PHYSICAL REVIEW VOLUME 81, NUMBER 2 JANUARY 15, 1951

Production of Mesons by Photons on Nuclei*

MELVIN LAX
Department of Physics, Syracuse University, Syracuse, New York

AND

HERMAN FESHBACH
Department of Physics

AND

Laboratory for Nuclear Science and Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received July 19, 1950)

The total cross section, energy spectrum, and angular distribution of mesons produced by photons incident on a target nucleus have been expressed in terms of the nucleon momentum distribution within the nucleus. The theory used here is valid for photon energies greater than approximately 1.2 times threshold, and for all but the lightest nuclei. These results have been applied to carbon employing Goldberger and Chew's momentum distribution for the protons in carbon. Excellent agreement is obtained for the meson energy spectrum at 90° and the decrease in efficiency of meson production compared to free protons as targets. The change in efficiency with photon energy, and the meson spectrum at various angles can be used to determine the internal nucleon momentum distribution. Positive mesons yield the proton distribution, negative mesons the neutron distribution.

I. INTRODUCTION

WE shall estimate here the total cross section, angular distribution, and energy spectrum of π-mesons produced in a photo-nuclear collision. We assume that the matrix elements for the basic process in which the target nucleus is a free proton or neutron are known. Actually this is not quite so; the nature of the π-meson is not yet definitely settled, and matrix elements in the available theories are known only in the Born approximation.[1] We shall therefore make use of these matrix elements in a schematic way, without committing ourselves to any particular form of meson theory. We shall, for example, express the cross section for positive meson production with a nuclear target in terms of the cross section with a free proton as target. In fact we shall show that the ratio of these cross sections is *essentially independent of the matrix elements* and depends primarily on the proton momentum distribution within the struck nucleus (providing only that the matrix elements have a smooth energy and angle dependence). Thus the measurement of the meson production cross section in hydrogen and other nuclei can be used as a tool for investigating the momentum distribution within these nuclei.

The angular dependence of the nuclear cross section for photo-meson production also yields fairly direct information concerning the momentum distribution with the nucleus. If the target nucleus is hydrogen, there is no appreciable internal momentum; the energy of the meson and its angle of production are precisely related by a "Compton law." For other target nuclei, the internal momenta of the nucleons leads to a broadening of the Compton line. The shape of the Compton line, half-width, etc., are measures of the momentum distribution with the nucleus. These results, of course, are also influenced by the meson matrix elements.

Experimental results for the total cross section and the meson spectrum at 90° are available for carbon and hydrogen.[2] Instead of using these experiments to calculate the internal momentum distribution, we prefer to check our theory by making use of an approximate distribution available from deuteron pick-up reactions.[3] Both the 90° spectrum and the total cross-section ratio are found to be in excellent agreement with experiment.

The experimental result that seems most startling at first sight is "that the cross section of six bound protons in a carbon atom is only about twice as large as that of a single free proton." In other words, the efficiency $\varepsilon=\sigma/(Z\sigma_p)$ ratio of the positive meson cross section to that for Z free protons is only $\frac{1}{3}$ in carbon. This result is explained by our theory. It will be shown that the efficiency is zero at threshold, rises to unity at high energies, and has intermediate values in good agreement with the experimental result quoted.

The reason for the decrease in efficiency at low energies is that only a portion of the proton momentum distribution can participate in the reaction. The region in momentum space energetically possible will generally intersect the momentum region occupied by the protons in the nucleus. As the photon energy increases, the common volume increases so that eventually the entire proton momentum distribution is covered and the efficiency becomes unity.

The evaluation of the efficiency requires some knowledge of the nuclear states. We shall see that sufficiently far above threshold the momentum transfer goes primarily to the struck proton so that only one particle in

* Assisted in part by the Joint Program of the ONR and AEC.
[1] H. Feshbach and M. Lax, Phys. Rev. 76, 134 (1949).

[2] J. Steinberger and A. S. Bishop, Phys. Rev. 78, 494 (1950); McMillan, Peterson, and White, Science 110, 579 (1949).
[3] G. F. Chew and M. L. Goldberger, Phys. Rev. 77, 470 (1950).

the nucleus changes its state. This "single-particle" cross section is then computed using the Hartree approximation to describe the nucleus. This procedure yields results which essentially could be surmised on the basis of the physical ideas given in the preceding paragraph.

It is not actually necessary to assume that the other particles do not change state, but merely that the most important residual states have a small energy spread. The summation over all possible residual states is then equivalent to closure over (A-1) variables. The result is then identical to that obtained by the Hartree approximation except for a binding energy correction.

II. MATRIX ELEMENTS. FREE PROTON CROSS SECTION

The matrix element for positive meson production can always be written in the form

$$\langle n|T(\mathbf{x},\mathbf{p},\boldsymbol{\sigma},\boldsymbol{\tau})|p\rangle, \tag{1}$$

where $|n\rangle$ represents the final neutron state, $|p\rangle$ the initial proton state and T is an operator which is a function of the common space coordinate $\mathbf{x}$, the momentum operator $\mathbf{p}=-ih\boldsymbol{\nabla}$ and the spin and isotopic spin operators $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. This matrix element can be assumed to be exact; i.e., the result of an infinite order relativistic perturbation calculation from which divergences have been removed. It will in general depend on the photon, meson, proton, and neutron energies as well as on the spins and polarizations of the various particles. The corresponding matrix element for a nuclear target can be written:

$$(\Psi_f, \textstyle\sum_r T_r\Psi_i), \tag{2}$$

where Ψ_i and Ψ_f are the initial and final nuclear states and T_r acts on the coordinates of particle r:

$$T_r=T(\mathbf{x}_r,\mathbf{p}_r,\boldsymbol{\sigma}_r,\boldsymbol{\tau}_r). \tag{3}$$

The operator T in a relativistic calculation will in general depend on all 16 Dirac operators; and this could be included in our notation, and subsequent procedure. Since both final and initial energy states involve positive energies only, only the terms in 1, $\boldsymbol{\sigma}$, β, and $\beta\boldsymbol{\sigma}$ will survive our evaluation of the matrix element. The matrix elements of the operators β and $\beta\boldsymbol{\sigma}$ may be expanded in a power series in $(1/M)$ with terms involving $\mathbf{p}$ and $\boldsymbol{\sigma}$. Equivalently, a contact transformation to nucleon states involving only the two spinors (1 0 0 0) and (0 1 0 0) may be applied to obtain a new operator T which may then be expressed in terms of 1, $\boldsymbol{\sigma}$ and $\mathbf{p}$ only.

