negligible at 40 Mev. The absorption data were obtained as soon after bombardment as possible when a minimum of Pr^{139} was present.

Beryllium absorption showed two particles of ranges 70 and 1560 mg Be/cm^2 corresponding to 0.28 and 3.1 Mev, respectively. These were shown to be a conversion electron and a positron on the magnetic counter, and the energies were verified, for the conversion electron, $H\rho=2010$ gauss-cm$=0.28$ Mev, and for the positron, $H\rho=1.19\cdot 10^4=3.1$ Mev. The sweeps were taken a number of hours after bombardment and the curious shape of curve B results from the 1.0-Mev Pr^{139} positron which had grown in, the presence of which in no way affects the determination of the Nd^{139} positron end point.

In addition to the K x-rays and the annihilation radiation, a gamma-ray of half-thickness 12.5 g Pb/cm^2 $=1.3$ Mev was detected by lead absorption.

The relative abundances of these radiations, subject to the uncertainty introduced by daughter growth, are as follows: 0.28-Mev e^-:3.1-Mev β^+:K x-rays:0.5-Mev γ:1.3-Mev $\gamma=0.03:0.11:1:0.11:0.10$.

From these ratios it can be postulated that Ne^{139} decays mainly by electron capture, and, assuming that each K x-ray quantum represents one disintegration by electron capture, the positron branching is approximately 10 percent.

Nd^{138}

A 22-min activity was formed in the bombardments of praseodymium with 40- and 50-Mev protons, and appeared in greater yield relative to Nd^{139} at 50 Mev. It was shown to be a rare earth by fluoride hydroxide cycles, and, since Nd^{139}, Pr^{139}, and Pr^{138} have been identified, the 22-min activity is tentatively assigned to Nd^{138}.

A beryllium absorption was taken of the 22-min activity, and, after it was gone, the absorption was repeated on the Nd^{139}. A point-by-point resolution shows a particle of approximate range 1160 mg Be/cm^2 corresponding to a maximum energy of 2.4 Mev (Fig. 5). It undoubtedly is a positron, and the shape of the curve indicates that there may also be a conversion electron.

I wish to thank Dr. B. B. Cunningham under whose direction the work was done, Mr. J. T. Vale and the crew of the 184-in. cyclotron, Mr. R. D. Watt and the crew of the linear accelerator, Mr. J. Conway and Mr. M. Moore of the spectrographic laboratory, and Dr. LeRoy Eyring who participated in the study of Nd^{139}.

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Energy Degeneration of Cosmic-Ray Primaries*

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Under the assumptions that a meson-producing collision between two nucleons with relativistic energies is completely inelastic and that the collision cross-section is independent of energy, one can calculate the energy degeneration of the primary cosmic rays (assumed to be nucleons) and the energy spectrum of high energy nucleons as a function of depth in the atmosphere for any primary spectrum. While this model is highly restricted, it is of considerable interest for comparison with experimental results, since it leads to the maximum possible rate of energy degeneration for nucleons. Binding of nucleons into nuclei is neglected (except for the "packing effect" on the mean-free-path for collision) and the results are further limited to energies greater than 1 Bev where energy loss due to ionization is negligible and the velocity of the nucleons may be taken to be the velocity of light. Analytic solutions are obtained for the energy spectrum at any depth for monoenergetic and power-law primary spectra. The effect of the geomagnetic cut-off on a primary power-law spectrum is also investigated.

I. INTRODUCTION

RECENT cosmic-ray investigations,[1] particularly those at high altitudes, have accumulated a body of experimental evidence favoring a coherent interpretation of cosmic-ray phenomena in terms of the identification of the majority of cosmic-ray primaries as *nucleons* (either free (protons) or bound in nuclei[2]) and for meson-producing collisions (or, more generally, nuclear interactions) as the principal agency for their rapid absorption in the atmosphere. The variety of

* Part of the work herein contained formed a portion of a thesis submitted by one of the authors (F.J.M.) in partial fulfillment of the requirements for the degree of Bachelor of Science in Physics. A preliminary report on this work was made at the semi-centennial meeting of the American Physical Society, Cambridge, Massachusetts, June 16–18, 1949.

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[1] See, for example, W. Heisenberg, Ed., *Cosmic Radiation* (Dover Publications, 1946); B. Rossi, Rev. Mod. Phys. **20**, 537 (1948); Proceedings of the Symposium on Cosmic Rays at the California Institute of Technology, June 21–25, 1948, Rev. Mod. Phys. **21** (1949).

