On Bound States and Scattering in Positron Theory

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The use of bound-state wave functions in calculations in positron theory is justified by the introduction of a new representation, in a certain sense intermediate between the Heisenberg and interaction representations. In the bound-state representation the definition of a stable vacuum state is possible only for a restricted class of external fields. Some attention is given to the problem of vacuum polarization, and it is shown that a very simple procedure accomplishes the charge renormalization with sufficient accuracy to be of use in certain scattering problems. The application to the scattering of radiation is discussed in some detail, in order to show the relation between the different points of view that may be adopted in problems of the coherent scattering by a bound electron and the "Delbrück scattering" by virtual electron pairs.

I. INTRODUCTION

IN the early days of the Dirac theory of the electron, strong evidence for the importance of the states with negative energy was found in the theory of scattering of radiation. A very significant part of the scattering had to be ascribed to "intermediate states" in which the electron's energy was negative. Such states indeed provided all of the scattering of light of longer wavelengths ($h\nu \ll mc^2$) by free electrons. In the nonrelativistic theory this scattering came from the term $(e^2A^2/2mc^2)\psi$ in the Schrödinger equation and was called by Dirac "true scattering."

Dirac's suggestion that the negative energy states are "filled up," which is the underlying idea of the present positron theory, was seen to contradict the idea of negative energy intermediate states. It was pointed out, however, by Dirac[1] and by Waller[2] that the same result for the scattering could still be obtained. Instead of intermediate states with the electron's energy negative, one has to consider intermediate states in which a pair is present in addition to the original electron, the electron of the pair being in its final state. The transition to the final state of the system then occurs with the annihilation of the positron and the original electron. The contribution to the probability amplitude for scattering is just equal to that calculated earlier by using negative energy intermediate states.

During the development of positron theory stimulated by the experimental discovery of the positron, it was realized that the possibility of intermediate states with a pair present has another connection with scattering. An electromagnetic field alone can scatter light by processes in which electrons appear only virtually, in such intermediate states. The possibility of this kind of coherent scattering by the Coulomb field of a nucleus was pointed out by Delbrück,[3] and results inferred from incomplete calculations by the Born approximation were published by Kemmer and Ludwig[4] and by Achieser and Pomerantchuk.[5]

More recently Halpern and Hall[6] have raised questions about scattering in positron theory and have suggested that, when bound states of the electron are involved, the conclusions of Dirac and Waller require modification. Particularly in the case of coherent scattering by a bound electron, the usual idea of an intermediate state with a pair present cannot be used: the electron of the pair would be created in its final state, but this state is the same as the initial state and is already occupied. On the other hand, the presence of the electron in a bound state will affect the possibility of some of the transitions involved in the Delbrück scattering. The question as to whether the interaction of electrons in an atom has such an effect that Waller's results would have to be modified has been considered by Arnous; in a brief note[7] based on extensive calculations he concludes that such a modification is not required. In the present situation, in which various questions have been raised but only brief notes have been published, it appears that a general discussion of the relations of incoherent and coherent scattering, Delbrück scattering, and vacuum polarization may be of interest.

The writer has been particularly concerned to consider these matters in connection with the general question of bound states in positron theory. It has usually been taken for granted that calculations in positron theory could be based on either free-electron wave functions or on the wave functions of a bound electron in a given field, as might be convenient. Complete but rather involved proofs of the equivalence of the two methods for a number of important cases were given by Furry and Oppenheimer.[8] It afterward[9] ap-

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[1] P. A. M. Dirac, Proc. Roy. Soc. **126A**, 360 (1930).
[2] I. Waller, Z. Physik **61**, 837 (1930).
[3] M. Delbrück, Z. Physik **84**, 144 (1933).
[4] N. Kemmer and G. Ludwig, Helv. Phys. Acta **10**, 182 (1937).
[5] A. Achieser and I. Pomerantchuk, Physik. Z. Sowjetunion **11**, 478 (1937).
[6] O. Halpern and H. Hall, Phys. Rev. **75**, 910, 1322A (1949).
[7] E. Arnous, Phys. Rev. **77**, 149 (1950).
[8] W. H. Furry and J. R. Oppenheimer, Phys. Rev. **45**, 245 (1934).
[9] W. H. Furry and J. R. Oppenheimer Phys. Rev. **45**, 343 (1934).

peared possible to dispense with such proofs by remarking that the simple subtraction procedure of primitive positron theory could have no effect on the validity of the usual transformation theory. It turned out, however, that the simple subtraction procedure was neither so harmless nor so adequate as at first supposed, since it gave rise to difficulties with the Lorentz- and gauge-transformation properties of the theory. Moreover, there are clearly differences between the free-particle and bound-particle methods in regard to the phenomena of vacuum polarization. Thus it seems desirable to reconsider the question of the bound-particle method from the point of view of Schwinger's reformulation of the theory, in which the invariance properties are kept manifest and no subtraction procedure such as those of early positron theories is used. Such a reconsideration not only provides for the usual cases of interest a derivation that is rigorous without being excessively involved; it also provides a framework for the treatment of exceptional cases that may arise.

The section immediately following is, accordingly, devoted to a development of the bound-state method as a modification of Schwinger's method of the interaction representation. This is accomplished by a canonical transformation, and the various formulas of the theory are obtained in the resulting bound-state representation. The third section is concerned with the problem of vacuum polarization as it appears in the bound-state representation, and with charge-renormalization. In its present context the task of renormalization has a difficult aspect and is dealt with in a rather crude and summary fashion; the procedure followed is adequate for the treatment of scattering of light with $h\nu \ll mc^2$ by an electron moving in the field of one or more arbitrary small sources and also for scattering with an arbitrary value of $h\nu$ if the field is a Coulomb field. The last section contains the application to scattering and a discussion of the physical meaning of the result.