The dependence of T on space must be of the plane wave form $\exp[i(\boldsymbol{\nu}-\boldsymbol{\mu})\cdot\mathbf{x}]$ so that insertion of the neutron and proton wave functions in (1) leads to the usual conservation of momentum factor $\delta(\boldsymbol{\mu}+\mathbf{n}-\mathbf{p}-\boldsymbol{\nu})$. Here $\mathbf{u}$, $\mathbf{n}$, $\mathbf{p}$, $\boldsymbol{\nu}$ are the momenta of the meson, neutron, proton, and photon respectively. The dependence on $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ can at most be linear. The isotopic spin dependence must be given by $\tau^+=(\tau_x+i\tau_y)/2$ since it converts a proton into a neutron. Thus we can write

$$T=\exp[i(\boldsymbol{\nu}-\boldsymbol{\mu})\cdot\mathbf{x}]\tau^+(\mathbf{K}\cdot\boldsymbol{\sigma}+L), \tag{4}$$

where $\mathbf{K}$ and L are abbreviations for matrices $\langle n|\mathbf{K}|p\rangle$ and $\langle n|L|p\rangle$ that depend on the photon and meson momenta and polarizations in addition to the nucleon momenta. The matrix element in the negative meson case is similar: τ^+ is replaced by τ^-, and $\mathbf{K}$ and L are replaced by other functions approximately but not exactly equal to the positive meson matrix elements.[4] This difference is important in calculating the ratio of negative to positive meson production. If scalar, pseudoscalar or vector mesons are treated in the first Born approximation, and the nucleons are treated non-relativistically, $\mathbf{K}$ and L will be independent of the nucleon momenta. This assumption is unnecessary for the subsequent theory but is utilized in the numerical comparison with experiment. We also mention that to terms of order meson mass/nucleon mass $\mathbf{K}=0$ in the scalar case, and $L=0$ in the pseudoscalar case, but neither $\mathbf{K}$ nor L vanishes in the vector meson case.

The matrix elements (4) are similar to those in beta-decay problems with a combination of Fermi and Gamow-Teller selection rules. However, the phase $(\boldsymbol{\nu}-\boldsymbol{\mu})\cdot\mathbf{x}$ is much larger than in the beta-decay case, and the exponential cannot be approximated by a power series.

The matrix element in the free proton case (1) can be integrated to give momentum conservation. Take the absolute square of (1) and perform the spin sums. The free proton cross section is then:

$$\sigma_p=(2\pi)^{-2}\int(K^2+L^2)d\boldsymbol{\mu}I_p, \tag{5}$$

$$I_p=\int d\mathbf{n}\delta(\mathbf{n}+\boldsymbol{\nu}-\boldsymbol{\mu}-\mathbf{p})\delta(n_0+\mu_0-\nu_0-p_0) \tag{6}$$

in units $\hbar=c=$ meson mass $=$ unity. Here $\mu_0=(1+\mu^2)^{\frac{1}{2}}$ is the meson energy, $n_0=(M^2+n^2)^{\frac{1}{2}}$ is the neutron energy, $p_0=(M^2+p^2)^{\frac{1}{2}}$ is the proton energy and $\nu_0=\nu$ is the photon energy. In what follows we shall take the struck proton to be at rest: $p=0$, $p_0=M$.

The momentum-energy conditions expressed by (6) reduce after integrating over $\mathbf{n}$ to a single condition

$$I_p=\delta[(M^2+(\boldsymbol{\nu}-\boldsymbol{\mu})^2)^{\frac{1}{2}}+\mu_0-\nu_0-M] \tag{7}$$

equivalent to the "Compton" relation between the meson momentum and its angle θ relative to ν:

$$2\nu(\mu_0-\mu\cos\theta)=1+2M(\nu_0-\mu_0). \tag{8}$$

This relation reduces to the usual one for photons if we set $\mu\simeq\mu_0=$ degraded photon energy and omit the 1 (the meson rest mass).

The integrations over $d\boldsymbol{\mu}=2\pi d(\cos\theta)\mu\mu_0 d\mu_0$ can be performed in either order according to whether the final

[4] K. A. Brueckner and M. L. Goldberger, Phys. Rev. **76**, 1725 (1949).

result is to be expressed as a meson spectrum or as an angular distribution. For the meson spectrum we need

$$\int I_p d(\cos\theta) = (M+\nu_0-\mu_0)/\mu\nu. \quad (9)$$

For the angular distribution we need

$$\int I_p d\mu_0 = (\partial\mu_0/\partial E_f)_\theta = \mu n_0/[\mu n_0+\mu_0(\mu-\nu\cos\theta)]. \quad (10)$$

It is understood that μ is to be eliminated from (5) and (10) with the help of the Compton Law. For future reference, we note that if I_p is replaced by (6) and the integrations in (10) over $d\mathbf{n}$ and $d\mu_0$ are performed in reverse order:

$$\int I_p d\mu_0 = |\partial\mathbf{n}/\partial\mathbf{k}|_{E_f=E_i}, \quad (11)$$

where $\mathbf{k}=\mathbf{n}+\boldsymbol{\mu}-\boldsymbol{\nu}$ and the Jacobian does not take on the value unity because μ_0 depends on n_0 through energy conservation.