[2] Freier, Lofgren, Ney, Oppenheimer, Bradt, and Peters, Phys. Rev. **74**, 213 (1948); Freier, Lofgren, Ney, and Oppenheimer, Phys. Rev. **74**, 1818 (1948); H. L. Bradt and B. Peters, Phys. Rev. **74**, 1828 (1948).

cosmic-ray phenomena at lower altitudes are then to be interpreted as arising from the numerous progeny through several generations of the primary particles. If this picture is correct, it is of interest to investigate the energy degeneration of these primaries as they pass through the atmosphere and the consequent variation of the energy spectrum of high energy nucleons with atmospheric depth. A detailed theoretical discussion of this process is made difficult by the lack of any thoroughly satisfactory theory of meson production in collisions of high energy nucleons.[3] Experimental observations bearing on this problem, because of difficulties and ambiguities in the interpretation of the experimental results, have not yet yielded a very clear or coherent picture of the meson production process, itself, at high energies. There appears, however, to be general agreement that the primary cosmic-ray particles, as a result of meson-producing collisions, are approximately exponentially absorbed in the atmosphere[4] with an absorption mean-free-path of the order of 120 g/cm^2, and that mesons are produced multiply[5] in such collisions with a multiplicity which probably increases with energy.

In view of the lack of more detailed information, it might be considered premature to attempt a theoretical investigation of the proposed problem at the present time; on the other hand, an investigation of this problem, even on the basis of a highly oversimplified model of the meson production process, may have considerable value as a basis on which to analyze forthcoming experimental results. In the spirit of this latter view we have investigated theoretically the energy degeneration process on the simplest model of meson-producing collisions consistent with the experimental evidence presented above. In particular, this model may have considerable value from the point of view of an interpretion of the experimental results, since it represents an extreme limiting case in which the energy degeneration process occurs at the maximum possible rate.

The features of this model are:

(1) The cross section for a meson-producing collision between two nucleons is assumed to be independent of energy from 1 Bev to the highest energies encountered in the cosmic rays.

(2) The collision of two high energy nucleons resulting in meson production is assumed to be a completely inelastic collision in which all of the available energy in the frame in which the total momentum of the two nucleons is zero goes into the production of mesons.

The first of these assumptions is probably not unreasonable and perhaps quite close to the truth. The second is dictated more by tractability of the theory than by physical arguments. The principal objection which may be raised to it is that the theoretical results, especially at ultra-relativistic energies, may depend critically upon it. As will be seen in the next section, there is a considerable difference between a totally inelastic collision and one which is say 99 percent inelastic. In defense of the second assumption, the only evidence presently available is very indirect. The application of this assumption by Wouthuysen[6] to the study of the production of cosmic-ray stars by the primary cosmic radiation at high altitudes (95,000 feet) has made possible the explanation of the experimental curve of number of stars of given prong number *versus* prong number up to prong numbers of the order of 10. Evidence of this character is, of course, far from conclusive.

The convenience of the second assumption lies in the fact that the energy of the two nucleons after a collision (in the laboratory frame) is uniquely determined by the primary energy, independently of any further details of the process, and the two nucleons emerge from the collision with equal energies. This is not true for partially inelastic collisions, since the energy distribution between the residual nucleons then depends on the angle of emergence of the nucleons from the collision relative to their initial direction in the reference frame in which the total momentum is zero.

In the present work we shall be concerned only with nucleons of relativistic energies (specifically, kinetic energies greater than 1 Bev) for which the assumptions above might be expected to have some validity and for which there is in addition further simplification of the theory. When a nucleon of such kinetic energy (whether free or bound in a primary nucleus) strikes a nucleus, it is reasonable to assume that the binding forces in both the incident and target nuclei play no part in the collision and that the collisions may be considered to take place between effectively free nucleons. Hence, all effects of binding of nucleons into nuclei will be neglected.[7]

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[3] Reference should be made, however, to the following work: Lewis, Oppenheimer, and Wouthuysen, Phys. Rev. **73**, 127 (1948); Hamilton, Heitler, and Peng, Phys. Rev. **64**, 78 (1943); Heitler and Peng, Proc. Ir. Ac. **49**, 101 (1944); W. Heitler and P. Walsh, Rev. Mod. Phys. **17**, 252 (1945); L. Janossy, **64**, 345 (1943); R. P. Feynman and H. A. Bethe, Phys. Rev. **70**, 786 (1946); see also, G. Wataghin, Phys. Rev. **70**, 787 (1946); **74**, 975 (1948); **75**, 693 (1949).

[4] See for example, B. Rossi, Rev. Mod. Phys. **20**, 537 (1948).

