{\infty} [\rho_3(\partial u_3 / \partial t)^2 + T(\partial u_3 / \partial x)^2] dx. \end{aligned} \tag{5a}$$

Assuming that u_1, u_3 vanish appropriately for $x = \pm \infty$, and using the wave equations (4), we obtain

$$\begin{aligned} dH/dt = T(\partial u_1 / \partial x)_{x=0}(\partial u_1 / \partial t)_{x=0} \\ + M_2(d^2 u_2 / dt^2)(du_2 / dt) \\ - T(\partial u_3 / \partial x)_{x=0}(\partial u_3 / \partial t)_{x=0} = 0. \end{aligned} \tag{5b}$$

If the coupling between the wires and the mass point **II** is such that the displacement of the wires is equal to that of **II**, we have

$$u_1(0, t) = u_2(t) \quad \text{and} \quad u_2(t) = u_3(0, t). \tag{6}$$

From Eqs. (5b) and (6), the equation of motion of the particle **II** becomes

$$M_2(d^2 u_2 / dt^2) + T(\partial u_1 / \partial x)_{x=0} - T(\partial u_3 / \partial x)_{x=0} = 0. \tag{7}$$

We see that the equation of motion of the particle is modified by the presence of terms due to the action of the higher dimensional media. Furthermore, the interaction between **I** and **III** takes place through the

boundary conditions (6) and (7) at $x=0$ and in no way through the equation of motion of the strings. In fact, with the help of Eq. (6), we can eliminate u_2 from Eq. (7) and obtain direct boundary conditions between u_1 and u_3 and their derivatives at $x=0$.

The properties of the vibration problem, suggest the following procedure for the Fock wave function in configuration space:

(a) The wave functions for the initial and final particles satisfy free-particle Schrodinger wave equations everywhere except at the point of coincidence of the particles.

(b) The wave equation for the compound system ψ_2 will be modified by the wave functions of the higher dimensional configuration spaces, similarly to what happens in Eq. (7).

(c) The theorem of conservation of probability could guide us in the determination of the boundary conditions of the problem.

We will proceed to apply these considerations to several types of nuclear reaction problems, starting with elastic scattering.

III. ELASTIC SCATTERING

For the description of elastic scattering, our Fock wave function will have only two components corresponding to the initial particles and the compound system, respectively, as those are the only stages in which our state can be found. We restrict our state to zero total momentum, i.e., we choose the center-of-mass reference frame.

The Fock wave function becomes then

$$\Psi = \begin{bmatrix} \psi_1(\mathbf{r}, t) \\ \psi_2(t) \end{bmatrix}, \quad (8)$$

where $\psi_1(\mathbf{r}, t)$ is the wave function representing the two initial particles with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ being the relative position vector, and $\psi_2(t)$ is the wave function for the compound particle.

When there is no interaction between the components of the Fock wave function, they should obey the free particle Schrodinger equations:

$$-i\hbar\partial\psi_1/\partial t + (m_1 + m_2)c^2\psi_1 = (\hbar^2/2\mu)\nabla^2\psi_1, \quad (9)$$

$$-i\hbar(\partial\psi_2/\partial t) + Mc^2\psi_2 = 0, \quad (10)$$

where m_1, m_2 are the masses of the two initial particles, $\mu = m_1m_2/(m_1 + m_2)$ the reduced mass, and M the mass of the compound particle. The energy operator $E = i\hbar\partial/\partial t$ represents here, for reasons which will be apparent later, the total energy and not, as usual, the kinetic energy alone.

In case there are interactions between the configuration spaces of different dimensions of ψ_1 and ψ_2 , we can still consider that ψ_1 satisfies Eq. (9) as long as $r \neq 0$. On the other hand, the analogy with the vibration problem suggests that ψ_2 can no longer satisfy Eq. (10), and also that the theorem of conservation of probability could give a clue to the modifications we have to impose on Eq. (10).

