On the Quantization of Angular Momenta in Heavy Nuclei

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The individual particle model of nuclear structure fails to account for the observed large nuclear quadrupole moments. It is possible, however, to allow for the existence of the quadrupole moments, and still retain the essential features of the individual particle model, by assuming the average field in which the nucleons move to deviate from spherical symmetry. The assumptions underlying such an asymmetric nuclear model are discussed; this model implies, in particular, a quantization of angular momenta in analogy with molecular structure. The asymmetric model appears to account better than the extreme single particle model for empirical data regarding nuclear magnetic moments.

I. INTRODUCTION

THE individual particle model, which describes the stationary state of a nucleus in terms of the motion of the individual nucleons in an average nuclear field, has accounted successfully for a large number of nuclear properties.[1] In the simplest form of this model the nucleons are assumed to move in a field of spherical symmetry and the quantization of angular momenta is similar as in atomic structures.

This extreme model meets with the difficulty, however, that nuclei are found to have very large electric quadrupole moments. Even in the case of nuclei with one particle more or one particle less than the number required for closed shells, the quadrupole moments are, in general, several times larger than can reasonably be attributed to a single particle.[2]

This circumstance has a direct bearing on the problem of the quantization of angular momenta in the nucleus, quite apart from what other modifications of the model may be necessary. In fact, while the extreme single particle model, in the case of odd nuclei, assumes the total angular momentum of the nucleus to be possessed by the single odd particle, the magnitude of the quadrupole moments, in certain cases, demands that at least 20 or 30 nucleons somehow share the nuclear angular momentum.

Little is known regarding the coupling of angular momenta in heavy nuclei, and we may have to do with complicated coupling schemes. However, it is possible to allow for the existence of the large quadrupole moments, and still retain many essential features of the individual particle model by assuming that the average nuclear field in which the nucleons move deviates from spherical symmetry.

This model involves certain assumptions regarding the dynamical properties of the nucleus and has a number of direct implications regarding nuclear moments which form the subject of the present paper.

A model of this type has recently been considered independently by J. Rainwater,[3] who found it possible to account for the order of magnitude of the quadrupole moments by estimating the nuclear deformation produced by the centrifugal pressure of the odd particle. The following considerations, however, are largely independent of the origin of the nuclear deformations responsible for the large quadrupole moments.

II. THE ASYMMETRIC NUCLEAR MODEL

The model of the asymmetric nucleus suggests an analogy with molecular structure, in which the electrons move in average force fields which deviate from spherical symmetry. As is well known, the possibility of a simplified treatment of molecular states is based on the fact that the masses of the nuclei are very large compared with the mass of the electron. Consequently the motion of the nuclei, their vibration and rotation, only adiabatically influences the electronic motion, whose frequencies are very large compared to those characteristic of the nuclear motion.

The model of a nucleon moving in an average field of a deformed nucleus assumes, in a similar manner, that the shape and orientation of the nucleus is approximately fixed over time intervals comparable with the periods of the single particle motion. In contrast with the molecular case, there are here no heavy particles to provide the necessary rigidity of the structure. However, nuclear matter appears to have some of the properties of coherent matter which makes it capable of types of motion for which the effective mass is large as compared with the mass of a single nucleon.

Thus it is generally assumed that nuclei may vibrate under the influence of surface tension, and since such oscillations involve the whole nuclear mass, the shape of the nucleus will change very slowly compared with the single particle motion. Moreover, the magnitude of the quadrupole moments implies that in general a large number of nucleons are involved in the deformation of the nucleus, and therefore suggests that the frequencies associated with the rotation of the deformed nucleus

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[1] M. G. Mayer, Phys. Rev. 78, 16, 22 (1950). These articles contain references to previous literature regarding the individual particle model. E. Feenberg, Phys. Rev. 77, 771 (1950).

[2] Townes, Foley, and Low, Phys. Rev. 76, 1415 (1949).

[3] J. Rainwater, Phys. Rev. 79, 432 (1950). I am indebted to Dr. Rainwater for informing me of his results prior to publication.

may be considered to be small compared with single particle frequencies.

It may be noted that the nucleus is not assumed to rotate like a rigid structure; this assumption would imply the existence of very closely spaced rotational energy levels in heavy nuclei, and such levels have not been observed. It seems quite possible that a rotation of the nucleus involves the motion of only a fraction of the nuclear particles, due to an incomplete rigidity of the nuclear structure, or due to exchange effects of the type considered by Teller and Wheeler.[4] The effective moment of inertia to be associated with the rotation may thus be considerably smaller than that of the whole nucleus. The rotational level spacing, although small compared with that of a single nucleon, may therefore still be compatible with the absence of very low lying nuclear excited states.