[5] See reference 1. There is still considerable controversy as to whether the meson production is truly multiple or "plural." For a discussion of the latter interpretation see the papers of Heitler and his co-workers under reference 3 as well as L. Jánossy, *Cosmic Rays* (Oxford University Press, 1948) and references contained therein. Criticisms of the theoretical foundations of the work of Hamilton, Heitler, and Peng (reference 3) are presented in H. A. Bethe, Phys. Rev. **70**, 785 (1946). Recent photographic evidence favoring truly multiple production (with co-existent plural production) is presented in Kaplon, Peters, and Bradt, Phys. Rev. **76**, 1735 (1949). The model of the meson production process employed in this paper assumes the existence of true multiple production of mesons in an elementary collision between nucleons.

[6] S. A. Wouthuysen, unpublished work.

[7] If either the target nucleons or the incident nucleons are bound into nuclei instead of being uniformly distributed in space, the collision mean-free-path will be increased. This "packing

A picture of the energy degeneration process on the basis of the above assumptions may be formulated in the following way. An incident nucleon of total energy E makes a collision with a nucleon at rest. If the collision is completely inelastic, then in the frame in which the total momentum is zero the two nucleons are at rest after the collision. Transforming to the terrestrial frame, one then has two nucleons travelling in the direction of the incident primary, each with a total energy which one can easily calculate to be

$$\bar{E}=[Mc^2(E+Mc^2)/2]^{\frac{1}{2}}. \tag{1}$$

These two nucleons can then make further collisions of the same character with consequent multiplication of the number of nucleons but with degradation of their energies according to the relation (1) at each step. This process continues until energies at which our assumptions surely break down are reached. The meson-producing proclivities of the nucleons are then reduced and other processes, such as ionization energy loss, nuclear excitation, etc., complete the absorption of the nucleons. At energies above 1 Bev, energy loss by ionization of a proton is negligible compared to the energy loss entailed in meson-producing collisions (see Fig. 1) and so can be safely neglected.

In the following, after a brief discussion of the collision between two nucleons, we derive the equation for the change in the differential energy spectrum of high energy nucleons with thickness of matter traversed. Various solutions to this equation are then examined: (1) solutions for which the energy spectrum does not change with depth, (2) the solution corresponding to an initially monoenergetic flux of nucleons incident on the matter, (3) by application of the preceding result, solutions corresponding to incident energy spectra of the form of a power law in both total energy and kinetic energy, and (4) solutions for an incident energy spectrum corresponding to a geomagnetic cut-off at the lower end of a power law initial spectrum.

One general result of these calculations may be readily anticipated. If the incident spectrum of nucleons falls off monotonically with increasing energy at a sufficiently rapid rate, then there exists for every thickness of the material traversed an energy E_d above which the original spectral distribution is maintained but exponentially absorbed. This is a consequence simply of the fact that the number of particles leaving some small energy interval by collisions is proportional to the number present in this interval, while the number entering this interval as a result of collisions of higher energy particles is small compared with the number leaving because of the relative scarcity of particles of higher energy. Below the indicated energy E_d there are deviations of the lower energy intervals. This latter effect is responsible, then, for a difference between the mean-free-path for collision and the mean-free-path for absorption of the lower energy nucleons. The dividing energy E_d increases with increasing depth at a rate which depends on the initial energy spectrum.

effect" on the mean-free-path can be taken into account in our calculations simply by employing the effective mean-free-path for a collision in all of the formulas we develop.

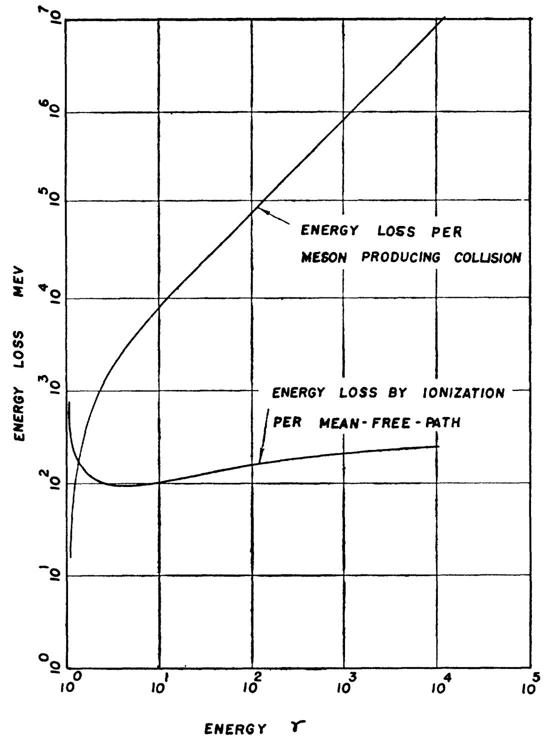

FIG. 1. Energy loss per meson-producing collision and energy loss by ionization per mean-free-path as functions of energy γ.