II. THE BOUND-STATE REPRESENTATION

We proceed to introduce a representation in which the action on the electrons and positrons of a classically prescribed external field with potentials $A_\mu^{(e)}(x)$ is included in the equations of motion of the wave function ψ and its adjoint $\psi^\dagger$, but the interaction between the particles and the dynamical electromagnetic field is taken into account in the dependence of the Schrödinger functional $\Psi'[\sigma]$ on the space-like surface σ. This is a modified type of interaction representation, differing from the standard interaction representation in that bound states of the one-electron problem, rather than free-particle states, provide the bases for the construction of the wave function operators ψ and $\psi^\dagger$.

The arguments used are for the most part closely analogous to those of Sec. 2 of Schwinger's first paper[10] on quantum electrodynamics, and much detail can, accordingly, be omitted here. The notation is that of Schwinger's paper, which is referred to as **I**.

The equation of motion and supplementary condition for the Schrödinger functional $\Psi[\sigma]$ of the interaction representation can be taken in the forms

$$i\hbar c\delta\Psi[\sigma]/\delta\sigma(x) = \{-(1/c)j_\mu(x)[A_\mu(x)+A_\mu^{(e)}(x)]\}\Psi[\sigma], \quad (1)$$

$$\left\{[\partial A_\mu(x')/\partial x_\mu']-(1/c)\times\int_\sigma D(x'-x)j_\mu(x)d\sigma_\mu\right\}\Psi[\sigma]=0. \quad (2)$$

It is here conveniently assumed that the classically prescribed potentials $A_\mu^{(e)}$ satisfy

$$\partial A_\mu^{(e)}/\partial x_\mu=0. \quad (3)$$

It has been shown by Schwinger[11] that these equations are related by a canonical transformation to a perhaps more naturally accepted set in which the classically prescribed quantities are the source-currents $j_\mu^{(e)}(x)$ instead of the potentials $A_\mu^{(e)}(x)$.

We now make the canonical transformation

$$\Psi[\sigma]=V[\sigma]\Psi'[\sigma], \quad (4)$$

where $V[\sigma]$ is taken to satisfy the equation

$$i\hbar c\delta V[\sigma]/\delta\sigma(x)=-(1/c)j_\mu(x)A_\mu^{(e)}(x)V[\sigma] \quad (5)$$

and commutes with $A_\mu(x)$. The new equation of motion and supplementary condition are found to be

$$i\hbar c\delta\Psi'[\sigma]/\delta\sigma(x)=-(1/c)\mathbf{j}_\mu(x)A_\mu(x)\Psi'[\sigma], \quad (6)$$

$$\left\{[\partial A_\mu(x')/\partial x_\mu']-(1/c)\times\int_\sigma D(x'-x)\mathbf{j}_\mu(x)d\sigma_\mu\right\}\Psi'[\sigma]=0, \quad (7)$$

where

$$\mathbf{j}_\mu(x)=V^{-1}[\sigma]j_\mu(x)V[\sigma]=-(iec/2)[\psi_\alpha,\psi_\beta^\dagger](\gamma_\mu)_{\beta\alpha} \quad (8)$$

with

$$\psi(x)=V^{-1}[\sigma]\psi(x)V[\sigma],\quad \psi^\dagger(x)=V^{-1}[\sigma]\psi^\dagger(x)V[\sigma]. \quad (9)$$

Here the point x lies on the surface σ.

In obtaining the equations of motion of ψ and $\psi^\dagger$ we shall depart from exact analogy with the procedure of **I**. It has been widely recognized that the argument of **I** lacks generality in that it is restricted to cases in which the operator occurring in the equation of motion of the transformation matrix—Eq. (5), or (**I**, 2.5)—does not contain derivatives. This condition is of course satisfied in the case of **I** and in our present case, though not in meson theories. It has been pointed out, however, by

[10] J. Schwinger, Phys. Rev. **74**, 1439 (1948), referred to in the text as **I**.

[11] J. Schwinger, Phys. Rev. **76**, 790 (1949).

Belinfante[12] that, even in the case of **I**, the equation (**I**, 2.9) used in obtaining the new equations of motion of the operators is not of really general validity. We here set up an equation not subject to this objection.

It is proved in **I** that the validity of an equation of the form of (5) (or (**I**, 2.5)) has as a consequence that the transformed quantity

$$\mathbf{F}(x)=V^{-1}[\sigma]F(x)V[\sigma], \tag{10}$$

like the original quantity $F(x)$, depends only on x and not on the particular choice of space-like surface σ containing x; this result actually depends on the absence of derivatives in the operator appearing in Eq. (5) (or (**I**, 2.15)). It follows that derivatives of $\mathbf{F}(x)$ can be calculated from

$$\partial\mathbf{F}/\partial x_\nu=\lim_{a\to 0}(1/a)\times\{V^{-1}[(\sigma')_{\nu a}]F(x_\mu+\delta_{\mu\nu}a)V[(\sigma')_{\nu a}]-V^{-1}[\sigma]F(x)V[\sigma]\}, \tag{11}$$

where $(\sigma')_{\nu a}$ is the surface obtained by displacing σ rigidly by the amount a in the direction of x_ν. The change in $V[\sigma]$ produced by such a displacement is

$$\begin{aligned}\delta V[\sigma]&=V[(\sigma')_{\nu a}]-V[\sigma]\\&=a\int_\sigma\{\delta V[\sigma]/\delta\sigma(x')\}d\sigma_\nu'+O(a^2)\\&=(ia/\hbar c^2)\int_\sigma j_\mu(x')A_\mu^{(e)}(x')V[\sigma]d\sigma_\nu'+O(a^2).\end{aligned}$$