From Eq. (2) we have

$$P(t) \equiv (\Psi, \Psi) = \int \psi_1^*(\mathbf{r}, t)\psi_1(\mathbf{r}, t)d\mathbf{r} + \psi_2^*(t)\psi_2(t), \quad (11)$$

which represents the probability of finding at time t the state (8) in any of its stages. This probability should be a constant, and equal to unity in case the Fock wave function is properly normalized.

We assume now that the wave function $\psi_1(\mathbf{r}, t)$ is regular at $\mathbf{r} = \infty$, and we surround the point $\mathbf{r} = 0$ where the wave function is singular by the sphere $r^2 = a^2$ whose radius $a \rightarrow 0$. If we evaluate dP/dt , making use of the wave equation (9) in the region outside the sphere $r^2 = a^2$, we obtain

$$\begin{aligned} \frac{\hbar}{i} \frac{dP}{dt} = \lim_{a \rightarrow 0} \int_0^\pi \int_0^{2\pi} \frac{\hbar^2}{2\mu} \left[\left(\psi_1 \frac{\partial \psi_1^*}{\partial r} - \psi_1^* \frac{\partial \psi_1}{\partial r} \right) r^2 \right]_{r=a} \\ \times \sin\theta d\theta d\varphi + \psi_2^* \left(\frac{\hbar}{i} \frac{\partial \psi_2}{\partial t} + Mc^2\psi_2 \right) \\ - \psi_2 \left(\frac{\hbar}{i} \frac{\partial \psi_2}{\partial t} + Mc^2\psi_2 \right)^* = 0. \quad (12) \end{aligned}$$

The Fock wave function that satisfies Eq. (12) must represent a state for which the conservation of probability holds. We restrict ourselves to an initial state in which the particles have zero relative angular momentum, as otherwise the law of conservation of angular momentum would require that the compound particle represented by ψ_2 should have a spin. Furthermore, if relation (12) is satisfied for the wave function

$$\Psi = \begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} \quad \text{and} \quad \Psi' = \begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix},$$

the general principles of quantum mechanics require that it should be satisfied by any linear superposition of Ψ and Ψ' , and this implies that Eq. (12) becomes the bilinear relation:

$$\begin{aligned} \frac{2\pi\hbar^2}{\mu} \left(\frac{\partial r \psi_1}{\partial r} \right)_{r=0}^* (r\psi_1')_{r=0} - (r\psi_1)_{r=0}^* \frac{2\pi\hbar^2}{\mu} \left(\frac{\partial r \psi_1'}{\partial r} \right)_{r=0} \\ + \psi_2^* \left(\frac{\hbar}{i} \frac{\partial \psi_2'}{\partial t} + Mc^2\psi_2' \right) \\ - \left(\frac{\hbar}{i} \frac{\partial \psi_2}{\partial t} + Mc^2\psi_2 \right)^* \psi_2' = 0. \quad (13) \end{aligned}$$

This bilinear form is of the familiar type:

$$x_1^*y_3 - x_3^*y_1 + x_2^*y_4 - x_4^*y_2 = 0, \quad (14)$$

which appears in many boundary value problems,⁵ and

⁵ E. L. Ince, *Ordinary Differential Equations* (Dover Publications, New York, 1944), Chapter IX.

it is known⁶ that the general linear self-adjoint equations which make Eq. (14) vanish are:

$$x_3 = C_{11}x_1 + C_{12}x_2, \quad x_4 = C_{21}x_1 + C_{22}x_2, \quad (15)$$

where $\mathbf{C} = \|C_{ij}\|$ is a constant Hermitian matrix.

