holes are immobile, being bound to the impurity atoms. Under these circumstances the polarization is large and temperature independent. Absorption of radiation in the long wavelength region of the fundamental absorption band is expected to result in the production of excitons. If these are thermally dissociated at room temperature, free electrons and holes are produced, and the polarization is small. At low temperatures, where the excitons are not thermally dissociated, they wander to impurity atoms where dissociation does occur, the electron becoming free but the hole remaining bound to the impurity atom. Under these conditions of excitation, as the temperature is reduced, the polarization increases.

The dissociation energy of an exciton can be estimated from the relation[16]

$$E_n = -\pi^2 me^4/\mu^4 h^2 n^2 \tag{4}$$

by taking $n=1$. The index of refraction, μ, of diamond is 2.42 and $E_1 \simeq 0.2$ ev. The lifetime of the exciton is given by

$$\tau = \tau_0 \exp(+E_1/kT). \tag{5}$$

Estimating $\tau_0 \simeq 10^{-12}$ sec, one obtains $\tau(100°\mathrm{K}) \simeq 10^{-2}$ sec and $\tau(300°\mathrm{K}) \simeq 10^{-9}$ sec. If the cross section for collision with an impurity atom is taken to be 10^{-14} cm^2, the concentration of impurity atoms of the order of 10^{17} cm^{-3}, and the velocity of the exciton as 10^5 cm/sec, then at 100°K an exciton will make a million collisions with impurity atoms during its thermal lifetime. At room temperature, the corresponding number of collisions is less than unity. The behavior of the exciton is therefore in agreement with the preceeding interpretation.

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[16] F. Seitz, Phys. Rev. 76, 1376 (1949).

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Relativistic Quantum Theory for Finite Time Intervals

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If transition probabilities are evaluated for transitions occuring during a finite time interval, additional divergencies occur different from those commonly encountered for infinite time intervals. The expressions obtained can however be made convergent, if an indeterminacy of time is attributed to each epoch of observation. The method is applied to the emission of a photon by a free electron.

I. INTRODUCTION

THE convergent results in the relativistic quantum theory of elementary particles, which have been recently obtained by different authors,[1] apply only to time periods of infinite duration between two observations. If one tries to evaluate transition probabilities for processes which are localized in space-time by a *sharply defined boundary* (for example two time-like hypersurfaces specifying an initial and final observation), one obtains divergent results. These divergences arise from regions near the boundary, where processes occur without conservation of the momentum-energy component normal to the hypersurface. However, we show here that one can obtain convergent results if *diffuse boundaries* are introduced. We show in Sec. I that this generalization is possible without affecting the unitarity and causality of the array of probability amplitude forming the S-matrix. In Sec. III, we evaluate, in second-order approximation, the time-independent probability for the emission of a photon by an electron.

Time, with these unsharp limits, no longer appears as a *parameter*, t, whose values $t=t'$ and $t=t''$ are fixed for the two limits of the *period of evolution*, $t''-t'=2T$, during which the photon emission takes place. The initial and final epochs themselves, $t \cong t'$ and $t \cong t''$, are now of finite duration, $\Delta t'$ and $\Delta t''$, and must be given in terms of *two probability amplitudes for time*, $f'(t)$ and $f''(t)$, describing the precision with which t' and t'' have been determined. In the probability $dw(\omega)$ that an electron has emitted a photon of frequency between ω and $\omega+d\omega$ during the period considered, the Fourier transforms, $g'(\omega)$ and $g''(\omega)$, of the two probability amplitudes figure as convergence factors for the integral.[2] We have:

$$dw(\omega) = d\omega(|g''|^2 + |g'|^2)(\omega) n(\omega) \equiv dw''(\omega) + dw'(\omega). \tag{1}$$

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[1] S. Tomonaga, Progr. Theor. Phys. 1, No. 2, 27 (1946); J. Schwinger, Phys. Rev. 74, 1439 (1948); 75, 651 (1949); 76, 790 (1949); R. P. Feynman, Phys. Rev. 76, 749, 769 (1949); F. Dyson, Phys. Rev. 75, 486, 1736 (1949).