We find that

$$\partial\mathbf{F}/\partial x_\nu=V^{-1}[\sigma]\Big\{(\partial F/\partial x_\nu)+(i/\hbar c^2)\times\int_\sigma[F(x),j_\mu(x')A_\mu(x')]d\sigma_\nu'\Big\}V[\sigma]. \tag{12}$$

This equation differs from an exact analogue of Eq. (**I**, 2.9) essentially by the interchange of the arguments x and x' in the integrand. It is generally valid as a consequence of Eqs. (5) and (10), whereas Eq. (**I**, 2.9) gives incorrect results[12] in some cases in which $F(x)$ contains derivatives.

From Eqs. (9) and (12) and the equations of motion and commutation relations of the operators ψ, $\psi^\dagger$ in the interaction representation, the equations of motion of ψ, $\psi^\dagger$ are readily obtained. The argument resembles in a general way that of Eqs (**I**, 2.13–16) and gives the results

$$\{\gamma_\mu[(\partial/\partial x_\mu)-(ieA_\mu^{(e)}/\hbar c)]+\kappa_0\}\psi=0, \tag{13}$$

$$[(\partial/\partial x_\mu)+(ieA_\mu^{(e)}/\hbar c)]\psi^\dagger\gamma_\mu-\kappa_0\psi^\dagger=0. \tag{14}$$

[12] F. J. Belinfante, Phys. Rev. **76**, 66 (1949), especially footnotes 5 and 23.

The energy-momentum four-vector of the fields ψ and A_μ, taken apart from their mutual interaction, may be written $P_\mu{}^B$, where the superscript emphasizes that ψ satisfies Eq. (13) for bound electrons. The expression for $P_\mu{}^B$ is formally the same as is given in Eqs. (**I**, 1.55) and (**I**, 1.26), with ψ, $\psi^\dagger$ replacing ψ, $\psi^\dagger$. The property of a time-like component of this vector as a displacement operator is used in the definition of the vacuum state.[13] The complete energy-momentum four-vector, including the effects of interaction, is

$$P_\mu[\sigma]=P_\mu{}^B+(1/c^2)\int_\sigma \mathbf{j}_\nu A_\nu d\sigma_\mu. \tag{15}$$

In establishing these facts it is convenient to use the transformation from the constant state-functional Φ of the Heisenberg representation to the functional $\Psi'[\sigma]$:

$$\Psi'[\sigma]=R[\sigma]\Phi, \tag{16}$$

$$i\hbar c\delta R[\sigma]/\delta\sigma(x)=-(1/c)\mathbf{j}_\mu A_\mu R[\sigma]. \tag{17}$$

The consistency of these relations with Eqs. (1)–(5) is readily established. The argument leading to Eq. (15) is then similar to that of Eqs. (**I**, 2.45–52); the use of an equation analogous to Eq. (12), rather than to (**I**, 2.9), removes the need for a doubtful step like that in (**I**, 2.51).

The commutation relations of the operators ψ and $\psi^\dagger$ are not the same as those of the operators ψ, $\psi^\dagger$ of the interaction representation. It is through the difference in the commutation relations that the effects of the external potentials $A_\mu^{(e)}$ appear in subsequent calculations. The physical reason for the difference in the commutation relations is that ψ and $\psi^\dagger$ contain the effects of the external field, as shown in Eqs. (13) and (14). Formally, it is seen that the commutation relations of ψ, $\psi^\dagger$ at points x and x' are not the same as those of ψ, $\psi^\dagger$ because in the transformation of Eq. (9) one has in general to use different surfaces σ, σ' for the points x, x': thus, the transformation is not a mere similarity transformation.

In the construction of the commutation relations of ψ, $\psi^\dagger$ it is convenient to start with the special case in which x and x' are on the same surface σ, so that the transformation is just one of similarity. Here and in following calculations we need consider only plane surfaces σ, with $x_4=ict=$constant. For $t=t'$, then, the relations for ψ, $\psi^\dagger$ are the same as those for ψ, $\psi^\dagger$, and we have

$$\{\psi_\alpha(x),\psi_\beta^*(x')\}=\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'),\qquad t=t', \tag{18}$$

$$\{\psi_\alpha(x),\psi_\beta(x')\}=\{\psi_\alpha^*(x),\psi_\beta^*(x')\}=0,\qquad t=t'. \tag{19}$$

The operator ψ for time $t=0$ can be expanded in terms of any complete orthonormal set of functions of the position $\mathbf{r}$; for convenience we use the notation of the dis-

[13] J. Schwinger, Phys. Rev. **75**, 651 (1949). Here particularly Sec. 1. Referred to in the text as **II**.

crete case:

$$\psi_\alpha(\mathbf{r};0)=\sum_{(s)} a_{(s)}\psi_{(s)\alpha}(\mathbf{r});$$
$$\psi_\alpha^*(\mathbf{r};0)=\sum_{(s)} a_{(s)}^*\psi_{(s)\alpha}^*(\mathbf{r}). \tag{20}$$

The coefficients $a_{(s)}$, $a_{(s)}^*$ are operators; from Eqs. (18)–(20) it follows that

$$\{a_{(r)}, a_{(s)}^*\}=\delta_{(r)(s)}; \quad \{a_{(r)}, a_{(s)}\}=\{a_{(r)}^*, a_{(s)}^*\}=0. \tag{21}$$