III. CLOSURE APPROXIMATION TO THE CROSS SECTION

Using matrix (2) the cross section for a nuclear target can be written as

$$\sigma=(2\pi)^{-2}\int d\boldsymbol{\mu}\, \textstyle\sum_f |(\Psi_f, \sum_r T_r\Psi_i)|^2\delta(E_f-E_i). \quad (12)$$

E_f and E_i are the final and initial energies of the entire system. In comparison with the free particle case (5) $\mathbf{p}+\boldsymbol{\nu}\neq\mathbf{n}+\boldsymbol{\mu}$ and the integration over $\mathbf{n}$ is now replaced by a summation over the final states of the residual nucleus. These final states are limited by energetic considerations as indicated by the δ-function in (12). To obtain a closure approximation to the cross section we relax the energy conservation sufficiently to permit the summation to be made over all final states of the nucleus. This is performed by dropping the energy of the initial and final nucleus from the argument of the δ-function in (12) and replacing the final meson energy μ_0 by some average value over the emergent meson energy distribution. Sufficiently far above threshold this average is approximately given by the "Compton" relation (8); i.e., the replacement given below is approximately valid:

$$\delta(E_f-E_i)\rightarrow\delta\{[M^2+(\boldsymbol{\nu}-\boldsymbol{\mu})^2]^{\frac{1}{2}}+\mu_0-\nu_0-M\}. \quad (13)$$

The closure approximation should be accurate when the photon energy exceeds the threshold energy by a sufficiently large amount. (At lower energies, particularly near threshold, the closure approximation should give an upper bound to the cross section, inasmuch as it includes many final states which are not energetically possible.) We can then employ closure to obtain

$$M=\textstyle\sum_f|(\Psi_f, \sum T_r\Psi_i)|^2=(\Psi_i, \sum_{r,s} T_r^\dagger T_s\Psi_i). \quad (14)$$

We now consider the diagonal terms and the non-diagonal terms in this sum separately. The diagonal terms involve just one nuclear particle at a time so that we shall call their contribution to the cross section the single particle cross section and the corresponding matrix element M_1. The non-diagonal terms involve two nucleons at a time, and are therefore called the two particle contribution; the corresponding matrix element is M_2. It is clear that the two-particle matrix element can have a contribution only inasmuch as there is some correlation in the location of nucleons in the nucleus. Correlation is of course a necessary but not a sufficient condition for $M_2\neq 0$.

Consider now the single-particle terms. These are given by

$$\langle i|\textstyle\sum_r T_r^\dagger T_r|i\rangle=\langle i|\sum_r(\mathbf{K}\cdot\boldsymbol{\sigma}_r+L)^2\tau_r^-\tau_r^+|i\rangle. \quad (15)$$

This result may be averaged over all orientations of the nucleus:

$$\langle(\mathbf{K}\cdot\boldsymbol{\sigma}_r+L)^2\rangle_{\text{Av}}=K^2+L^2. \quad (16)$$

The equivalence between an orientation average and a spin sum can also be regarded as a consequence of the principle of spectroscopic stability. Furthermore $\langle i|\sum_r \tau_r^-\tau_r^+|i\rangle=Z$, the number of protons in the nucleus. Combining Eqs. (12) to (16) we find that in the region where (13) is valid, the one-particle contribution is simply

$$\sigma_1=Z\sigma_p, \quad (17)$$

the result to be expected from Z free protons. The off-diagonal or two-particle contributions are given by:

$$M_2=\sum_{r\neq s}\langle i|\tau_r^-\tau_s^+(\mathbf{K}\cdot\boldsymbol{\sigma}_r+L)(\mathbf{K}\cdot\boldsymbol{\sigma}_s+L) \times\exp[i(\boldsymbol{\nu}-\boldsymbol{\mu})\cdot(\mathbf{x}_r-\mathbf{x}_s)]|i\rangle. \quad (18)$$

M_2 can be evaluated exactly if the pair correlation function $\rho(\mathbf{x}_r, \mathbf{x}_s)$ is known for all spin, isotopic spin states. In general, however, we do not have such detailed information concerning the nuclear state. We shall therefore make the following assumptions.

(1) There are only two independent correlation functions, one for space symmetric, and one for space antisymmetric states. In other words, the spin, isotopic spin state of a pair of particles influences their space correlation only through the Pauli principle.

(2) These correlation functions depend only on the separation $|\mathbf{x}_r-\mathbf{x}_s|$ between particles. Thus, any odd dependence on $\mathbf{x}_r-\mathbf{x}_s$ which is present in (18) will disappear on averaging.

Making use of the second assumption we can replace the exponential by an orientation average:

$$V_{rs}=\langle\exp[i(\boldsymbol{\nu}-\boldsymbol{\mu})\cdot(\mathbf{x}_r-\mathbf{x}_s)]\rangle_{\text{Av}} =\sin(|\boldsymbol{\nu}-\boldsymbol{\mu}||\mathbf{x}_r-\mathbf{x}_s|)/(|\boldsymbol{\nu}-\boldsymbol{\mu}||\mathbf{x}_r-\mathbf{x}_s|).$$

It is now permissible to average over the spin orientation of the nucleus

$$\langle(\mathbf{K}\cdot\boldsymbol{\sigma}_r+L)(\mathbf{K}\cdot\boldsymbol{\sigma}_s+L)\rangle_{\text{Av}}=\tfrac{1}{3}K^2(\boldsymbol{\sigma}_r\cdot\boldsymbol{\sigma}_s)+L^2$$

and to symmetrize on indices r and s:

$$M_2=\sum_{r\neq s}\langle i|O_{rs}V_{rs}|i\rangle,$$

$$O_{rs}=\tfrac{1}{2}(\tau_r^-\tau_s^++\tau_s^-\tau_r^+)[\tfrac{1}{3}K^2(\boldsymbol{\sigma}_r\cdot\boldsymbol{\sigma}_s)+L^2].$$