[2] To $g'(\omega) = \exp(i\omega t')$; $g''(\omega) = \exp(i\omega t'')$ correspond the epochs $f'(t) = \delta(t-t')$ and $f''(t) = \delta(t-t'')$ of sharply determined time values, for which the integral of Eq. (1) diverges.

$g'(\omega)$ and $g''(\omega)$ are normalized to $|g'(0)|=|g''(0)|=1$. Their absolute squares are time independent. We are thus led to think of (1) as the sum of two probabilities for processes which take place *only during the epochs of the initial and of the final observations*. Furthermore, these processes show no conservation of energy: ω is the surplus energy in the final state over that in the initial state. This excess energy can be interpreted as having been furnished by the measuring apparatus during either of the two epochs of observation. Then $|g'(\omega)|^2$ and $|g''(\omega)|^2$ indicate the probability that such an energy is available.

II. THE CONVERGENCE CONDITIONS FOR PROBABILITY AMPLITUDES

We describe a process taking place in a given space-time (x-space) region, V, as the *annihilation of momentum-energy* out of the *incoming* matter or radiation waves and the *creation of momentum-energy* into the *outgoing* waves. We represent the incoming waves of electrons, positrons and photons by the wave packets $u'(x)$, $v'(x)$ and $\varphi'(x)$ with a positive frequency spectrum and the outgoing waves by their conjugate complex $u''^{\dagger}(x)$, $v''^{\dagger}(x)$ and $\varphi''^{\dagger}(x)$ with a negative frequency spectrum. The particular packets

$$u_A'(x)=(2\pi)^{-\frac{3}{2}}\pi_A(k'n')\exp(ik'x) \tag{2}$$

are plane electron waves. A packet (2) represents a quantum of sharply defined *momentum-energy* k', lying in the *momentum-energy space* (p-space) on the hypersurface of rest mass κ, $p=k'$, where

$$k'^4=+(\kappa^2+|\mathbf{k}'|^2)^{\frac{1}{2}} \tag{3}$$

n' numbers the two spin orientations, π^A in (3) is normalized to $\pi_A{}^{\dagger''}\pi'^A=2\kappa i\delta(n''/n')$ for electrons. For photons of rest mass[3] μ we write $p=l'$ and number their polarizations by $n'=1$, 2, 3 (normalization $\pi_\alpha{}''^{\dagger}\pi'^\alpha=\delta(n''/n')$). Then the nth order contribution to the probability amplitude of a process is the n-fold space-time integral over V:

$$\epsilon^n S_n[V](u''\cdots\varphi''\cdots/u'\cdots)$$
$$=i\epsilon^n\int dx''V(x'')\cdots\int dy''V(y'')\cdots\int dxV(x)\cdots$$
$$\times\int dx'V(x')\cdots u_A{}''^{\dagger}(x'')\cdots\varphi_\alpha{}''^{\dagger}(y'')\cdots$$
$$\times\Delta_{B\cdots}^{(c)A\cdots\alpha\cdots}(x''-y'',\cdots x''-x,$$
$$\cdots x''-x',\cdots)u'^B(x')\cdots. \tag{4}$$

For sharply defined boundaries, $V(x')$ is a *discontinuous function*, with the two values 0 or 1 for events x' outside or inside V. The causal[4] function, $\Delta^{(c)A\cdots}(x''-y'',\cdots)$ is a covariant function of the $n-1$ relative displacements of the n events. It is contragradient in its indices to the vector, $\alpha\cdots\beta\cdots$, or spinor, $A\cdots B\cdots$, indices of the packets.