In the analysis of nuclear stationary states we may now first consider the motion of a nucleon in the approximately fixed field produced by the core of the nucleus. In a non-central field the angular momentum of the particle will not be a constant, but for cylindrical symmetry will precess around the symmetry axis of the nucleus. If this precession takes place rapidly compared with nuclear rotation (a point to be considered in more detail in Sec. III), the component of the nucleon angular momentum along the symmetry axis will be a constant of the motion. In analogy with the case of diatomic molecules, we designate this quantity by Ω which may take the values j, $j-1$, $\cdots$, $\frac{1}{2}$, where j is the magnitude of the single particle angular momentum.

Next, we must consider the motion of the nuclear core, its vibration and rotation. For our purpose the rotation is of particular significance. Even in the ground state, the nuclear axis will not stay fixed in space, but will undergo a precession around the total nuclear angular momentum vector. This motion, which is like that of a symmetrical top, may be characterized by the quantum numbers I, M, and K, representing, respectively, the magnitude of the total angular momentum, its projection on a fixed axis in space, and its projection on the nuclear symmetry axis. For the ground state, it is to be expected that $I=K=\Omega$ in which case the nuclear rotation energy is smallest.

The value of Ω corresponding to the lowest single particle state will depend on the couplings involved; but at any rate, for nuclei with one particle more or one particle less than required for a closed shell, the value $\Omega=j$ seems probable. In fact, in the two cases mentioned, the signs of the quadrupole moments show that the nuclear shape is of the type of an oblate and prolate spheroid respectively. In both cases therefore, the state of maximum Ω will be energetically favored.

Assuming $\Omega=j$, we have $I=j$ for the ground state. Thus the modification of the single particle model here considered need not impair the success of the model in predicting nuclear spins.

[4] E. Teller and J. A. Wheeler, Phys. Rev. **53**, 778 (1938).

III. COUPLING OF ANGULAR MOMENTA AND DETERMINATION OF MAGNETIC MOMENTS

In a more detailed classification of nuclear states, which is required for the evaluation of magnetic moments in particular, it is necessary to compare the strengths of the couplings between the various angular momentum vectors of the nucleus.

In the first place, the coupling between the orbital and spin angular momenta, **l** and **s**, of the single particle, can be characterized by the frequency ν_{ls} with which the two vectors precess around each other. In the second place, the coupling of **l** to the nuclear axis may be described by the precession frequency ν_{lA} of **l** around the axis. Finally, it will be convenient to introduce the frequency ν_{AI} with which the axis precesses around **I**.

We consider first the case $\nu_{AI}>\nu_{lA}$, in which the coupling of the nucleon motion to the nuclear axis is weak. In this case, the single particle angular momentum and the rotation **R** of the nuclear core must first be quantized separately. For the ground state, R must be assumed to be zero and thus we have again essentially the extreme single particle model which does not allow for the existence of large quadrupole moments. We shall therefore assume $\nu_{AI}<\nu_{lA}$, which seems justifiable since, for the nuclear deformations in question, one expects values for $h\nu_{lA}$ of the order[3] of one Mev.

For $\nu_{AI}<\nu_{lA}$, a number of different cases arise, depending on the magnitude of the spin-orbit coupling:

(A). If the spin-orbit coupling is weak ($\nu_{ls}<\nu_{AI}$), the vector **l**, but not **s**, is coupled to the nuclear axis. The resultant of **l** and **R** forms a total orbital angular momentum **L** which is to be compounded with **s** by simple vector addition. For the lowest nuclear rotational state, the total orbital g-factor g_L will deviate only slightly from the single-particle value g_l, and the magnetic moment of the nucleus will therefore be close to the single particle value.

(B). For $\nu_{ls}>\nu_{AI}$, the total angular momentum of the single particle is coupled to the nuclear axis along which it has the component Ω. Assuming $\Omega=I$, one finds for the resultant nuclear g-factor:

$$g_I=g_\Omega\frac{I}{I+1}+g_R\frac{1}{I+1}, \tag{1}$$

where g_Ω and g_R are the g-factors associated with the angular momenta Ω and R. For g_R it seems most plausible to assume a value of about Z/A, although deviations from this value may occur, since only a fraction of the nucleons is expected to take part in the nuclear rotation. The value of g_Ω will depend on the relative magnitude of ν_{ls} and ν_{lA}.