Employing methods similar to those of Wigner and Feenberg[5] we split the state $|i\rangle$ into its space symmetric and antisymmetric parts:

$$|i\rangle=|s\rangle+|a\rangle,$$
$$|s\rangle=\tfrac{1}{2}(1+P_{12})|i\rangle;\quad |a\rangle=\tfrac{1}{2}(1-P_{12})|i\rangle, \qquad (19)$$

where P_{12} is the space exchange operator

$$-P_{12}=\tfrac{1}{4}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2).$$

Thus a typical term in the sum (18) is:

$$\langle i|O_{12}V_{12}|i\rangle=\langle s|O_{12}V_{12}|s\rangle+\langle a|O_{12}V_{12}|a\rangle$$

and cross terms vanish because O_{12} commutes with P_{12}. Making use of assumption (1) and (19) we can simplify this matrix element:

$$\begin{aligned}\langle i|O_{12}V_{12}|i\rangle&=V_s\langle s|O_{12}|s\rangle+V_a\langle a|O_{12}|a\rangle\\&=\tfrac{1}{2}(V_s+V_a)\langle i|O_{12}|i\rangle\\&\quad+\tfrac{1}{2}(V_s-V_a)\langle i|O_{12}P_{12}|i\rangle.\end{aligned}$$

The spin matrix elements are evaluated by methods similar to those of Wigner:[6]

$$\begin{aligned}\langle i|\sum_{r\neq s}O_{rs}|i\rangle=&-Z(K^2+L^2)+L^2[T(T+1)-T_z(T_z+1)]\\&+\tfrac{1}{3}K^2[P^2+(P')^2+(P'')^2+4P+2P'-S(S+1)\\&-T(T+1)-3T_z-S-\langle Y_z^2\rangle]\simeq-Z(K^2+L^2),\end{aligned}$$

$$\begin{aligned}\langle i|\sum_{r\neq s}O_{rs}P_{rs}|i\rangle=&-\tfrac{1}{8}(K^2+L^2)(A^2-4T_z^2)\\&-\tfrac{1}{6}(3L^2-2K^2)(S^2-\langle Y_z^2\rangle)\simeq-\tfrac{1}{8}(K^2+L^2)A^2\end{aligned}$$

where $\langle Y_z^2\rangle$ is the mean value of the operator $(\frac{1}{2}\sum_r\sigma_{rz}\tau_{rz})^2$ in the state with quantum numbers, P, P', P'', T, T_z, S, $S_z=S$ (compare reference 5 for notation). The approximate values contain all terms of order A^2 and A. Thus the complete matrix element is given approximately by:

$$M_1+M_2\simeq(K^2+L^2)\{Z[1-\tfrac{1}{2}(V_s+V_a)]-\tfrac{1}{16}A^2(V_s-V_a)\}$$

and the ratio of the two particle to the one-particle contribution is approximately

$$M_2/M_1\simeq-\tfrac{1}{2}(V_s+V_a)-\tfrac{1}{2}(A^2/8Z)(V_s-V_a). \qquad (20)$$

The order of magnitude of V_s and V_a can be obtained from a simple model in which the two particle densities $\rho_a{}^s$ are proportional to $\exp[-(x_{12}/R)]\pm\exp[-(x_{12}/r_0)]$ respectively, where R is the nuclear radius and r_0 the range of nuclear forces. The results neglecting the fixed position of the center of mass of the nucleus are given below. The more precise calculations in which this approximation has not been made have been performed and give very similar numerical results.

$$V_a{}^s=\left\{\frac{R^3}{[1+(|\boldsymbol{\nu}-\boldsymbol{\mu}|R)^2]^2}\pm\frac{r_0{}^3}{[1+(|\boldsymbol{\nu}-\boldsymbol{\mu}|r_0)^2]^2}\right\}(R^3\pm r_0{}^3)^{-1},$$

$$\tfrac{1}{2}(V_s+V_a)\simeq[1+(|\boldsymbol{\nu}-\boldsymbol{\mu}|R)^2]^{-2},$$

$$\tfrac{1}{2}(V_s-V_a)\simeq(r_0/R)^3[1+(|\boldsymbol{\nu}-\boldsymbol{\mu}|r_0)^2]^{-2}. \qquad (21)$$

In our units $r_0\simeq1$ and $R^3\simeq Ar_0{}^3$. Except in the forward direction $|\boldsymbol{\nu}-\boldsymbol{\mu}|R\gg1$ and the two-particle contributions are small. An estimate of their effect on the integrated cross section can be obtained by averaging the V's over the angular distribution associated with free proton targets. The result is

$$\langle(V_s+V_a)/2\rangle=[(R^2-1)^2+(2\nu R)^2]^{-1}\sim[A^{4/3}+4\nu^2A^{2/3}]^{-1}$$

$$\langle(V_s-V_a)/2\rangle=(r_0/R)^3[(r_0{}^2-1)^2+(2\nu r_0)^2]^{-1}\sim(4\nu^2A)^{-1},$$

$$|M_2/M_1|\sim(A^{4/3}+4\nu^2A^{2/3})^{-1}+(16\nu^2)^{-1}.$$

Thus the one-particle contribution is an adequate description away from the threshold, in all but the lightest nuclei. How closely may threshold be approached before M_2 starts to make significant contributions? In the next section we shall make a closer estimate of M_1. Combining this with the estimate of M_2 contained herein, it is estimated that M_2 may be neglected for $\nu>1.2$ or $\nu>170$ Mev. This estimate, which is rigorous only at a considerable energy above threshold, has been checked by the calculation of the two-particle contribution for the deuteron as target nucleus.[7]

IV. THE SINGLE-PARTICLE CONTRIBUTIONS IN THE HARTREE APPROXIMATION

The closure procedure used in the preceding section is not based on any model of the nucleus, or on any assumptions concerning nuclear forces. However, it overestimates the cross section by extending the sum over final states to include some which are not energetically possible. In order to estimate the energies at which the closure approximation is valid as well as the decrease in cross section below these energies we shall introduce the Hartree approximation. We shall consider in detail the one-particle contributions, the main term in the cross section.