Let us now see how the *generalization to a continuous real function* $V(x)$ related to the time uncertainties of the initial and final observation epochs is possible without affecting the unitarity of the theory or changing the causal function $\Delta^{(c)}$. The unitarity of the S-matrix corresponding to (4) implies that the hermitian part of S_n is determined in terms of the S_m for $m<n$. Therefore S_1 is antihermitian and is given in terms of a single space-time integral of the hermitian interaction energy density and the real function $V(x)$. In electrodynamics the typical element is:

$$\epsilon S_1[V](u''\varphi''/u')$$
$$=i\epsilon 2^{-\frac{1}{2}}\int dxV(x)(\varphi_\alpha{}''^{\dagger}u''^{\dagger}\gamma^\alpha u')(x). \tag{5}$$

In terms of this S_1, we *define* the hermitian part of S_2 by means of the unitarity relation. We obtain for this part expressions of the type (4) (with $n=2$), where $i\Delta^{(c)}$ is replaced by non-causal functions $-\frac{1}{4}(\Delta^{(+)}+\Delta^{(-)})$. The anti-Hermitian part of S_2 is then determined in terms of a causality correction $i\Delta^{(s)}$, such that the sum of the hermitian and antihermitian parts gives $i\Delta^{(c)}$. The higher order approximations are obtained in exactly the same way. This procedure shows that the unitarity and the causality of the S-matrix are independent of the particular form given to $V(x)$, as long as it is a real function.

In order to find the convergence conditions of (4), we transform it into p-space. The transformed integral is $n-1$ fold. If m is the total number of incoming and outgoing packets, then

$$\epsilon^n S_n[V](k''n'',\cdots l'',\cdots/k'n',\cdots)$$
$$=i\epsilon^n(2\pi)^{4n-\frac{3}{2}m}\int dp\cdots\int dq\cdots\int dr\cdots$$
$$\times V(k''-p-q-r-\cdots)\cdots V(l''+p)\cdots$$
$$\times V(0+q)\cdots V(-k'+r)\cdots\pi_A{}''^{\dagger}\cdots\pi_\alpha{}''^{\dagger}\cdots$$
$$\times\Delta_{B\cdots}^{(c)A\cdots\alpha\cdots}(p,\cdots q,\cdots r,\cdots)\pi'^B\cdots. \tag{6}$$

This formula involves the Fourier transforms of $V(x)$ and $\Delta^{(c)}$. We shall use the same letters to describe these functions in both x- and p-spaces. We see at once that the existence of the Fourier transform of the causal

[3] In order to avoid the difficulties connected with zero rest mass photons, we suppose the photon to have a finite rest mass, small compared to that of the electron.

[4] By the term "causal" we imply that $\Delta^{(c)}$ in Eq. (4) is a network of causal functions $\Delta^{(c)}(x-y)$ [given in Eq. (9)] describing an outgoing wave at y and an incoming wave at x, if x is later than y (the "creation at y" precedes the "annihilation at x"), and *vice versa* if y is later than x. See Stueckelberg and Rivier, Phys. Rev. 74, 218 (1948); Helv. Phys. Acta 23, 215 (1950); 23, supp. III, 236 (1930).

function $\Delta^{(c)}(p,\cdots)$ is a *necessary condition for the convergence of the amplitude* S_n. For the particular case $V(p)=\delta(p)$ one obtains the usual result:

$$\epsilon^n S_n[\infty](k''n''\cdots l''\cdots/k'n'\cdots)$$
$$=i\epsilon^n(2\pi)^{4n-\frac{3}{2}m}\delta(-k''-l''-\cdots+k'+\cdots)$$
$$\times\pi_A''^{\dagger}\cdots\pi_\alpha''^{\dagger}\cdots$$
$$\times\Delta^{(c)A\cdots\alpha\cdots}{}_{B\cdots}(-l'',\cdots 0,\cdots k',\cdots)\pi'^B\cdots,\quad (7)$$

which is commonly used to define transition probabilities for a region V extending over "infinite" physical space-time.