(B_1). For $\nu_{ls}<\nu_{lA}$, the vector **l** is first coupled to the nuclear axis and, due to the spin-orbit coupling, the spin moment is subsequently quantized along the axis with a component $\pm\frac{1}{2}$. Consequently, we have:

$$Ig_\Omega = \pm\tfrac{1}{2}g_s + (I \mp \tfrac{1}{2})g_l \tag{2}$$

corresponding to the two possible types of states for a given I.

(B_2). If $\nu_{ls} > \nu_{lA}$, the vectors $\mathbf{l}$ and $\mathbf{s}$ must first be compounded to a resultant $\mathbf{j}$, which is then coupled to the axis; in this case,

$$g_\Omega = g_j \tag{3}$$

where g_j is the Landé factor for l-s coupling.

In all three cases, A, B_1 and B_2, the model allows for the existence of large quadrupole moments. However, it should be noted that the quadrupole moment Q referred to fixed axes in space, for the substate $M=I$, will be smaller than the intrinsic nuclear quadrupole moment Q_0, defined with respect to the axis of the nucleus.

In the case A one thus finds:

$$Q = Q_0 \frac{I-1}{I+1}\frac{2I-1}{2I+1} \quad (I = L+\tfrac{1}{2}) \tag{4}$$

and

$$Q = Q_0 \frac{I}{I+1}\frac{2I-1}{2I+3} \quad (I = L-\tfrac{1}{2}) \tag{5}$$

whereas the cases B always lead to expression (5).

For small values of I, the ratio Q/Q_0 may be quite small; for $I=3/2$, the expressions (4) and (5) give values of 1/10 and 1/5 respectively.

The formulas given in the present section can be derived immediately from the vector model for compounding angular momenta, and from simple considerations regarding the motion of the symmetrical top. Wave functions corresponding to the different coupling cases can also be constructed, as regards their angular and spin dependence, in complete analogy to molecular wave functions.

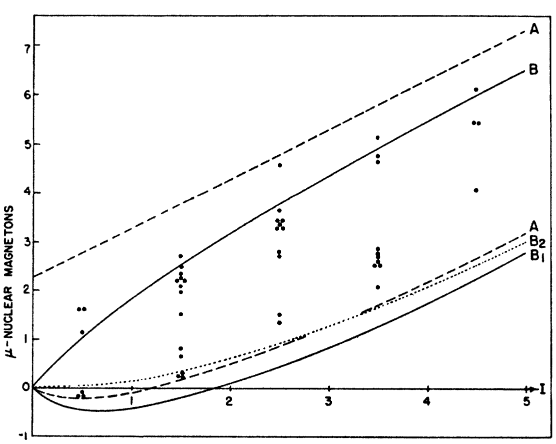

FIG. 1. Magnetic moments of odd proton nuclei.

IV. INTERMEDIATE COUPLING CASES

In Figs. 1 and 2, the magnetic moments of the odd nuclei are given.[5] Only nuclei with mass-number greater than 20 are included, since the models here discussed can only be assumed to have relevance for fairly heavy nuclei. The curves A represent the moments derived from the extreme single particle model (the Schmidt limits). As mentioned above, these values do not differ appreciably from those expected in coupling case A. The curves corresponding to the cases B_1 and B_2 are also shown in the figures, for a value of g_R of 0.45. For parallel $\mathbf{l}$ and $\mathbf{s}$ the expressions (2) and (3) lead to the same value for the magnetic moment.

It was one of the first outstanding successes of the single particle model that it accounted, in a general way, for the dependence of the moments on the value of I. Nevertheless, it is a major defect of the model that it fails to explain why the moments do not coincide with the values corresponding to pure single particle states, but lie in between the values for the two states possible for a given value of I.

This difficulty is especially conspicuous for nuclei having one particle more or one particle less than is required for a closed shell. In fact, the moments of these nuclei show no general tendency to approach the pure single particle values.

Since the two states in question have opposite parity, and therefore do not combine by means of conventional types of nuclear interaction, it is possible to alter the magnetic moments only by exciting additional particles. Such configuration perturbations would, however, have to be so strong that the single particle model would represent only a crude approximation. Moreover, it seems to be difficult to explain on this basis why the moments fall in between the two single particle values, rather than deviate more irregularly from these values.

The situation is essentially modified by the introduction of an asymmetric nuclear core. A large number of states now become possible for a given value of I. Thus, in the coupling case B the particular states considered in Sec. III were characterized by $\Omega=I$, and by the largest possible component of the single particle angular momentum along the nuclear axis. If these two restraining conditions are not imposed, it is possible to construct states having a wide range of moment values.