From Eqs. (13) and (15) we obtain the linear equations:

$$(r\psi_1)_{r=0} = C_{11}(2\pi\hbar^2/\mu)(\partial r\psi_1/\partial r)_{r=0} + C_{12}\psi_2, \quad (16a)$$

$$\begin{aligned} (\hbar/i)(\partial\psi_2/\partial t) + Mc^2\psi_2 \\ = C_{21}(2\pi\hbar^2/\mu)(\partial r\psi_1/\partial r)_{r=0} + C_{22}\psi_2. \end{aligned} \quad (16b)$$

Equations (16a) and (16b) give us the coupling between the wave functions of the initial particles and the compound particle. The constant C_{12} is a measure of the coupling, because the interaction between ψ_1 and ψ_2 disappears if $C_{12} = C_{21}^* = 0$.

When there is no coupling, Eq. (16b) should reduce to Eq. (10); and this implies that $C_{22} = 0$. If we now assume that our state corresponds to a definite energy E , we have $-i\hbar\partial\psi_2/\partial t = -E\psi_2$ and eliminating ψ_2 between Eqs. (16a) and (16b) we obtain

$$(r\psi_1)_{r=0} = [4\pi\hbar C_{12}^2/(Mc^2 - E) + 4\pi\hbar C_{11}] \times (\hbar/2\mu)(\partial r\psi_1/\partial r)_{r=0}. \quad (17)$$

This equation is the boundary condition which characterizes the elastic scattering, and the term in the square bracket corresponds to the single term of Wigner's³ \mathbf{R} -matrix.

The energy E corresponds to the total energy of the initial pair of particles, so that:

$$E - Mc^2 = E_k - [M - (m_1 + m_2)]c^2 = E_k - E_0 \quad (18)$$

where E_k is the kinetic energy of the relative motion of the two initial particles, and E_0 is the binding energy of the compound system. Thus the \mathbf{R} -matrix, which is in this case one-dimensional, becomes

$$\mathbf{R} = [4\pi\hbar |C_{12}|^2/(E_0 - E_k)] + 4\pi\hbar C_{11}. \quad (19)$$

This result is identical to the form of the \mathbf{R} -matrix for single-level elastic scattering, and we see that in the present phenomenological description it is only a consequence of the theorem of conservation of probability.

IV. MANY-LEVEL ELASTIC SCATTERING

The formation of a single compound particle has led to the single-level Breit-Wigner formula for elastic scattering. We shall obtain the many-level formula by assuming that the collision process may give rise to any of $n-1$ compound particles. The Fock wave function then becomes

$$\Psi = \begin{bmatrix} \psi_1(\mathbf{r}, t) \\ \psi_2(t) \\ \dots \\ \psi_n(t) \end{bmatrix}. \quad (20)$$

This implies that our system can be in the state in

⁶ This will be proved in the Appendix.

which the two initial particles are present, or in the $n-1$ other states in which single particles of masses M_2, M_3, \dots, M_n are present instead.

By applying the theorem of the conservation of probability to the Fock wave function (20), we arrive at the bilinear relation:

$$\sum_{i=1}^n (x_i^* y_{n+i} - x_{n+i}^* y_i) = 0, \quad (21)$$

where for all $i = 2, 3, \dots, n$,

$$\begin{aligned} x_1 &= (2\pi\hbar^2/\mu)(\partial r\psi_1/\partial r), & x_{n+1} &= (r\psi_1)_{r=0}, \\ x_i &= \psi_i, & x_{n+i} &= -i\hbar(\partial\psi_i/\partial t) + M_i c^2 \psi_i \end{aligned}$$

and the y 's have a similar significance with respect to a second wave function.

The linear self-adjoint equations which make this bilinear form vanish, are

$$x_{n+i} = \sum_{j=1}^n C_{ij} x_j, \quad (22)$$

where $\mathbf{C} = \|C_{ij}\|$ is, as before, a constant Hermitian matrix.

There should be no coupling between the wave functions of the compound particles themselves, but only between them and the wave function for the initial two particles. This implies that only $C_{i1} = C_{1i}^* \neq 0$ for $i = 2, 3, \dots, n$ and the linear relations (22) become

$$x_{n+i} = \sum_{j=1}^n C_{1j} x_j$$

and $x_{n+i} = C_{i1} x_1$ for $i = 2, 3, \dots, n$.