III. PHOTON EMISSION BY A FREE ELECTRON

In the second approximation of quantum electrodynamics, the probability amplitude of an electron-electron transition inside of V is

$$\epsilon^2 S_2[V](u''/u')$$
$$=\epsilon^2\int dxV(x)\int dyV(y)u''^{\dagger}(x)$$
$$\times(\gamma\partial\Delta_2^{(c)}+\kappa\Delta_1^{(c)})(x-y)u'(y)$$
$$\equiv\delta(u''/u')(-\tfrac{1}{2}W(u')+\tfrac{1}{2}C(u')-i\Phi(u')).\quad (8)$$

Its Hermitian and anti-Hermitian parts correspond to the separation of the two scalar, causal functions into real and imaginary parts according to:

$$i\Delta^{(c)}=-\tfrac{1}{4}(\Delta^{(+)}+\Delta^{(-)})+i\Delta^{(s)}.\quad (9)$$

$\Delta^{(\pm)}$ are functions involving only definite frequencies larger than the sum of the two rest masses κ and μ.

$$\Delta^{(-)}(-p)=\Delta^{(+)}(p)=\begin{Bmatrix}0\\2\Delta^{(1)}(p^2)\end{Bmatrix}\text{ for } p^4\lesseqgtr 0,\quad (10)$$

$$\Delta^{(1)}(p^2)=0\text{ for }-p^2<(\kappa+\mu)^2.\quad (11)$$

$\Delta^{(s)}$ is the causality correction, defined in terms of $\Delta^{(1)}$ by the following differential equation

$$\left(-\frac{d}{d(p^2)}\right)^n\Delta^{(s)}(p^2)=\frac{n!}{2\pi}\int_0^\infty d(z)^2\frac{\Delta^{(1)}(-z^2)}{(p^2+z^2)^{n+1}}\quad (12)$$

for the lowest n, for which the principal value converges.[5]

Let us now recall briefly the physical meaning of the three parts in Eq. (8) due to $\Delta^{(\pm)}$ and $\Delta^{(s)}$ in (9):

(1) $W(u')$ is a *decrease of the probability* of observing only an emerging electron u'', due to the process of photon emission (in which case we should observe the outgoing waves u'' and φ'').

(2) $C(u')$ is an *increase of this probability*, due to the diminution of the probability for the spontaneous process of three quantum creation (photon, electron and positron), when the electron state u' is occupied.

(3) $\Phi(u')$ is a phase change, different for different wave packets, and giving rise to a *change of the dispersion law for electron waves* [i.e., differing from (2)].

We evaluate these quantities for the space-time region V bounded by two time-like hyperplanes $t\cong t'$ and $t\cong t''$ defined by the amplitudes $f'(t)$ and $f''(t)$. In terms of the Fourier transforms, normalized to $|g'(0)|=|g''(0)|=1$, the transform $V(p)$ in (6) (Appendix I) is

$$V(p)=\delta(\mathbf{p})i(2\pi\omega)^{-1}(g'-g'')(\omega),\ \omega=p^4.\quad (13)$$

Omitting the δ-symbol in (8) (because it is evidently diagonal in the momentum and spin space of the electron), we have the following p-space representation of $S_2[V]$ in terms of W, C and Φ:

$$W(k')=\int_{-\infty}^{+\infty}d\omega|g''-g'|^2(\omega)\frac{(2\pi)^3\epsilon^2}{k'^4\omega^2}$$
$$\times(k'p\Delta_2^{(+)}+\kappa^2\Delta_1^{(+)})(p),\quad (14)$$
$$\mathbf{p}=\mathbf{k}';\quad p^4=k'^4+\omega.\quad (14a)$$

C is the same expression involving $-\Delta^{(-)}$ instead of $\Delta^{(+)}$, and in the phase $\Phi(u')$, $4\Delta^{(s)}$ has to be substituted for $\Delta^{(+)}$. In the limit, where the period $2T=t''-t'$ between the two epochs t'' and t' is long with respect to their indeterminacies $\Delta t''$ and $\Delta t'$, a frequency ω_0 may be defined

$$2T\gg\omega_0^{-1}\gg\Delta t'',\ \Delta t'\quad (15)$$

allowing in the integrand of (14) the substitution:

$$\omega^{-2}|g''-g'|^2(\omega)=\begin{cases}\omega^{-2}(2\sin\omega T)^2,\text{ for }\omega^2<\omega_0^2 & (16)\\ \omega^{-2}(|g''|^2+|g'|^2)(\omega), & \\ \qquad\text{for }\omega^2>\omega_0^2. & (17)\end{cases}$$

If the intergrand is different from zero for $\omega=0$, W, C, and Φ consist of a *time proportional* part plus a *time independent* part.