These values lie between definite limits. In fact, for a given value of the total orbital g-factor, g_L, the moment is uniquely determined by the component of the intrinsic spin along the axis of space-quantization. For the substate, $M=I$, the average value of this spin component S_Z satisfies the condition $-\frac{1}{2}I/(I+1) < \bar{S}_Z < +\frac{1}{2}$. Since the difference between g_L and g_l is usually

[5] The data are taken from a compilation of moments by H. L. Poss, published by Brookhaven National Laboratory, October 1949, supplemented by a few more recent moment determinations published in Phys. Rev.

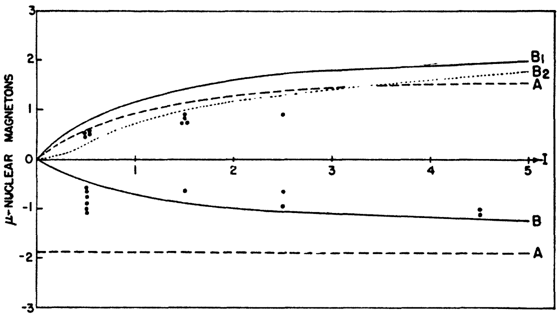

FIG. 2. Magnetic moments of odd neutron nuclei.

of minor importance, the moment is therefore confined within the limiting values A.

If the nucleus can be represented to a high approximation by one of the simple coupling cases discussed in Sec. III, the ground state would seem to have the quantum numbers which were assumed in that section, and which lead to the moment values given by the curves in Figs. 1 and 2. Not only is this choice of quantum numbers suggested by energetic considerations, but it is also required to make the nuclear spin equal to the single particle angular momentum.

However, when some of the characteristic frequencies introduced in Sec. III are comparable in magnitude, we have to do with mixed coupling types. The nuclear ground state will then no longer be a pure state in either of the representations A, B_1 or B_2, but will instead correspond to a mixture of states. Consequently the magnetic moments will, in general, fall in between the limiting values.

As an illustration, we may consider the particularly simple nuclei which have a single particle in addition to the number required for closed shells; e.g., $_{83}\mathrm{Bi}^{209}$, and $_{51}\mathrm{Sb}^{123}$. According to the shell model, which assumes that a strong spin-orbit coupling is a determining factor for the order of nuclear levels, these nuclei have $j=l-\frac{1}{2}$ and should be represented by the coupling case B_2. Nevertheless, the moments are found to lie very far from the predicted pure state values.

The influence of an asymmetric core implies, however, a coupling case intermediate between B_2 and B_1. The single particle state in question, having $\Omega=l-\frac{1}{2}$, will no longer be a pure $j=l-\frac{1}{2}$ state, but will have an admixture of $j=l+\frac{1}{2}$. The negative sign of the quadrupole moment, for the nuclei in question, indicates that the asymmetry favors states with large components of $\mathbf{l}$ along the nuclear axis. In this case, the magnetic moment for the state in question increases with increasing ν_{lA}. For $\nu_{lA}\gg\nu_{ls}$, the moment actually approaches the opposite limit B, corresponding to a spin-component of $+\frac{1}{2}$ along the nuclear axis. In order to account for the observed moments, comparable values of ν_{lA} and ν_{ls} are required. This appears to be consistent with the evidence[1] for a spin-orbit coupling of the order of 2 Mev for large values of l.

A detailed estimate of such perturbations is made difficult by the fact that a large asymmetry of the core will affect the ordering of the single particle levels to such an extent that the nuclei considered can no longer simply be characterized as having a single particle in addition to closed shells. Of course, this circumstance need not interfere with the explanation of the particular stability of the closed shell nuclei themselves, since these nuclei may be of spherical form.

If the deviation of the moments from the limiting values is ascribed to perturbations of the type considered, one would expect that nuclei having a j-value larger than that of any neighboring single particle level will have moments close to the pure state values. The known magnetic moments of nuclei of this type are given in Table I. There actually appears to be a rather close agreement with the values μ_B, predicted for the coupling case B. For comparison, the moments μ_A, corresponding to the extreme single particle model, are also listed. This model is not in good agreement with the empirical moments of these nuclei.

As regards the deviations of the moments from the μ_B-values, it must be noted that the moments depend to some extent on the value of g_R. As already mentioned, this value may vary from nucleus to nucleus, and a certain spread of the moments is to be expected. Moreover, even the simple nuclei considered will not represent completely pure states. In particular, the asymmetry of the core implies that l is not an exact quantum number. States of higher l, but with the same value of Ω, will therefore be admixed to some extent.