Since $a_{(s)}$, $a_{(s)}^*$ are independent of $\mathbf{r}$, it is natural to take them independent of t also; we then set

$$\psi_\alpha(\mathbf{r};t)=\sum_{(s)} a_{(s)}\psi_{(s)\alpha}(\mathbf{r};t);$$
$$\psi_\alpha^*(\mathbf{r};t)=\sum_{(s)} a_{(s)}^*\psi_{(s)\alpha}^*(\mathbf{r};t) \tag{22}$$

with the time-dependences of $\psi_{(s)}$ and $\psi_{(s)}^*=\psi_{(s)}^\dagger\gamma_4$ given by Eqs. (13) and (14). From these equations it follows that

$$(\partial/\partial x_\nu)\psi_{(r)}^\dagger\gamma_\nu\psi_{(s)}=0, \tag{23}$$

and thus by the usual argument that the orthonormal property of the functions $\psi_{(s)}$ persists in time; accordingly, the use of $a_{(s)}$, $a_{(s)}^*$ independent of time is consistent with the general validity of Eqs. (18) and (19). Then from Eqs. (21) and (22) we have in general

$$\{\psi_\alpha(x), \psi_\beta^\dagger(x')\}=\sum_{(s)} \psi_{(s)\alpha}(x)\psi_{(s)\beta}^\dagger(x'), \tag{24}$$

$$\{\psi_\alpha(x), \psi_\beta(x')\}=\{\psi_\alpha^\dagger(x), \psi_\beta^\dagger(x')\}=0. \tag{25}$$

The sum appearing on the right-hand side of Eq. (24) is a generalization of Schwinger's function $S(x-x')$, as changed by the effect of the external field $A_\mu^{(e)}$:

$$\sum_{(s)} \psi_{(s)\alpha}(x)\psi_{(s)\beta}^\dagger(x') \longrightarrow -iS_{\alpha\beta}(x-x') \text{ for } A_\mu^{(e)}\longrightarrow 0. \tag{26}$$

The sum also plays the same role as the function $S(x-x')$ in solving the boundary-value problem of the equations of motion; in analogy to Eqs. (I, 2.23) and (I, 2.27) we have

$$\psi(\mathbf{r};t)=\int \sum_{(s)} \psi_{(s)}(\mathbf{r};t)\psi_{(s)}^\dagger(\mathbf{r}';t')\gamma_4\psi(\mathbf{r}';t')dv', \tag{27}$$

$$\psi^\dagger(\mathbf{r};t)=\int dv'\psi^\dagger(\mathbf{r}';t')\gamma_4\sum_{(s)} \psi_{(s)}(\mathbf{r}';t')\psi_{(s)}^\dagger(\mathbf{r};t). \tag{28}$$

In the bound-state representation a gauge-transformation may affect either the external potentials $A_\mu^{(e)}$, the dynamical potentials A_μ, or both. The transformation equations are

$$A_\mu^{(e)}\longrightarrow A_\mu^{(e)}-\partial\Lambda^{(e)}/\partial x_\mu,$$
$$\psi\longrightarrow\exp(-ie\Lambda^{(e)}/\hbar c)\psi,$$
$$\psi^\dagger\longrightarrow\exp(ie\Lambda^{(e)}/\hbar c)\psi^\dagger \tag{29}$$

and

$$A_\mu\longrightarrow A_\mu-\partial\Lambda/\partial x_\mu, \quad \Psi'[\sigma]\longrightarrow e^{-iG[\sigma]}\Psi'[\sigma] \tag{30}$$

with

$$G[\sigma]=(1/\hbar c^2)\int_\sigma \mathbf{j}_\mu(x)\Lambda(x)d\sigma_\mu. \tag{31}$$

The proof of invariance under the latter transformation is entirely analogous to Schwinger's discussion, Eqs. (I, 2.39–43). The essential lemma, Eq. (I, 2.34), follows readily, for the sufficient case of plane surfaces, from Eqs. (24)–(28).

Charge-conjugate wave functions ψ', $\psi^{\dagger\prime}$ can be defined by the usual relations

$$\psi'(x)=C\psi^\dagger(x), \quad \psi^{\dagger\prime}(x)=C^{-1}\psi(x), \tag{32}$$

with

$$C^{-1}\gamma_\mu C=-\gamma_\mu{}^T. \tag{33}$$

The matrix C commutes with $V[\sigma]$, so that the relation to the operators of the interaction representation is

$$\psi'=V^{-1}[\sigma]\psi' V[\sigma], \quad \psi^{\dagger\prime}=V^{-1}[\sigma]\psi^{\dagger\prime}V[\sigma]. \tag{34}$$

The equations of motion of the charge-conjugate operators differ from Eqs. (13) and (14) by a change of sign of the factor e, and their commutation relations differ from those of the original operators in a corresponding way. The lack of absolute symmetry between ψ and ψ' brought about by the external field has physical consequences, among them vacuum polarization (following section).