In our particular case, we obtain a time proportional part for the phase alone. Its form, $2T\Delta\kappa^2(2k'^4)^{-1}$, shows that the dispersion law Eq. (2) is affected in the form of a rest mass change $\Delta\kappa^2$. The value of the invariant $\Delta\kappa^2$ is undetermined on account of the arbitrary constants in the definition of $\Delta^{(s)}$ in (12). The integrand of the time-independent probability (14) decomposes, on account of (17), into the two contributions due to the two limits of the region V. The four vector, p, is the momentum-energy in the electron+photon state. The substitution of $2\Delta^{(1)}$ for $\omega\ll\kappa=k'^4$ (Appendix II) gives the following expression for the low frequency

[5] The integration of (12) introduces a finite series of terms of the type $a_0+a_1p^2+\cdots+a_{n-1}(p^2)^{n-1}$ with n arbitrary constants. A detailed discussion of this arbitrariness will be published in the *Helvetica Physica Acta.*

probability spectrum to find a photon within ω and $\omega+d\omega$ created at $t\cong t'$.

$$dw'(\omega)=d\omega\,|g'(\omega)|^2\frac{\epsilon^2}{(2\pi)^2}\frac{(\omega^2-\mu^2)^{\frac{1}{2}}}{\omega^2\mu^2}. \tag{18}$$

This expression may be compared to the frequency spectrum of the total energy of a spherically symmetric classical radiation field $\chi^\alpha(x)$ of longitudinal photons ($n'=3$). Writing $\omega=l'^4$, the total energy is in such a case:

$$P^4=\int d\sigma(l')l'^4\sum_{n'=1}^{3}(b^\dagger b)(l'n')$$
$$=\int d\omega 4\pi\omega(\omega^2-\mu^2)^{\frac{1}{2}}(b^\dagger b)(l3). \tag{19}$$

$b^\dagger$ and b are the coefficients of the field $\sqrt{2}\chi^\alpha(x)$, developed in terms of plane wave packets $\varphi'^\alpha(x)$ and their conjugates of the form (2) with the three polarizations n' normalized to $\pi_\alpha''^\dagger\pi'^\alpha=\delta(n''/n')$.

We now determine χ^α from the condition at $t=t'$

$$\chi(x')=0;\quad \chi^4(x')=-\frac{\epsilon}{4\pi}\frac{\exp(-\mu|\mathbf{x}'|)}{|\mathbf{x}'|}+\frac{\epsilon}{\mu^2}\delta(\mathbf{x}');$$
$$\partial_4\chi(x')=-\frac{\epsilon}{\mu^2}\,\mathrm{grad}\delta(\mathbf{x}');\quad \partial_4\chi^4(x')=0. \tag{20}$$

At $t=t'$, this field compensates the static field everywhere except "inside" of the point particle. We find

$$b^\dagger b(l3)=\frac{1}{2}\frac{\epsilon^2}{(2\pi)^3}\frac{\omega^2-\mu^2}{\omega^2\mu^2}. \tag{21}$$

Comparing (18) with (19) and (21), we see that the classical radiation field corresponding to $dw'(\omega)$ compensates the field of the point charge ϵ, except within a sphere with a radius of the order of $\Delta t'$. The energy density of the total field (static+radiation) is strictly zero outside of this sphere.

IV. CONCLUSIONS

These arguments show, that the introduction of a finite period of evolution in current quantum electrodynamics produces no difficulty of convergence, if diffuse time boundaries are used.