If we now assume a definite energy E for the state, we can eliminate, as in the one-level case, the ψ_i ($i = 2, 3, \dots, n$) and obtain

$$(r\psi_1)_{r=0} = \left(\sum_{i=2}^n \frac{4\pi\hbar |C_{i1}|^2}{E_{0i} - E_k} + 4\pi\hbar C_{11} \right) \frac{\hbar}{2\mu} \left(\frac{\partial r\psi_1}{\partial r} \right)_{r=0}, \quad (23)$$

where $E_{0i} = [M_i - (m_1 + m_2)]c^2$ represents the binding energy of the i th compound particle.

Equation (23) is the many-level formula for elastic scattering.

V. TWO-PARTICLE NUCLEAR REACTIONS

The nuclear reaction in which the compound particle can disintegrate into either of two pairs of particles can be described by the Fock wave function:

$$\Psi = \begin{bmatrix} \psi_1(\mathbf{r}, t) \\ \psi_2(t) \\ \psi_3(\mathbf{r}, t) \end{bmatrix}. \quad (24)$$

Both wave functions ψ_1, ψ_3 , for the initial and final particles, satisfy the free particle wave equation (9) with reduced masses μ_1 and μ_3 , as long as $\mathbf{r} \neq 0$. If we apply the principle of conservation of probability, the same reasoning which we used in the case of elastic scat-

tering leads to the bilinear form

$$\sum_{i=1}^3 (x_i^* y_{3+i} - x_{3+i}^* y_i) = 0, \quad (25)$$

where x_1, x_2, x_4, x_5 are given as in Eq. (21) for $n=3$, and

$$x_3 = (2\pi\hbar^2/\mu_3)(\partial r\psi_3/\partial r)_{r=0}, \quad x_6 = (r\psi_3)_{r=0}. \quad (26)$$

The linear self-adjoint equations which guarantee the vanishing of Eq. (25) have again the form (22), with $n=3$. We shall assume that in the present problem there is no direct coupling between the initial and final wave functions, but only a coupling through the compound state. The Hermitian matrix then has the form

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{21} & 0 & C_{23} \\ 0 & C_{32} & C_{33} \end{pmatrix}. \quad (27)$$

For a definite energy E of the system, we can again eliminate ψ_2 from the linear equations (22). We thus obtain the relations

$$(r\psi_i)_{r=0} = R_{i1}(\hbar/2\mu_1)(\partial r\psi_1/\partial r)_{r=0} + R_{i3}(\hbar/2\mu_3)(\partial r\psi_3/\partial r)_{r=0} \quad (\text{with } i=1, 3), \quad (28)$$

where the matrix $\mathbf{R} = \|R_{ij}\|$, $i, j=1, 3$ has the form:

$$\mathbf{R} = \frac{4\pi\hbar}{E_0 - E_k} \begin{bmatrix} C_{12}C_{21} & C_{12}C_{23} \\ C_{32}C_{21} & C_{32}C_{23} \end{bmatrix} + 4\pi\hbar \begin{bmatrix} C_{11} & 0 \\ 0 & C_{33} \end{bmatrix}.$$

The matrix \mathbf{R} is clearly Hermitian and equivalent to the one obtained by Wigner for the single-level two-alternative reaction.

Our phenomenological description could deal with many-level, many-alternative, two-particle nuclear reactions if we assumed a Fock wave function which could take into account all of the possible stages of the nuclear reaction.