As an additional indication of the influence of the nuclear asymmetry on the magnetic moments, it may be noted that the following empirical rule seems to hold: of two isotopes, which differ by 2 neutrons and have the same value of I, the nucleus with the numerically smallest quadrupole moment has a magnetic moment closest to the pure state value.[6] Examples are

TABLE I. Magnetic moments of selected nuclei.

Nucleus	I	μ_{obs}	μ_B	μ_A
$_{12}\mathrm{Mg}^{25}$	5/2	−0.96	−1.04	−1.91
$_{13}\mathrm{Al}^{27}$	5/2	3.64	3.74	4.79
$_{21}\mathrm{Sc}^{45}$	7/2	4.75	4.86	5.79
$_{23}\mathrm{V}^{51}$	7/2	5.14	4.86	5.79
$_{27}\mathrm{Co}^{59}$	7/2	4.64	4.86	5.79
$_{36}\mathrm{Kr}^{83}$	9/2	−0.97	−1.19	−1.91
$_{38}\mathrm{Sr}^{87}$	9/2	−1.1	−1.19	−1.91
$_{41}\mathrm{Cb}^{93}$	9/2	6.17	5.92	6.79
$_{49}\mathrm{In}^{113}$	9/2	5.46	5.92	6.79
$_{49}\mathrm{In}^{115}$	9/2	5.47	5.92	6.79

[6] This rule is somewhat different from the one suggested by W. Gordy (Phys. Rev. **76**, 139 (1949)). In many cases the two rules coincide but, although the evidence is not conclusive, Gordy's rule appears not to hold for the Cu-isotopes (P. Brix, Zeits. f. Physik **126**, 725 (1949)).

the isotopes of Cl, Cu, Ga, Br, Eu, and Re. The only known exception is In, whose isotopes, however, have almost identical moments.

V. CONCLUDING REMARKS

The most adequate coupling cases to represent nuclear states may be expected to vary from nucleus to nucleus depending, in particular, on the quadrupole moment and the value of the single particle angular momentum.

As regards the question of predominant coupling types, some information can be obtained from Figs. 1 and 2.

In the first instance, a large part of the moments for small values of I fall outside the curves B_2. Apart from the case $I=\frac{1}{2}$ with parallel $\mathbf{l}$ and $\mathbf{s}$, it appears, however, that practically all moments of heavy nuclei lie within the limits B_1, although these limits are considerably narrower than the A curves. In view of the uncertainty of g_R, the only definite exception is the moment of Pr^{141} ($I=5/2$), which lies very close to the A-limit.

It is not surprising that the moments for $I=\frac{1}{2}$ with parallel $\mathbf{l}$ and $\mathbf{s}$ tend to fall outside the B curves. In fact, just in this particular case, the projection of $\mathbf{l}$ on the nuclear axis vanishes (Σ states in molecular terminology). There is thus no net spin-orbit coupling, and the coupling of $\mathbf{s}$ to an axis of the nucleus is presumably very weak. Consequently, an approach to the model A is to be expected.

Evidence regarding the structure of nuclear magnetic moments has been obtained from a study of the anomalies in atomic hyperfine structures caused by the finite size of the nucleus.[7,8] The analysis of such effects has given indications that the orbital g-factor of the nucleus is close to the single particle value. This evidence is in agreement with all of the models considered here, since for the lowest rotational state of the nucleus the total orbital g-factor, g_L, deviates only slightly from g_l. It may be added that this value for g_L would not be obtained in general if the large nuclear quadrupole moments were ascribed to more complicated departures from the single particle model than those discussed here.

The hfs anomaly for the odd K-isotopes[8] allows a more detailed test of the nuclear model. These nuclei contain 19 protons, one less than the number 20 required for closed shells, and are therefore expected to be well represented by the model of a single particle moving in an asymmetric nuclear core. Assuming a coupling case between B_1 and B_2, the nuclear wave functions can be determined from the empirical values of the moments. An estimate of the hfs anomaly on this basis[9] gives a result in close agreement with the observed value.

In many respects it thus appears that the modifications of the extreme single particle model, which are necessary in order to allow for the large quadrupole moments, at the same time improve the agreement of the model with empirical data on nuclear magnetic moments.

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[7] A. Bohr and V. Weisskopf, Phys. Rev. **77**, 94 (1950).
[8] Ochs, Logan, and Kusch, Phys. Rev. **78**, 184 (1950).
[9] A. Bohr, Phys. Rev. (to be published).