We shall show that the relation between the bound and the free-particle cross sections depends primarily on the momentum distribution in the nucleus, and not on a more detailed knowledge of the nuclear wave functions. This result undoubtedly has a validity greater than that of the Hartree model with which it will be obtained.

[5] E. P. Wigner and E. Feenberg, "Reports on progress in physics," Phys. Soc., London 8, 308 (1941).
[6] E. P. Wigner, Phys. Rev. 56, 519 (1939).
[7] M. Lax and H. Feshbach, unpublished.

The initial nuclear state can be represented by the anti-symmetrized product:

$$\Psi_i=(A!)^{-\frac{1}{2}}\sum \epsilon_\alpha\varphi_{\alpha1}(x_1)\varphi_{\alpha2}(x_2)\cdots\varphi_{\alpha n}(x_n), \quad (22)$$

and the final nuclear state is represented by a similar product. The one-particle contributions are obtained by using a final wave function in which only one state, say $\varphi_j(x)$ has been changed. The ejected particle may be represented approximately by a plane wave $(2\pi)^{-\frac{3}{2}}\exp(i\mathbf{n}\cdot\mathbf{x})$. In doing so we have of course neglected the interaction of the emergent neutrons and the residual nucleus. We have also assumed that the final neutron has sufficient energy to leave the nucleus. Terms involving the nucleons remaining in the nucleus are, except for small energy transfers, included in the two particle terms; an estimate for these was obtained in Sec. III.

The spin sum yields the same result as in the free case if we neglect the variation of the space dependence of φ_j with spin direction. It is easy to include this dependence, but in view of the large uncertainties in the nuclear wave function, the choice of an average seems to be more appropriate. In terms of the normalized momentum wave function

$$c_j(\mathbf{k})=(2\pi)^{-\frac{3}{2}}\int \exp(-i\mathbf{k}\cdot\mathbf{x})\varphi_j(\mathbf{x})d\mathbf{x},$$

the matrix element takes the form $c_j(\mathbf{n}+\boldsymbol{\mu}-\boldsymbol{\nu})$. Squaring and summing over the contributions of the various one-particle transitions we find that

$$M_1=\sum_j \int |c_j(\mathbf{n}+\boldsymbol{\mu}-\boldsymbol{\nu})|^2 d\mathbf{n} \quad (23)$$

takes the place of the closure expression (14). Because of the isotopic spin operator τ^+ the sum over j includes only the proton states. In effect, M_1 replaces the conservation of momentum factor $\delta(\mathbf{n}+\boldsymbol{\mu}-\boldsymbol{\nu})$ in the free particle case (6). Introducing the normalized momentum density:

$$\rho(\mathbf{k})=Z^{-1}\sum_j |c_j(\mathbf{k})|^2,$$

$$M_1=Z\int d\mathbf{n}\int d\mathbf{k}\rho(\mathbf{k})\delta(\mathbf{n}+\boldsymbol{\mu}-\boldsymbol{\nu}-\mathbf{k}),$$

the one-particle contributions to the nuclear cross section can now be written in a form directly comparable with the free particle result (5):

$$\sigma_1=Z(2\pi)^{-2}\int(K^2+L^2)d\boldsymbol{\mu}I, \quad (24)$$

$$I=\int\rho(\mathbf{k})d\mathbf{k}\int d\mathbf{n}\delta(\mathbf{n}+\boldsymbol{\mu}-\boldsymbol{\nu}-\mathbf{k})\times\delta(n_0+\mu_0-\nu_0-M+\epsilon)$$
$$=\int\rho(\mathbf{k})d\mathbf{k}\delta\{[(\boldsymbol{\nu}+\mathbf{k}-\boldsymbol{\mu})^2+M^2]^{\frac{1}{2}}-M+\mu_0-\nu_0+\epsilon]\}. \quad (25)$$

By inserting the correct (rather than Hartree) energy conservation condition in (25) we obtain the same cross section as that yielded by the partial closure method discussed in the introduction. The correct condition is $\delta(n+\mu_0+$*mass of residual nucleus*$-\nu_0-$*mass of struck nucleus*). We may therefore interpret $\epsilon=$(*mass of residual nucleus*$+M-$*mass of struck nucleus*) as approximately (*binding energy of the proton*$+$*average excitation energy of the residual nucleus*). Since the struck proton energy does not appear in the energy conservation we cannot interpret the cross section as simply that due to a distribution of free protons.

The choice $\rho(\mathbf{k})=\delta(\mathbf{k})$ corresponding to a proton at rest reduces I to its free proton value I_p in (6) if we neglect binding energy corrections (set $\epsilon=0$). In the free case, I_p vanishes unless the Compton law relating μ and θ is obeyed—i.e. at each angle θ the meson energy takes a definite value. If a distribution of momenta $\rho(\mathbf{k})$ is available, the Compton line will be broadened. The extent to which this occurs can be investigated by integration, over the directions of $\mathbf{k}$ using $\boldsymbol{\nu}-\boldsymbol{\mu}$ as a polar axis and $d\mathbf{k}=k^2dk2\pi d(\cos\chi)$. The integration over $\cos\chi$ can then be evaluated as:

$$I=2\pi\frac{M+\nu_0-\mu_0}{a}\int_{(b-a)\text{ or }(a^2-b^2)^{\frac{1}{2}}}^{a+b}\rho(k)kdk, \quad (26)$$
$$a=|\boldsymbol{\nu}-\boldsymbol{\mu}|,\quad b=[(\nu_0'-\mu_0)(2M+\nu_0'-\mu_0)]^{\frac{1}{2}},$$

where $\nu_0'=\nu_0-\epsilon$. Limits on k are set by the conservation condition given by (25) subject to $|\cos\chi|=1$. Equation (26) exhibits the direct relationship between the shape of the Compton line and the momentum distribution in the nucleus.