The definition of the vacuum state given by Schwinger in his second paper[13] on quantum electrodynamics can reasonably be used in the bound-state representation only for certain kinds of external fields. Already from the possibility of gauge transformations of the type (29), which do not exist in the interaction representation, it is seen that the separation of the operators ψ and $\psi^\dagger$ into parts,

$$\psi=\psi^{(+)}+\psi^{(-)}, \quad \psi^\dagger=\psi^{\dagger(+)}+\psi^{\dagger(-)}, \tag{35}$$

containing respectively the positive-frequency and negative-frequency parts of the time-dependences, can have no absolute significance. A natural physical meaning can be given to this separation only if there is a gap in the spectrum of the time dependences of the operators. Such a gap can be expected to have width about 2 mc^2, in the energy scale. As a reasonable choice[14] of the gauge $\Lambda^{(e)}$ we take such potentials $A_\mu^{(e)}$ that the location of the gap is about the same for $\psi^\dagger$ as for ψ, whereupon it can be expected that zero frequency falls about the middle of the gap. The vacuum state with Schrödinger functional Ψ_0' is then defined by

$$\psi^{(+)}(x)\Psi_0'=0, \quad \psi^{\dagger(+)}(x)\Psi_0'=0. \tag{36}$$

The energy of the vacuum state is then lower than that of any other state by about the amount mc^2. The fact that the definition of a stable vacuum state is possible for only a restricted class of fields was pointed out in the early days of positron theory.[15]

[14] Of course, this is by no means a unique specification of the gauge. Moreover, while convenient in that it makes possible the definition of the vacuum state as that of minimum energy, the restriction is not necessary: for example, the fact that the addition of a large constant to the scalar potential puts a premium on states having large net charge has a natural and clear physical meaning.

[15] Reference 8, p. 255.

We must from now on restrict our attention to fields for which this procedure is available. Having chosen some suitable gauge for any particular given field, we can choose the basic sets of functions $\psi_{(s)}$ so that the sequence of labels (s) can be divided into two separate lists, written s and σ, such that

$$\psi^{(+)}=\sum_s a_s\psi_s;\quad \psi^{(-)}=\sum_\sigma a_\sigma\psi_\sigma. \tag{37}$$

It then follows that

$$\psi^{\dagger(+)}=\sum_\sigma a_\sigma^*\psi_\sigma^\dagger;\quad \psi^{\dagger(-)}=\sum_s a_s^*\psi_s^\dagger. \tag{38}$$

The states s are positive energy states of the electron in the given field; the states σ, negative energy states.

The way in which the interpretation of the operators $a_{(s)}$, $a_{(s)}^*$ presents itself naturally in the bound-state representation is shown in the following section, in connection with the evaluation of vacuum expectation values.

III. POLARIZATION OF THE VACUUM: RENORMALIZATION

The current four-vector, Eq. (8), can be written in either of two equivalent forms:

$$\begin{aligned}\mathbf{j}_\mu&=-(iec/2)[\psi_\alpha,\psi_\beta^\dagger](\gamma_\mu)_{\beta\alpha}\\&=(iec/2)[\psi_\alpha',\psi_\beta^{\dagger\prime}](\gamma_\mu)_{\beta\alpha}.\end{aligned} \tag{39}$$

Because of the differences in the behavior of ψ and ψ', it does not follow, as it did in the interaction representation (Eq. (II, 1.72))[16] that the vacuum expectation value $\langle\mathbf{j}_\mu(x)\rangle_0$ must vanish; it can only be concluded that, if an expansion in powers of e can be used, the series for $\langle\mathbf{j}_\mu(x)\rangle_0$ contains only terms with the even powers.

The formal evaluation of $\langle\mathbf{j}_\mu(x)\rangle_0$ in terms of the definition of the vacuum state, Eq. (36), involves just the algebraic steps used by Schwinger, Eqs. (II, 1.66–67). The result is

$$\langle\mathbf{j}_\mu(x)\rangle_0=-(iec/2)\langle\{(\psi_\alpha^{(+)}-\psi_\alpha^{(-)}),\psi_\beta^\dagger\}\rangle_0(\gamma_\mu)_{\beta\alpha}. \tag{40}$$

Because the condition (36) holds for every point x, we can conclude from Eqs. (36)–(38) that

$$a_s\Psi_0'=0,\quad a_\sigma^*\Psi_0'=0. \tag{41}$$

From this and Eq. (21) we find that

$$\begin{aligned}&\langle a_{(r)}^*a_{(s)}\rangle_0=\langle a_{(s)}a_{(r)}^*\rangle_0=0,\quad (r)\neq(s),\\&\langle a_s^*a_s\rangle_0=\langle a_\sigma a_\sigma^*\rangle_0=0,\\&\langle a_sa_s^*\rangle_0=\langle a_\sigma^*a_\sigma\rangle_0=1.\end{aligned} \tag{42}$$

If the operators $a_{(s)}$, $a_{(s)}^*$ are given their original interpretation[17] for a many-electron theory, these formulas (42) correspond to the original Dirac suggestion that the vacuum has all negative energy states filled. It is to be noted, however, that Eq. (42) follows, without any special assumption, simply from the definition of the vacuum state as the state which in a definite physical sense has minimum energy, and that the formulas hold in the bound-state representation as well as in the free-particle or interaction representation.

States $a_s^*\Psi_0'$ and $a_\sigma\Psi_0'$ have energies higher than that of the state Ψ_0' by at least about mc^2; on the other hand $a_sa_s^*\Psi_0'=a_\sigma^*a_\sigma\Psi_0'=\Psi_0'$. The interpretation of these facts is that the operators a_s^* and a_σ refer to the creation of an electron and a positron, respectively, while a_s and a_σ^* are the corresponding destruction operators. That the effect of these operators on the charge agrees with this interpretation follows from arguments of a familiar kind, given for instance by Schwinger in Eqs. (II, 3.54–55).[18] The construction of corresponding arguments in the bound-state representation, by the use of Eqs. (24)–(28), presents no difficulty.