II. RELATIVISTIC COLLISION OF TWO NUCLEONS

Consider a collision between two nucleons, each of mass M, one of the nucleons being initially at rest and the other having a total energy E_1 and momentum p_1. Let the frame of reference in which this situation holds be denoted by S, and let the reference frame in which the total momentum is zero be denoted by S'. One finds readily that the velocity of S' relative to S is given by

$$v=c^2p_1/(E_1+Mc^2).$$

The energies and momenta of the nucleons before the collision in the frame S' will be denoted by $E_1'=E_2'$ and $p_1'=-p_2'$. Using bars to denote the corresponding quantities after the collision, we can define a parameter K to measure the elasticity of the collision by

$$K=(\bar{E}_1'+\bar{E}_2'-2Mc^2)/(E_1'+E_2'-2Mc^2), \tag{2}$$

where $K=1$ for a completely elastic collision and $K=0$ for a completely inelastic collision. Since in the frame S', $\bar{E}_1'=\bar{E}_2'$, it follows from the last equation that

$$\bar{E}_1'=\bar{E}_2'=KE_1'+(1-K)Mc^2.$$

The total energy of the two nucleons in the frame S can then be found from the total energy in the S' frame by a Lorentz transformation using the velocity found above with the result

$$\bar{E}_1+\bar{E}_2=K(E_1+Mc^2)+(1-K)[2Mc^2(E_1+Mc^2)]^{\frac{1}{2}}. \quad (3)$$

Throughout this paper it will be convenient to measure all energies in units Mc^2, whence defining

$$\gamma=E/Mc^2,$$

Eq. (3) becomes

$$\bar{\gamma}_1+\bar{\gamma}_2=K(\gamma_1+1)+(1-K)[2(\gamma_1+1)]^{\frac{1}{2}}. \quad (4)$$

We may note that the total energy of the two particles after the collision is given by a relatively simple expression; in order to find the individual energies, however, it would also be necessary to know the angle at which the two particles emerged after the collision in the S' frame relative to their original direction.

It is obvious from Eq. (4) that for relativistic collisions $\gamma_1\gg 1$ there is a great difference between a completely inelastic collision and a highly, but not totally, inelastic one. Thus, for an incident nucleon of 10^{15}-ev energy, the total energy after a totally inelastic collision would be 1.4×10^{12} ev, while for a collision with K as small as 0.01, the total energy after the collision would be 10^{13} ev. The great simplicity of the completely inelastic case lies not only in the fact that the angle of scattering need not be known but also in the fact that the two particles after the collision each have the same energy given by

$$\gamma=[(\gamma_1+1)/2]^{\frac{1}{2}}. \quad (5)$$

It should also be pointed out that if the actual collisions are not completely inelastic, there is no reason to believe that the quantity K is independent of the energy.

III. THE EQUATION FOR THE ENERGY DISTRIBUTION FUNCTION

We now derive the equation for the variation of the energy spectrum of nucleons as they penetrate matter, under the assumption of completely inelastic collisions and a collision cross-section independent of energy. As a consequence of the latter assumption, the mean-free-path for collisions will also be independent of energy and we may use the mean-free-path as the unit of distance in the material. Let $n(\gamma,t)d\gamma$ be the number of nucleons per unit volume at a depth t (measured in mean-free-paths) in the material having an energy lying between γ and $\gamma+d\gamma$. The flux of nucleons in this energy range per unit area at a depth t in the material is then given by

$$n(\gamma,t)v(\gamma)d\gamma,$$

where v is the velocity corresponding to the energy γ. The flux at a depth $t+dt$ is then given by

$$n(\gamma,t+dt)v(\gamma)d\gamma=n(\gamma,t)v(\gamma)d\gamma-n(\gamma,t)v(\gamma)d\gamma dt +2n(\gamma_1,t)v(\gamma_1)d\gamma_1 dt, \quad (6)$$

that is, the flux at a depth t minus the number leaving the energy range under consideration as a result of collisions plus the number entering the energy range under consideration as a result of collisions in the thickness dt. The energy γ_1 is related to the energy γ by