We have assumed that $\rho(\mathbf{k})$ is spherically symmetric. It will, in general, be large out to some momentum α and decrease rapidly thereafter. The closure approximation will therefore be valid only if both conditions

$$|\boldsymbol{\nu}-\boldsymbol{\mu}|-[2M(\nu_0'-\mu_0)]^{\frac{1}{2}}<\alpha,$$
$$|\boldsymbol{\nu}-\boldsymbol{\mu}|+[2M(\nu_0'-\mu_0)]^{\frac{1}{2}}>\alpha,$$

can be met. The first condition with α and ϵ equal to zero is the Compton condition. This condition can be met if one particle can absorb all the available momentum $|\boldsymbol{\nu}-\boldsymbol{\mu}|$. This is always possible above the free particle threshold. Above the free particle threshold this condition simply describes the width of the Compton line. The second condition $(a+b)>\alpha$ is sufficient to make the integral independent of its upper limit, since it then covers the significant region of integration. In this case, the integrated intensity of the broadened Compton line will be the same as the corresponding unbroadened line.

A quantitative investigation of the decrease in total cross section when these conditions are not fulfilled requires an integration over $d\boldsymbol{\mu}$. This integration cannot be performed unless the dependence of K and L is specified, and some form is given to $\rho(k)$. An alternative

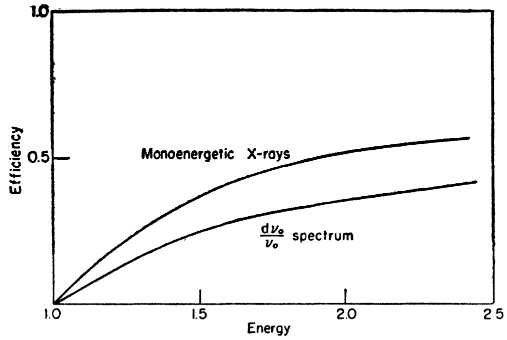

FIG. 1. Efficiency of meson production. The abscissa gives (1) photon energy for the upper curve (2) maximum photon energy for $d\nu_0/\nu_0$ bremsstrahlung spectrum.

procedure is to perform the integration over $d\boldsymbol{\mu}$ *before* that over $d\mathbf{k}$. We shall therefore introduce the following definition:

$$\sigma(\boldsymbol{\nu}+\mathbf{k},\nu_0')=(2\pi)^{-2}Z\int(K^2+L^2)d\boldsymbol{\mu}\times\delta\{[(\boldsymbol{\nu}+\mathbf{k}-\boldsymbol{\mu})^2+M^2]^{\frac{1}{2}}-M+\mu_0-\nu_0'\},\quad(27)$$

so that the total cross section can be rewritten as:

$$\sigma=\int\sigma(\boldsymbol{\nu}+\mathbf{k},\nu_0')\rho(\mathbf{k})d\mathbf{k}.\quad(28)$$

Roughly speaking, the situation is equivalent to having a distribution of photons with momenta $\boldsymbol{\nu}+\mathbf{k}$ and a fixed energy ν_0'. Only a certain portion of these "photons," however, can contribute to the cross section, since the momentum the neutron can absorb is limited by energy considerations. Since $\boldsymbol{\mu}$ has already been integrated over, the limit on $|\boldsymbol{\nu}+\mathbf{k}|$ is to be set using the most favorable value of $\boldsymbol{\mu}$. The most favorable direction for $\boldsymbol{\mu}$ is parallel to $\boldsymbol{\nu}+\mathbf{k}$ since the meson then helps to absorb the "excess" momentum. With this direction $|\boldsymbol{\nu}+\mathbf{k}|\leq\mu+b$. The most favorable value of μ_0 is then $\mu_0=1+(\nu_0-1)/M$ and we obtain the condition

$$|\boldsymbol{\nu}+\mathbf{k}|\lesssim[2M(\nu_0'-1)]^{\frac{1}{2}}=d.$$

The quantity d is just the nucleon momentum available if all of the incident photon energy is absorbed by the nucleon. When $\sigma(\boldsymbol{\nu}+\mathbf{k},\nu_0')$ varies slowly with $\mathbf{k}$, the decrease in cross section due to lack of closure is then simply:

$$\sigma_1/(Z\sigma_p)=\int_{|\boldsymbol{\nu}+\mathbf{k}|\leq d}\rho(\mathbf{k})d\mathbf{k}.\quad(29)$$

In short, the cross section is reduced because only a portion of the momentum distribution is energetically capable of contributing to the cross section. The region of integration is a sphere of radius $[2M(\nu_0-1)]^{\frac{1}{2}}$ centered at $-\boldsymbol{\nu}$. However, $\rho(\mathbf{k})$ is appreciable over the sphere $|\mathbf{k}|<p_F$ and small outside, where $p_F=(2M\epsilon_F)^{\frac{1}{2}}$ is the Fermi momentum. These two spheres will overlap fairly completely if

$$[2M(\nu_0-1)]^{\frac{1}{2}}>\nu_0+p_F.\quad(30)$$

Thus the cross section may be expected to approach the closure cross section at an energy of about 320 Mev, $\nu_0\simeq 2.3$.

Near threshold the sphere of integration shrinks to a point at $\mathbf{k}=-\boldsymbol{\nu}=-1$. $\sigma(\boldsymbol{\nu}+\mathbf{k},\nu_0)$ becomes $\sigma(0, 1)$ and the closure correction is simply

$$\sigma_1/(Z\sigma_p)\simeq\frac{4\pi}{3}[(2M)^{\frac{1}{2}}(1+(1/2M))]^3\rho(-1)\times\frac{\sigma(0,1)}{\sigma(1,1)}(\nu_0-1)^{\frac{3}{2}}.\quad(31)$$

Since the free proton cross section behaves like $(\nu_0-1)^{\frac{1}{2}}$ the nuclear cross section varies as $(\nu_0-1)^2$. We note that the threshold is actually 1 (aside from a small binding energy correction). The reason for this is that protons of momentum $\mathbf{k}=-1$ are available to absorb the photon impact so that no recoil energy need be absorbed.