From Eqs. (40) and (42) we find that

$$\langle\mathbf{j}_\mu(x)\rangle_0=-(iec/2)\{\textstyle\sum_s\psi_s^\dagger\gamma_\mu\psi_s-\sum_\sigma\psi_\sigma^\dagger\gamma_\mu\psi_\sigma\}. \tag{43}$$

These sums are divergent. The possibility of obtaining a definite result by making active use of the postulate that the theory is gauge invariant is illustrated in the work of Schwinger.[19] Another way to make the result definite is to use a program of modifying the formulas and then obtaining the value in a suitable limit, a procedure known as "regularization."[20] The result contains in order e^2 a logarithmically divergent but gauge-invariant term, whose removal must be accomplished by charge-renormalization; that is, by its being regarded as physically indistinguishable from the classically prescribed current $j_\mu^{(e)}=-c\partial^2A_\mu^{(e)}/\partial x_\nu^2$. We shall not attempt a generalization of these methods for application to Eq. (43), but shall only show, in order e^2, how the use of the Born approximation to evaluate $\langle\mathbf{j}_\mu(x)\rangle_0$ leads to the results already known in the interaction representation. We shall then concern ourselves with the problem of effecting the renormalization.

Before proceeding to show the calculation in Born approximation we must remark on the actual content of Eq. (43) and its relation to other quantities that have been calculated. To the order e^2, the expression given here contains the whole of the vacuum polarization. In higher orders part of the polarization comes from interaction between the electron-positron field and the dynamical field A_μ. This part is not included in the present expression; it has been studied in the approximation e^4 by Jost and Luttinger,[21] who find that there is also in this order a logarithmically divergent gauge-invariant term that requires removal by charge-renormalization. The higher order contributions that are included in Eq. (43) are non-linear in $A_\mu^{(e)}$, being of orders $e^4(A_\mu^{(e)})^3$, $e^6(A_\mu^{(e)})^5$, $\cdots$. These terms have been

[16] The symbol II is used to refer to Schwinger's second paper, reference 13.

[17] P. Jordan and E. Wigner, Z. Phys. 47, 631 (1928); W. Pauli, *Handb. d. Physik* 24/1, p. 199.

[18] Through a misprint Eq. (II, 3.54) has in the integrand a square bracket which should be a curly bracket (anticommutator).

[19] Reference 16, Sec. 2; also reference 11, Appendix.

[20] W. Pauli and F. Villars, Rev. Modern Phys. 21, 434 (1949).

[21] R. Jost and J. M. Luttinger, Helv. Phys. Acta 23. 201 (1950).

studied by Källén,[22] who finds that in order $e^4(A_\mu^{(e)})^3$ there are divergences completely removable by "realistic regularization," and that in higher orders there are no divergences. It should be remarked that Eq. (43) is a valid expression for this part of the vacuum polarization also in cases in which the Born approximation is not admissible, although the actual evaluation without use of the Born approximation could be expected to be very laborious.

In order to apply the Born approximation to the evaluation of $\langle \mathbf{j}_\mu(x)\rangle_0$ as given in Eq. (43), we expand $\psi_{(s)}$ and $\psi_{(s)}^\dagger$ in powers of the charge e:

$$\begin{aligned}\psi_{(s)}&=\psi_{(s)}+e\psi_s^{(1)}+\cdots,\\ \psi_{(s)}^\dagger&=\psi_{(s)}^\dagger+e\psi_s^{\dagger(1)}+\cdots.\end{aligned}\tag{44}$$

The term of order e in $\langle \mathbf{j}_\mu(x)\rangle_0$ is obtained by replacing $\psi_{(s)}$, $\psi_{(s)}^\dagger$ by the free-particle functions $\psi_{(s)}$, $\psi_{(s)}^\dagger$, and this term vanishes, as shown by Schwinger in Eqs. (**II**, 1.72–73). The term of order e^2 is

$$\langle \mathbf{j}_\mu(x)\rangle_0^{(2)}=-(ie^2c/2)\{\textstyle\sum_s(\psi_s^\dagger\gamma_\mu\psi_s^{(1)}+\psi_s^{\dagger(1)}\gamma_\mu\psi_s)-\sum_\sigma(\psi_\sigma^\dagger\gamma_\mu\psi_\sigma^{(1)}+\psi_\sigma^{\dagger(1)}\gamma_\mu\psi_\sigma)\}.\tag{45}$$

The function $\psi_{(s)}^{(1)}$ is the solution of the inhomogeneous equation

$$[(\gamma_\mu\partial/\partial x_\mu)+\kappa_0]\psi_{(s)}^{(1)}=(i/\hbar c)\gamma_\nu A_\nu^{(e)}\psi_{(s)}\tag{46}$$

given by the expression

$$\psi_{(s)}^{(1)}(x)=-(i/\hbar c)\times\int\bar{S}(x-x')\gamma_\nu A_\nu^{(e)}(x')\psi_{(s)}(x')d\omega',\tag{47}$$

where the function[23]

$$\bar{S}(x-x')=[(\gamma_\mu\partial/\partial x_\mu)-\kappa_0]\bar{\Delta}(x-x')\tag{48}$$

satisfies the condition (Eq. (**II**, A.6)),

$$[(\gamma_\mu\partial/\partial x_\mu)+\kappa_0]\bar{S}(x)=-\delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4).\tag{49}$$

On substituting Eq. (47) and the corresponding formula for $\psi_{(s)}^{\dagger(1)}=\psi_{(s)}^{(1)*}\gamma_4$ into Eq. (45), we see that the result depends on a sum whose value is

$$\textstyle\sum_s\psi_s(x)\psi_s^\dagger(x')-\sum_\sigma\psi_\sigma(x)\psi_\sigma^\dagger(x')=-S^{(1)}(x-x').\tag{50}$$

That this sum of products of *free-particle* functions has the value given can be seen by comparing Eqs. (39) and (43) with Eq. (**II**, 1.68). The value of $\langle \mathbf{j}_\mu(x)\rangle_0^{(2)}$ is found to be

$$\langle \mathbf{j}_\mu(x)\rangle_0^{(2)}=(e^2/\hbar)\int d\omega' Tr\{S^{(1)}(x'-x)\gamma_\mu\bar{S}(x-x')\gamma_\nu+\bar{S}(x'-x)\gamma_\mu S^{(1)}(x-x')\gamma_\nu\}A_\nu^{(e)}(x').\tag{51}$$

This is identical with Schwinger's expression given by Eqs. (**II**, 2.19) and (**II**, 2.10).