$$\gamma_1=2\gamma^2-1, \quad (5')$$

and the factor 2 in the last term in Eq. (6) comes from the fact that each collision of a nucleon of energy γ_1 yields 2 nucleons of energy γ. From Eq. (6) follows

$$\partial n(\gamma,t)/\partial t=2n(\gamma_1,t)[v(\gamma_1)d\gamma_1/v(\gamma)d\gamma]-n(\gamma,t) \quad (7)$$

and using Eq. (5′) together with the relation

$$v(\gamma_1)/v(\gamma)=[2\gamma^2/(2\gamma^2-1)]^{\frac{1}{2}},$$

one obtains the desired equation

$$\partial n(\gamma,t)/\partial t=8\gamma[2\gamma^2/(2\gamma^2-1)]^{\frac{1}{2}}n(2\gamma^2-1,t) -n(\gamma,t). \quad (7')$$

While we could attempt to solve this equation, it is doubtful whether our assumptions have any validity for nucleons with kinetic energies less than 1 Bev, that is, $\gamma=2$. For values of γ greater than this, we can replace $2\gamma^2-1$ by $2\gamma^2$ with only small error. This simplifies the equation to

$$\partial n(\gamma,t)/\partial t=8\gamma n(2\gamma^2,t)-n(\gamma,t). \quad (8)$$

The remainder of our work then consists of finding suitable solutions of this functional-differential equation.

IV. STATIONARY ENERGY SPECTRA

It is rather interesting that Eq. (8) possesses solutions in which the energy spectrum of the nucleons, apart from an over-all exponential absorption, does not change as the nucleons penetrate matter. These solutions exist as a consequence of the fact that the variables in the equation can be separated. By writing

$$n(\gamma,t)=T(t)G(\gamma), \quad (9)$$

we obtain the two equations

$$dT/Tdt=k \quad (10a)$$

$$8\gamma^2G(2\gamma^2)/G(\gamma)=k+1. \quad (10b)$$

The solution of the first is simply

$$T=e^{kt}.$$

The second is a functional equation of a sufficiently simple form that it can be solved by inspection:

$$G(\gamma)=\frac{w(\gamma)}{\gamma}\exp\left\{\left[\frac{\ln(k+1)}{\ln 2}-2\right]\ln\ln 2\gamma\right\}, \quad (11)$$

where $w(\gamma)$ is an arbitrary function satisfying the functional equation $w(2\gamma^2)=w(\gamma)$. k is an arbitrary constant which may be complex. The solution is to be taken as the real part of the product $T(t)G(\gamma)$. Actually, however, none of these solutions can be realized physically. k must be real and greater than -1 if the spectrum is to be positive definite. For $k>1$, the solutions are unbounded in the number of particles at high energies. For all values of k the total energy carried by the particles is unbounded at high energies.

V. SOLUTION FOR MONOENERGETIC INITIAL SPECTRUM

It is obvious that by finding the solution of Eq. (8) for a monoenergetic initial spectrum of nucleons one can readily obtain the solution for an arbitrary initial spectrum by superposition. Hence, we first derive the solution of Eq. (8) which corresponds to the initial condition

$$n(\gamma,0)=\delta(\gamma-\gamma_0), \qquad (12)$$

where δ is Dirac's δ-function.

To accomplish this we first make a Laplace transformation on the variable t:

$$n(\gamma,t)=(1/2\pi i)\int_C \nu(\gamma,\alpha)e^{\alpha t}d\alpha,$$

where C is an as yet undetermined contour; this transformation reduces Eq. (8) to the functional equation

$$\nu(2\gamma^2,\alpha)=(1+\alpha)\nu(\gamma,\alpha)/8\gamma. \qquad (13)$$

The solution of this equation can be obtained by inspection and has the form

$$\nu(\gamma,\alpha)=[g(\gamma,\alpha)/\gamma\ln^2 2\gamma] \times\exp\{[\ln\ln 2\gamma/\ln 2]\ln(1+\alpha)\}, \qquad (14)$$

where $g(\gamma,\alpha)$ is an arbitrary function of its arguments which satisfies the condition

$$g(2\gamma^2\,\alpha)=g(\gamma,\alpha). \qquad (15)$$

The solution of the original equation is therefore

$$n(\gamma,t)=(1/2\pi i)\int_C [g(\gamma,\alpha)/\gamma\ln^2 2\gamma] \times\exp\{(\ln\ln 2\gamma/\ln 2)\ln(1+\alpha)+\alpha t\}d\alpha. \qquad (16)$$