A previous analysis of the nuclear cross section near threshold[8] led to a behavior of $(\nu_0-1)^{5/2}$; i.e., an additional factor of $(\nu_0-1)^{\frac{1}{2}}$. This calculation was based, however, on the Fermi statistical model of the nucleus. On this model $\rho(\mathbf{k})$ vanishes abruptly for $k>p_F$. Thus $\int\rho(\mathbf{k})d\mathbf{k}$ is the intersection volume of two spheres. And the latter will vary as $(\nu_0-\nu_t)^2$ in the neighborhood of the threshold energy ν_t at which the spheres begin to overlap.

It should be emphasized that energy dependence given by (31) may be modified if $\sigma(0, 1)$ should vanish. For the deuteron as target nucleus,[7] if $\mathbf{K}=0$, the Pauli exclusion principle yields $\sigma(0, 1)=0$, so that the energy dependence of the cross section becomes $(\nu_0-1)^3$.

V. COMPARISON WITH EXPERIMENT

At present the only available data[2] involve carbon as the target nucleus. The incident beam is the bremsstrahlung spectrum from the Berkeley synchrotron. The measurements include the meson energy distribution at 90° with respect to the incident photon beam, and the efficiency of production. This latter is (1/3); the energy distribution at 90° is given in Fig. 3.

Comparison with the theory of Sec. IV involves a knowledge of the normalized momentum density, which fortunately has been evaluated for similar momentum transfers by Goldberger and Chew.[3] These authors give

$$\rho(\mathbf{k})=\alpha_p/\pi^2(\alpha_p^2+k^2)^2.\quad(32)$$

This expression is not expected to be valid for the large values of k for one would expect that $\rho(\mathbf{k})$ decreases more rapidly than indicated by (32).

We now introduce (32) into (29) to obtain the efficiency of positive meson production, and into (24) and

[8] D. ter Haar, Science **108**, 57 (1948).

(26) for the energy distribution at 90°. We neglect binding energy effects. These results must then be integrated over the brehmsstrahlung spectrum to obtain values comparable with experiment. Strictly speaking, such a calculation gives only the one-particle contribution to the cross section. The two-particle terms, which give the coherence effects, are small except for ν_0 near threshold. However, the cross section will generally increase with energy more rapidly than the photon-spectrum decreases, so that the integration over the photon-spectrum will emphasize the larger values of ν_0. Hence the error involved in including only the single-particle contribution should be small.

Let us consider the efficiency of positive meson production. From (29) and (32)

$$\varepsilon=\frac{\sigma_1}{Z\sigma_p}=\frac{1}{\pi}\left[\tan^{-1}\frac{d-\nu_0}{\alpha_p}+\tan^{-1}\frac{d+\nu_0}{\alpha_p}\right]+\frac{\alpha_p}{2\pi\nu_0}\ln\frac{\alpha_p^2+(d-\nu_0)^2}{\alpha_p^2+(d+\nu_0)^2}. \quad (33)$$

A plot of ε as a function of ν_0 is given in Fig. 1. The relatively slow rate of increase for large ν_0 is directly attributable to the long tail of distribution (32) and is suspect.

If the incident photon spectrum is $f(\nu_0)d\nu_0$ the average efficiency is

$$\bar{\varepsilon}=\left(\int\sigma_1 f(\nu_0)d\nu_0\right)\Big/\left(Z\int\sigma_p f(\nu_0)d\nu_0\right). \quad (34)$$

In this paper, we have approximated the spectrum by $f(\nu_0)=1/\nu_0$ and have employed the σ_p calculated in an earlier publication.[1] The consequent $\bar{\varepsilon}$ is also plotted in Fig. 1. The value at $\nu_0=2.4$ is 0.4, which is 20 percent higher than the estimated experimental value, but well within the experimental error. It is quite clear that an accurate determination of the excitation curve for either monochromatic x-rays or for a spectrum will yield information about the nucleon momentum distribution.

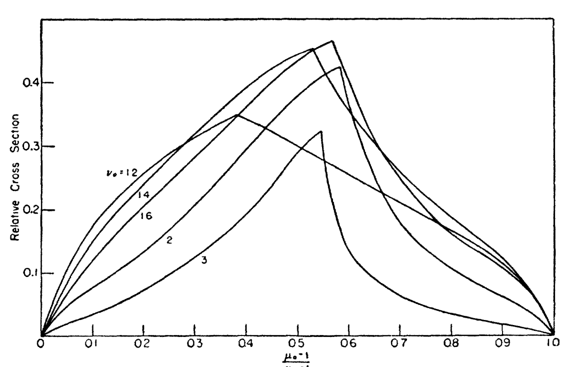

FIG. 2. Meson energy spectra at 90° with respect to photon beam for monoenergetic x-rays incident on carbon. The abscissa is the ratio of the meson kinetic energy to the excess of photon energy over threshold.

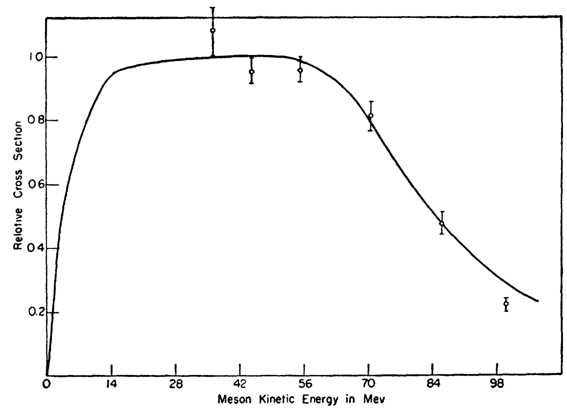

FIG. 3. Meson energy spectrum at 90° with respect to photon beam for a $d\nu_0/\nu_0$ bremsstrahlung spectrum incident on carbon. The experimental points are taken from Steinberger and Bishop.