It is also interesting to note that the emission of a photon by a free electron has a simple classical analog. The *photon field* expectation values correspond to a *classical* radiation field, which compensates (or "interferes away") the point electron's static field at the epoch of observation $t\cong t'$ everywhere, except within a small sphere with a radius of the order of the indeterminacy $\Delta t'$ of the time measurement.

APPENDIX I

If we wish to interpret the S-matrix as an operator operating on a state vector ψ, the state vector refers not to a time like hypersurface as in the Tomonaga-Schwinger theory, but to a time-like layer. We may describe such a layer by a real function $F(x)$, which has the values one or zero for events x lying in the future or in the past of the layer. The region V will then be given in terms of the two layers

$$V(x)=F'(x)-F''(x) \tag{I.1}$$

and the S-matrix transforms according to

$$\psi[F'']=S[V]\psi[F'] \tag{I.2}$$

the initial state into the final state. The probability amplitude for the time measurement is the gradient of $F(x)$:

$$f_\alpha'(x)=\partial_\alpha F'(x). \tag{I.3}$$

It reduces to a function of time alone in the case considered in Section III.

$$\mathbf{f}'(x)=0;\quad f_4'(x)\equiv f'(t)=(2\pi)^{-1}\int_{-\infty}^{+\infty}d\omega e^{-i\omega t}g'(\omega). \tag{I.4}$$

The Fourier transforms of $V(x)$ is then given by (13).

APPENDIX II

In terms of the invariant functions of given rest mass

$$D_\kappa^{(+)}(x)=(2\pi)^{-3}\int d\sigma(k)e^{ikx};\quad d\sigma(k)=(k^4)^{-1}(dk)^3 \tag{II.1}$$

the $\Delta^{(+)}$-functions in (8) are

$$\Delta_1^{(+)}(x)=-\tfrac{3}{4}D_\kappa^{(+)}(x)D_\mu^{(+)}(x)\equiv-\tfrac{3}{4}\Delta^{(+)}(x), \tag{II.2}$$

$$\partial_\alpha\Delta_2^{(+)}(x)=-\tfrac{1}{4}\partial_\alpha D_\kappa^{(+)}(x)\cdot D_\mu^{(+)}(x)-(1/2\mu^2)\times\partial_\beta D_\kappa^{(+)}(x)\cdot\partial^\beta\partial_\alpha D_\mu^{(+)}(x). \tag{II.3}$$

From the divergence of (II.3), we can explicitly evaluate the Fourier transforms for $p^4>0$ and their limiting values for $\mathbf{p}=0$, $p^4=\kappa+\omega$ and $\omega\ll\kappa$:

$$\Delta^{(+)}(p)=2\Delta^{(1)}(p^2)=(2\pi)^{-5}(-p^2)^{-1}[(p^2+(\kappa+\mu)^2)(p^2+(\kappa-\mu)^2)]^{\frac{1}{2}}\cong(2\pi)^{-5}\kappa^{-1}[2(\omega^2-\mu^2)^{\frac{1}{2}}+\sim\kappa^{-1}]. \tag{II.4}$$

$$\Delta_2^{(+)}(p)=2\Delta_2^{(1)}(p^2)=-\tfrac{1}{8}(-p^2)^{-1}(-p^2+\kappa^2+\mu^{-2}(p^2+\kappa^2)^2-2\mu^2)\Delta^{(1)}(p^2)\cong-(\tfrac{1}{4}+\tfrac{1}{2}\mu^{-2}\omega^2+\sim\kappa^{-1})2\Delta^{(1)}(p^2). \tag{II.5}$$

From these approximations, the expression

$$2(k'p\Delta_2^{(1)}+\kappa^2\Delta_1^{(1)})(p^2)\cong(2\pi)^{-5}\kappa(\mu^{-2}(\omega^2-\mu^2)^{\frac{3}{2}}+\sim\kappa^{-1}) \tag{II.6}$$

is obtained, leading from (14) to (18).