The expression (43) for $\langle \mathbf{j}_\mu(x)\rangle_0$ contains entangled in it the logarithmically infinite part, proportional to $j_\mu^{(e)}(x)$, which must be removed by renormalization, and other parts, finite and not just proportional to $j_\mu^{(e)}(x)$, which are regarded as having physical meaning. The task of disentangling these parts is a complicated one, and is scarcely to be undertaken at all except with the use of the Born approximation. We shall be content here to point out that for some purposes, to be illustrated by the application to scattering, the disentangling is not required.

The structure of $\langle \mathbf{j}_\mu\rangle_0$ can be symbolized by writing

$$\langle \mathbf{j}_\mu\rangle_0=J_\mu^{\log}+J_\mu',\tag{52}$$

where $J_\mu^{\log}$ is to be removed by renormalization. For any field from finite sources in the finite region of space the remaining part J_μ' satisfies

$$\int J_\mu' dv=0.\tag{53}$$

For a point source, or a source of dimensions small compared with $\hbar/mc$ the extent of the distribution of J_μ' is of the order of $\hbar/mc$. If there are a number of such sources separated from each other by distances large compared with $\hbar/mc$, Eq. (53) holds also with the integral extended over a suitable region around any single source. The emission, absorption, and scattering of radiation depend on quantities

$$\int \mathbf{j}_\mu e^{i\mathbf{k}\cdot\mathbf{r}}dv.\tag{54}$$

If $A_\mu^{(e)}$ is due to well-separated small sources, the contribution from J_μ' to such quantities can be calculated from a sum of integrals over regions of extent about $\hbar/mc$ around the separate sources. For light of long wavelength, $|\mathbf{k}|\ll mc/\hbar$, the exponential factor in each integral is essentially constant; thus by Eq. (53) all the integrals vanish, and J_μ' does not contribute to the scattering. In this case the renormalization is accomplished with sufficient accuracy by simply replacing $\mathbf{j}_\mu(x)$ with $\mathbf{j}_\mu(x)-\langle \mathbf{j}_\mu(x)\rangle_0$. For light of shorter wavelength the contribution of J_μ' to integrals such as Eq. (54) may not be negligible, and may have to be taken into account as a part of the Delbrück scattering.

For the case of the Coulomb field of a stationary charge, and indeed for any *truly static*[24] spherically symmetrical electric field, the contribution of J_μ' is zero for all wavelengths. Because of the transverse character of light waves, only the space-components of j_μ need be considered. By symmetry the direction of the vector $\mathbf{J}'$ is radial. In a static field the functions $\psi_{(s)}$ have strictly harmonic time-dependences, so that $\mathbf{J}'$ is constant in time. It is then zero, since a nonvanishing value would

[22] G. Källén, Helv. Phys. Acta **22**, 637 (1949).

[23] The choice of this particular solution of Eq. (49), which has a certain symmetry as between past and future, corresponds to the fact that the field $A_\mu^{(e)}$, belonging to the class specified at the end of Sec. II, is incapable of producing real pairs: see reference 13, pp. 659–660. A calculation like the present one, but approached from a different point of view, has just been published by G. Källén, Arkiv för Fysik **2**, 187 (1950).

[24] A field produced by screening is, of course, static only to a certain approximation.

mean nonconservation of charge. (J_4', on the other hand, is not zero.) Thus, for the Coulomb field the procedure we shall use gives correct results for the scattering at all frequencies, and there is no Delbrück scattering apart from that calculated in the next section.

According to Eqs. (39) and (22) the original expression for the current is

$$\mathbf{j}_\mu(x)=(iec/2)\times\sum_{(r)(s)}\psi_{(s)}{}^\dagger(x)\gamma_\mu\psi_{(r)}{}^\dagger(x)(a_{(s)}{}^*a_{(r)}-a_{(r)}a_{(s)}{}^*). \quad (55)$$

From this and Eq. (43) we get as the expression for the approximately renormalized current to be used in the calculations of the next section:

$$\mathbf{j}_\mu(x)-\langle\mathbf{j}_\mu(x)\rangle_0=(iec/2)\sum_{(r)(s)}\psi_{(s)}{}^\dagger(x)\gamma_\mu\psi_{(r)}(x)\times[a_{(s)}{}^*a_{(r)}-a_{(r)}a_{(s)}{}^*+\delta_{(r)(s)+}-\delta_{(r)(s)-}]. \quad (56)$$

Here $\delta_{(r)(s)+}$ equals 1 or 0, according as (r) and (s) do or do not take identical values belonging to the list s of positive energy states, and $\delta_{(r)(s)-}$ has an analogous definition. Then by Eq. (21)

$$\mathbf{j}_\mu(x)-\langle\mathbf{j}_\mu(x)\rangle_0=iec\sum_{(r)(s)}\psi_{(s)}{}^\dagger(x)\gamma_\mu\psi_{(r)}{}^\dagger(x)(a_{(s)}{}^*a_{(r)}-\delta_{(r)(s)-}). \quad (57)$$