To impose the initial condition, we make the change of variable $\alpha=e^\beta-1$ and let

$$g(\gamma,\alpha)=\rho(\gamma,\beta)\exp\{-\beta[(\ln\ln 2\gamma_0/\ln 2)+1]\} \qquad (17)$$
$$\rho(2\gamma^2,\beta)=\rho(\gamma,\beta).$$

The solution then becomes

$$n(\gamma,t)=(1/2\pi i)\int_{C'}[\rho(\gamma,\beta)/\gamma\ln^2 2\gamma] \times\exp\{\beta[(\ln\ln 2\gamma-\ln\ln 2\gamma_0)/\ln 2]+(e^\beta-1)t\}d\beta,$$

where C' is the mapping of C on the β-plane. Setting $t=0$, we see that the initial condition will be satisfied if C' is chosen to lie along the axis of imaginaries from $-i\infty$ to $i\infty$ and ρ is chosen as the constant

$$\rho(\gamma,\beta)=\ln(2\gamma_0)/\ln 2.$$

Hence, the desired solution is given in the form of a contour integral by

$$n(\gamma,t)=(e^{-t}\ln 2\gamma_0/\rho\ln 2\ln^2 2\gamma)(1/2\pi i) \times\int_{C'}\exp\{\beta[(\ln\ln 2\gamma-\ln\ln 2\gamma_0)/\ln 2]+te^\beta\}d\beta. \qquad (18)$$

If the integrand is now expanded in a power series in t and the integration performed term by term we obtain a series form of the solution:

$$n(\gamma,t)=\frac{e^{-t}\ln 2\gamma_0}{\gamma\ln 2\ln^2 2\gamma}\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{1}{2\pi i} \times\int_{C'}\exp\left\{\beta\left[\frac{\ln\ln 2\gamma}{\ln 2}-\frac{\ln\ln 2\gamma_0}{\ln 2}\right]+n\right\}d\beta$$
$$=\frac{e^{-t}\ln 2\gamma_0}{\gamma\ln 2\ln^2 2\gamma}\sum_{n=0}^{\infty}\frac{t^n}{n!}\delta\left(\frac{\ln\ln 2\gamma}{\ln 2}-\frac{\ln\ln 2\gamma_0}{\ln 2}+n\right). \qquad (19)$$

We see that an energy spectrum in the form of discrete lines is obtained, the sequence of energies being just those which follow from the sequence of energy degeneration steps generated by the law (5).

VI. INITIAL POWER LAW ENERGY SPECTRUM

From the solution obtained in the last section one can now obtain the solution corresponding to an arbitrary initial energy spectrum $n(\gamma,0)=f(\gamma)$ by integration:

$$n_f(\gamma,t)=\int f(\gamma_0)n(\gamma,t,\gamma_0)d\gamma_0. \qquad (20)$$

The cases of greatest interest are those in which the initial spectrum has the form of an inverse power law in either the total energy or the kinetic energy. Considering the former first, we have

$$f(\gamma_0)=\gamma_0^{-s}\quad(s>1). \qquad (21)$$

The integrations can be readily executed and the result is

$$n_s(\gamma,t)=\int\gamma_0^{-s}n(\gamma,t,\gamma_0)d\gamma_0$$
$$=\frac{e^{-t}2^{s-1}}{\gamma}\sum_{n=0}^{\infty}\frac{(4t)^n}{n!}\exp\{-2^n(s-1)\ln 2\gamma\}. \qquad (22)$$

We note that the result has the form of a series of inverse power spectra with increasingly negative exponents. The solution can also be obtained by using the contour integral representation (18) instead of the series solution, giving the result in the form of the

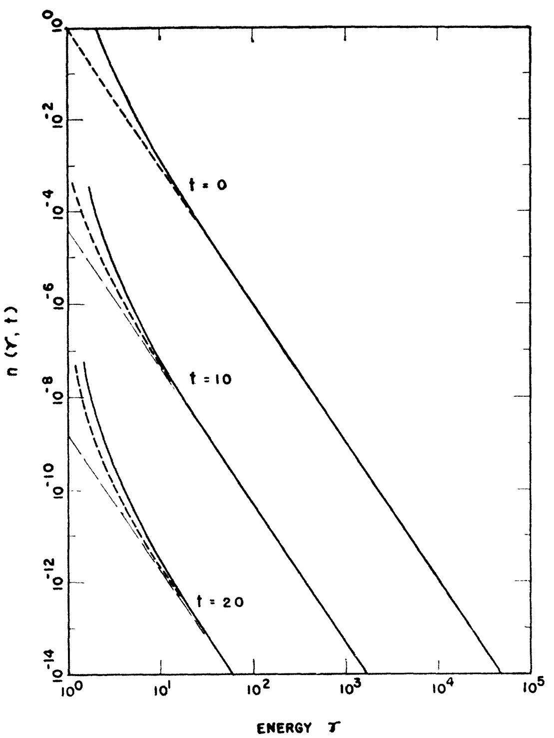

FIG. 2. Energy spectrum $n_3(\gamma, t)$ at various depths t. The solid curve is for an initial power law kinetic energy spectrum and the dashed curve for an initial power law total energy spectrum.