VI. β -DISINTEGRATION

As a last example we shall deal with a process in which one particle decomposes into three particles. A process of this nature is the β -decay such as the disintegration of a neutron into a proton-electron-neutrino triplet. Unfortunately, since the neutrino mass is zero, the energy distribution of the disintegration products given by the present nonrelativistic theory will not be directly comparable with the actual energy distribution. However, it will be possible to make the comparison on the basis of a remark of Uhlenbeck and Goudsmit.⁷

The vibration analog is illustrated in Fig. 2. The configuration space for the final particles is of higher dimensionality than that for the initial particle; and, in order to represent it, we need a vibrating medium of higher dimension, such as the plane **II** in Fig. 2. The original system, i.e., the neutron, is represented by the mass point **I**.

⁷ See G. E. Uhlenbeck and S. Goudsmit, article in "Pieter Zeeman" (M. Nijhoff, 'S-Gravenhage, 1935), p. 201.

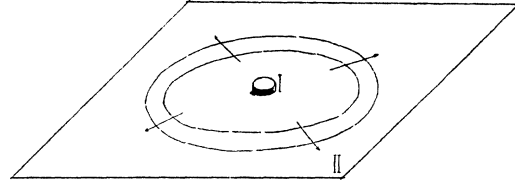


FIG. 2. Elastic system composed of a membrane (II), and a mass (I).

In Fig. 2, we would expect that the vibration set up in the region **II** will have circular symmetry; i.e., will correspond to the lowest "spherical harmonic" in two dimensions. Similarly, in the six-dimensional space of the relative coordinates of the disintegration products, the wave function will be an outgoing wave which is independent of the angular coordinates in that space,

$$\psi \approx R^{-2} H_2(kR) \exp(-iEt/\hbar). \quad (29)$$

In Eq. (29), $k = (2ME/\hbar^2)^{1/2}$ is the generalization of wave number for the six-dimensional space in which ψ is defined, H_2 is the outgoing Hankel function of order 2, and R is the analog of the radius vector of the particle; also, for six-dimensional configuration space,

$$R^2 = \sum_{s=1}^3 (m_s/M)(\mathbf{r}_s - \mathbf{r})^2, \quad (29a)$$

with \mathbf{r} the center of mass of the system,

$$\mathbf{r} = \sum m_s \mathbf{r}_s / \sum m_s, \quad M = \sum_{s=1}^3 m_s, \quad (29b)$$

The three values 1,2,3 of the index s refer to the three reaction products: proton, electron, and neutrino. Since the theory here presented is nonrelativistic, it was necessary to attribute a finite mass even to the neutrino. However, since the proton mass m_1 is very much larger than the other masses, we can use the approximation for Eq. (29a),

$$R^2 = (m_2 \mathbf{r}_2^2 + m_3 \mathbf{r}_3^2) / M. \quad (29c)$$

The same approximation is customarily made in β -theory; hence our result is made more easily comparable with the usual theory if we introduce the approximation involved in Eq. (29c).

For $kR \gg 1$, the wave function (29) becomes asymptotically

$$\psi \approx R^{-5/2} \exp(ikR - iEt/\hbar). \quad (30)$$

This asymptotic expression can be considered as a superposition of plane waves of equal intensity in all directions in six-dimensional space:

$$\psi \approx \int \exp\{ik\boldsymbol{\Omega} \cdot [(m_2/M)^{1/2} \mathbf{r}_2 + (m_3/M)^{1/2} \mathbf{r}_3] - iEt/\hbar\} d\boldsymbol{\Omega}. \quad (31)$$

In this, $\boldsymbol{\Omega}$ is a unit vector in six-dimensional space, and

$d\Omega$ indicates integration over the unit sphere in six dimensions. The integration over Ω can be replaced by integration over the six-component wave-number vector \mathbf{k} if \mathbf{k} is restricted to a very thin spherical shell $k < |\mathbf{k}| < k + \delta$. Further, this \mathbf{k} can be considered as a sum $\mathbf{k}_2' + \mathbf{k}_3'$ of two vectors, \mathbf{k}_2' and \mathbf{k}_3' , the last three components of \mathbf{k}_2' and the first three components of \mathbf{k}_3' being zero. This gives

$$\psi \approx \delta^{-1} \int d\mathbf{k}_2' d\mathbf{k}_3' \exp[i\mathbf{k}_2'(m_2/M)^{1/2} \cdot \mathbf{r}_2] \\ \times \exp[i\mathbf{k}_3'(m_3/M)^{1/2} \cdot \mathbf{r}_3] \exp(-iEt/\hbar).$$