The energy distribution at 90° is obtained by inserting momentum distribution (32) into (24) and (26). The integral I is given by

$$I=\frac{\alpha_p}{\pi}\frac{M+\nu_0-\mu_0}{a}\left[\frac{1}{\alpha_p^2+k_l^2}-\frac{1}{\alpha_p^2+(a+b)^2}\right], \quad (35)$$

where

$$k_l^2=\begin{cases}a^2-b^2 & a\geqslant b\\ (b-a)^2 & a\leqslant b\end{cases}.$$

I must then be multiplied by the statistical factors $\mu\mu_0$ and by the energy dependence of K^2+L^2. This must at least be $1/(\mu_0\nu_0)$ from the normalization of the meson and photon wave functions and indeed is closely that for pseudoscalar mesons though not precisely so. Hence the spectrum is given by

$$g(\mu_0)d\mu_0\sim[(\mu_0^2-1)^{\frac{1}{2}}I(\mu_0)/\nu_0]d\mu_0. \quad (36)$$

The function $g(\mu_0)$ is plotted in Fig. 2 for some values of ν_0 ranging from $\nu_0=1.2$ to $\nu_0=3$. The salient features are (1) the discontinuity in slope at $a=b$, i.e., at the meson energy obtained from Compton law (8); (2) the relatively rapid decrease of g with μ_0 for μ_0 greater than the value at the discontinuity. For these meson energies only those protons whose momenta are in a direction opposite to the direction of the incident photon can contribute.

In Fig. 3 the energy distribution average over a $d\nu_0/\nu_0$ spectrum is given, together with experimental values. The scale for the ordinate for the theoretical curve has been chosen as unity at a meson kinetic energy of 0.3. Comparison of the predicted absolute magnitude with experiment has already been made above. The agreement of the predicted and experimental energy distributions is remarkably close. Because of the many approximations and the experimental errors such agreement must be regarded as fortuitous. However, the agreement of prediction both with efficiency and the energy distribution indicates that the qualitative, and to some extent, the quantitative features of the simple theory employed here are correct; that one may hope to employ the produced mesons as a tool for investigation of nuclear structure.

VI. CONCLUSIONS

The cross section for the production of mesons by photons incident upon a nucleus has been shown to be directly related to the nucleon momentum distribution ρ. It was first necessary to demonstrate that sufficiently far above threshold the correlation effects are small except possibly in the forward direction and for the lightest nuclei. It was estimated that below $\nu \sim 1.2$, correlation effects would be an appreciable part of the cross section. These estimates were based on the closure approximation (Sec. II). Once this result was established, the efficiency of meson production was calculated, as well as the meson energy distribution at a given angle. In both of these, the assumption is made that the recoil nucleon can be described by a plane wave. From the efficiency it is possible to estimate the energy at which the closure approximation is valid and the efficiency is essentially one. In the absence of a tail to the momentum energy distribution, this photon energy is 320 Mev. Since the tail may still make an appreciable contribution, the efficiency will actually be one at some energy well above 320 Mev depending upon the shape of the tail. In calculating the efficiency it was assumed that the matrix elements varied smoothly with photon momentum; i.e., with the space phase of the photon wave. The dependence of the meson energy distribution on the momentum distribution ρ is given in (24, 26). This meson energy distribution has its maximum for each angle at the meson energy given by the Compton law for a free nucleon. The distribution is asymmetric, falling off more rapidly at higher energies. The width of the energy distribution is roughly determined by the nucleon momentum at which ρ starts to decrease rapidly. Finally the cross section at threshold was estimated. It was assumed that the photon energy dependence of the correlation contributions will be more rapid than the single particle term. The single particle contribution behaves as $(\nu-\nu_t)^2$ at threshold ν_t.

These calculations have been applied to carbon employing Goldberger and Chew's momentum distribution. Astonishingly good agreement with experiment is obtained for both the efficiency of meson production and the energy distribution of mesons produced at 90° with respect to the incident beam. This agreement, while partly accidental does indicate that at least a semi-quantitative understanding of the production is possible. More important, it strongly suggests that meson production may be employed as a tool for the investigation of nuclear structure, particularly of the momentum distribution inside the nucleus. *Indeed by observing the energy distribution at various angles, and by observing the efficiency as a function of photon energy, it should be possible to describe this momentum distribution very closely even without reference to any other experimental information.*

The interpretation at higher photon energies than are presently available would be somewhat simpler for here many of our assumptions become strictly valid. However, it would be necessary to have more complete information on the fundamental matrix element. This may be obtained empirically by employing hydrogen as a target nucleus. In the present paper the dependence of the matrix elements on the nucleon momenta have not been used in the calculations. Equation (24), however, is correct if K and L are replaced by matrices $\langle n|K|k\rangle$ and $\langle n|L|k\rangle$. If these matrices have a smooth energy dependence, for both negative and positive mesons, the negative positive cross-section ratio will be practically the same as for free nucleon targets. Some difference may be expected, however, in heavier nuclei where the neutron and proton momentum distributions are not equal.

We have also neglected the meson-nucleon interaction on the meson as it passes through the nucleus after its production. This should be unimportant for the lighter nuclei but may be significant for nuclei of large radius. Indeed if a nucleon momentum distribution should be known from other considerations, experiments with heavy nuclei might yield information on the meson-nucleon interaction.

Calculations[7] have been made for H^2 as a target nucleus where the correlation effects are important. These will be reported shortly. Calculations[9] are in progress in which the momenta dependent terms are included for both H^2 and heavier nuclei permitting then a calculation of the ratio of positive to negative meson production.

One point may be emphasized in closing. The energy distribution of mesons at 90° as calculated is sensitive to the meson energy dependence of the matrix elements and this nice check that we obtain is valid only for pseudoscalar mesons. However, this is not conclusive evidence that the observed mesons are pseudoscalar. Agreement might be possible for other mesons if the momentum distribution in carbon were modified.

We are indebted to Dr. Geoffrey Chew for some cogent comments.

[9] F. Villars, M. L. Goldberger, and H. Feshbach, private communication.