following contour integral:

$$n_s(\gamma, t) = [e^{-t2^{s-1}}/(s-1)^2\gamma \ln 2 \ln^2 2\gamma][1/2\pi i] \times \int_{C'} \exp\{\beta[(\ln \ln 2\gamma - \ln \ln 2\gamma_0)/\ln 2] + te^{\beta}\} \times \Gamma(2-\beta/\ln 2)d\beta. \quad (23)$$

For large t, this integral can be evaluated by the method of steepest descents, but the result is complicated and appears to be inferior for calculation to the rapidly converging series (22) even for relatively high values of t.

In the case where the initial spectrum is an inverse power law in the kinetic energy $f(\gamma_0) = (\gamma_0 - 1)^{-s}$, the calculations are similar and the result obtained is

$$n_s' = \frac{e^{-t2^{s-1}}}{\gamma} \sum_{n=0}^{\infty} \frac{(4t)^n}{n!} [(2\gamma)^{2^n} - 2]^{-s} (2\gamma)^{2^n}. \quad (24)$$

Graphs of the energy spectra at various depths are shown in Fig. 2 for both total energy and kinetic energy power law distributions with exponent $s=3$. It will be noted from these that at high energies, except for the change in normalization, the original power law character of the spectrum is preserved with the same exponent but that deviations toward higher populations occur for the lower energies. This behavior may be anticipated from our solutions (22) and (24), for by a comparison of successive terms in the series one finds that for energies

$$\gamma \gg 2^{(3-s)/(s-1)} t^{1/(s-1)}$$

the functions are accurately represented by the first term in each series. This behavior was anticipated on the basis of physical arguments in the introduction.

The above results allow one to calculate immediately a differential absorption coefficient which we define as $(-dn_s/n_s dt)$. This quantity is plotted in Fig. 3 as a function of t for various values of γ. If the absorption and collision mean-free-paths were equal, the differential absorption coefficient would simply be equal to unity. The curves here show the increase in absorption mean-free-path over collision mean-free-path as a result of the production of secondaries in the collisions.

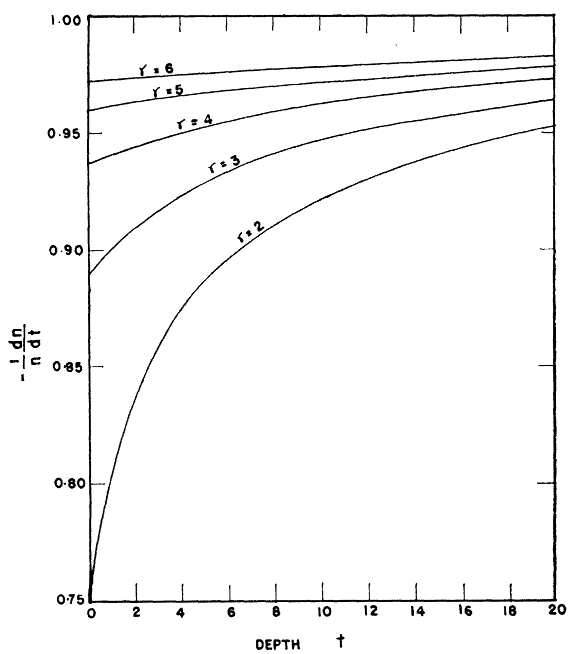

FIG. 3. Absorption coefficient for initial power law total energy spectrum as a function of depth for various energies ($s=3$).

VII. GEOMAGNETIC CUT-OFF

The effect of the earth's magnetic field on the primary energy spectrum can easily be included in our calculation for a power law spectrum by modifying the initial spectrum to the form

$$n(\gamma_0, 0) = \begin{cases} (\gamma_0 - 1)^{-s} & \gamma_0 > \gamma_c \\ 0 & \gamma_0 < \gamma_c \end{cases}$$

This spectrum approximates the more gradual geomagnetic cut-off on a power law spectrum by a sharp cut-off at a critical energy γ_c. For energies greater than this critical energy, there is no change in the results. However, for energies smaller than the critical energy, certain of the terms in the sum in Eq. (24) are absent.