The integration herein is to be extended over the shell $k^2 < \mathbf{k}_2'^2 + \mathbf{k}_3'^2 < (k + \delta)^2$. Introducing $\mathbf{k}_2 = (m_2/M)^{1/2} \mathbf{k}_2'$, $\mathbf{k}_3 = (m_3/M)^{1/2} \mathbf{k}_3'$, we have

$$\psi \approx \delta^{-1} \exp(-iEt/\hbar) \int \exp(i\mathbf{k}_2 \cdot \mathbf{r}_2) d\mathbf{k}_2 \\ \times \int \exp(i\mathbf{k}_3 \cdot \mathbf{r}_3) d\mathbf{k}_3, \quad (32)$$

where the integration is subject to the condition

$$k^2/M < (\mathbf{k}_2^2/m_2) + (\mathbf{k}_3^2/m_3) < (k + \delta)^2/M. \quad (32a)$$

We are interested in the probability that the energy of the electron $\epsilon_2 = \hbar^2 \mathbf{k}_2^2 / 2m_2$ is in unit neighborhood of ϵ_2 . This will be proportional to that part of the volume of the ellipsoidal shell (32a), for which

$$\epsilon_2 < \hbar^2 \mathbf{k}_2^2 / 2m_2 < \epsilon_2 + d\epsilon_2. \quad (32b)$$

This is easily calculated to be proportional to

$$[(\hbar^2 k^2 / 2M) - \epsilon_2]^{1/2} \epsilon_2^{1/2} d\epsilon_2. \quad (33)$$

Hence, the energy distribution of the β -disintegration products becomes proportional to the square roots of the energies of the two light particles which are emitted, i.e., proportional to the volume available in phase space. According to Uhlenbeck and Goudsmit,⁷ this is also the result in relativistic theory.⁸

⁸ The transition from Eq. (29) to Eq. (33) can be carried out also by means of an identity in the theory of Bessel functions.

VII. CONCLUSIONS

The principal new point raised in the present paper is the analogy between vibration problems in which bodies of different dimensionality are involved on the one hand, and quantum-mechanical problems in which the number of particles is not constant, on the other. The formalism here proposed, which amounts to a prescription of boundary conditions in Fock space,⁹ yields the same energy dependence for reaction cross sections as the ordinary theory and yields, for three particle disintegrations, an energy distribution between the product particles which is very similar to that to which the customary β -disintegration theory leads.

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APPENDIX

We are given the bilinear form:

$$\sum_{i=1}^n (x_i^* y_{n+i} - x_{n+i}^* y_i),$$

and we want to find n linear equations between the $2n$ variables x such that combined with the same linear equations for the y , they make the above bilinear form vanish identically.

Let us assume that the n linear equations are

$$x_{n+i} = \sum_{j=1}^n C_{ij} x_j.$$

Substituting this into the above bilinear form we obtain

$$\sum_{i=1}^n x_i^* (y_{n+i} - \sum_{j=1}^n C_{ji}^* y_j).$$

This will vanish if the y 's satisfy the equations

$$y_{n+i} = \sum_{j=1}^n C_{ji}^* y_j.$$

Since the x and y should satisfy the same equations, the \mathbf{C} matrix must be Hermitian. Conversely, every connection between the x_i and x_{n+i} of the above form, with an arbitrary Hermitian \mathbf{C} , guarantees the vanishing of our bilinear form.

⁹ M. Moshinsky and E. P. Wigner, Phys. Rev. **74**, 1212 (1948) and **75**, 1322 (1949), also M. Moshinsky, Princeton dissertation, 1949.