The result may be written in the same form as Eq. (24), except that the sum on n goes only from $n=n_1$ to infinity, where

$$n_1=\begin{cases}\text{smallest integer}>(\ln\ln 2\gamma_c-\ln\ln 2\gamma)/\ln 2 & \gamma<\gamma_c\\ 0 & \gamma>\gamma_c.\end{cases}$$

This spectrum, as would be anticipated from solution in δ-functions, contains a series of steps occurring at the critical energy γ_c, and at the energies to which nucleons of energy γ_c are successively degraded.

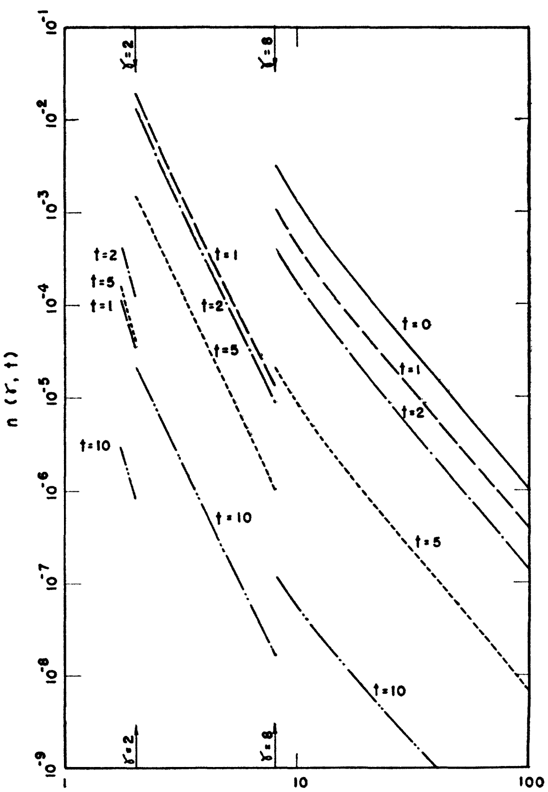

FIG. 4. Energy spectrum for initial power law total energy spectrum cut-off at $\gamma=8$ as a function of energy γ at various depths.

Graphs of this spectrum are shown in Fig. 4 for the exponent $s=3$. The critical energy has been taken to be 7 Bev, which corresponds to the energy above which

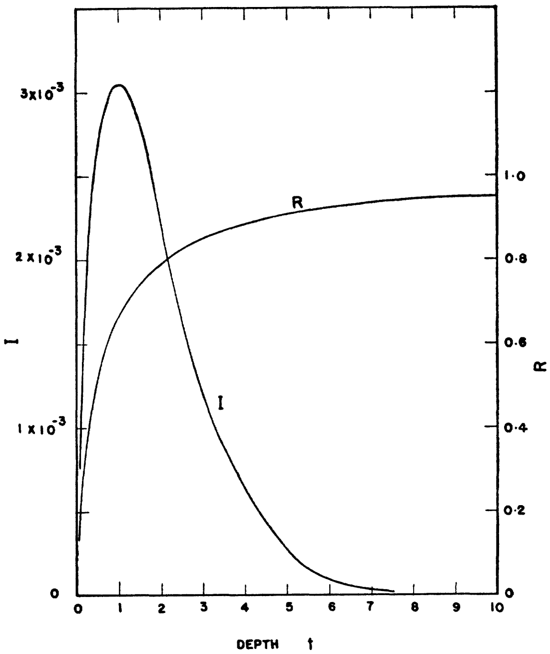

FIG. 5. $I=\int_2^8 n(\gamma,t)d\gamma$, $R=\int_2^8 n(\gamma,t)d\gamma/\int_2^\infty n(\gamma,t)d\gamma$ for initial cut-off power law kinetic energy spectrum as a function of depth.

all protons reach the earth at a geomagnetic latitude of about 40°. As the nucleons penetrate the atmosphere, the originally empty energy interval from $\gamma=2$ to $\gamma=8$ is filled in by nucleons from higher energies. This filling-in process is shown in Fig. 5, where the total number of nucleons in this energy interval, as well as the ratio of this number to the total number of nucleons with energies greater than $\gamma=2$, is plotted as a function of depth in the atmosphere. These two curves are closely approximated by the functions $2te^{-t}$ and $2t/(2t+1)$, respectively. The close similarity of this behavior with the growth and decay of a radioactive daughter nucleus whose half-life is the same as its parent nucleus may be noted; the extra factor of 2 results simply from the fact that two secondaries are produced in each collision.

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