In calculating it is convenient to write the factor $(a_{(s)}{}^*a_{(r)}-\delta_{(r)(s)-})$ as a single term. The way of doing this depends on whether (s) takes a positive-energy value s or a negative-energy value σ, and also on whether (r) takes a value r or a value ρ. The four possibilities are

$$a_{(s)}{}^*a_{(r)}-\delta_{(r)(s)-}=\begin{cases}a_s{}^*a_r\\ a_s{}^*a_\rho\\ a_\sigma{}^*a_r\\ -a_\rho a_\sigma{}^*\end{cases}. \quad (58)$$

The expressions written in Eqs. (57) and (58) are formally the same as those obtained by the simple subtraction procedure of primitive positron theory.[25] The meaning of these expressions and the status assigned to them are, however, entirely different from what they were in the earlier version. In primitive positron theory such formulas were to be applied only to free-particle states, or states defined in an "even" field in the sense of Schrödinger;[26] here they are written for particle states defined in any field that permits a reasonable definition of the vacuum state of the system. In primitive positron theory the formulas were obtained by a postulated subtraction procedure, and were originally believed to be definitively correct; the introduction of more elaborate subtraction procedures[27] removed some inconsistencies, but genuine consistency and adequacy were not achieved by the subtraction methods.[28] In the present argument the expressions in question are obtained as an approximation to the results of a physically reasonable process of renormalization, and there are strong reasons[29] to believe in the general consistency of the method. The expressions given represent only a crude way of estimating the results of renormalization; but we have been able to designate a class of cases for which they should be adequate, and a definite theoretical basis exists for improving them.

[25] For example, reference 8, pp. 250–251.
[26] E. Schrödinger, Berl. Ber., Jahrgang 1931, p. 63; W. Pauli, *Handb. d. Physik* 24/1, p. 229.
[27] W. Heisenberg, Z. Phys. **90**, 209 (1934).

IV. APPLICATION TO SCATTERING

When the formula (57) is used as an approximately correct expression for the renormalized current, the second-order term of the S-matrix[30] is written in the form

$$(-ie/\hbar)^2\int_{-\infty}^{\infty}dt\int_{-\infty}^{t}dt'\int\int dvdv'\times\sum_{(m)(n)(r)(s)}\psi_{(n)}{}^\dagger(x)\gamma_\kappa\psi_{(r)}(x)A_\kappa(x)\cdot\psi_{(s)}{}^\dagger(x')\gamma_\lambda\psi_{(m)}(x')A_\lambda(x')\cdot\{a_{(n)}{}^*a_{(r)}-\delta_{(n)(r)-}\}\{a_{(s)}{}^*a_{(m)}-\delta_{(s)(m)-}\}. \quad (59)$$

The quantities $a_{(l)}{}^*$, $a_{(l)}$ are operators affecting the occupation numbers of the various bound states of electrons and positrons, and the potentials A_μ contain linear combinations of operators affecting the number of photons.[31] The expression (59) thus comprises a list of various second-order processes of interaction between the particles and radiation, and indicates how the corresponding probability amplitudes are to be calculated. The calculation for any particular kind of transition is to be performed according to the ordinary method of Dirac for time-dependent perturbations;[32] when the customary use is made of states having harmonic time-dependence, the integration over dt secures conservation of energy and the integration over dt' leads to the usual expression of second-order perturbation theory, with an energy-difference in the denominator.

The expression (59) contains not only an account of scattering, double emission, and double absorption of radiation, but also of the interaction of particles, and of self-energies and the electromagnetic shift of energy-levels. It is not to be used as a basis for calculating these latter effects, however; the approximate way in which the effects of charge-renormalization have been taken into account is adapted primarily only to the problems of scattering, emission and absorption. This type of problem is obtained by choosing in the operators $A_\kappa(x)$

[28] R. Serber, Phys. Rev. **49**, 545 (1936).
[29] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).
[30] F. J. Dyson, Phys. Rev. **75**, 486 (1949), especially p. 489.
[31] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), Chapter IV.
[32] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, 1947), third edition, pp. 172–180.

and $A_\lambda(x')$ terms corresponding to the emission and/or absorption of two distinct photons.

In order to list the various processes it is convenient to separate the expression (59) into its two-particle, one-particle, and vacuum parts. To do this we need consider only the last two factors of the expression, and for each the transcription shown in Eq. (58) can be used. There are then $4\times4=16$ kinds of terms to be considered. The two-particle part is obtained by arranging the four factors so that all creation operators are to the left, and annihilation operators to the right: this assures that the process involves only transitions of existing particles and/or creation of new particles. One-particle parts contain only two factors a and/or a^*, with creation operators to the left, annihilation to the right. Such parts describe two successive transitions of an existing particle, or else two-step pair production or annihilation; the subscripts on a δ-symbol indicate the intermediate state of a particle. Vacuum parts contain no factors a or a^*, and involve particles only virtually; subscripts on δ-symbols indicate the states of particles appearing in the intermediate state of the system.

It suffices to illustrate the process of separation for the kind of term containing the greatest variety of parts:

$$a_\nu^* a_r a_s^* a_\mu = a_s^* a_\nu^* a_r a_\mu + a_\nu^* a_\mu \delta_{rs} = a_s^* a_\mu a_\nu^* a_r - a_s^* a_r \delta_{\mu\nu} - a_\mu a_\nu^* \delta_{rs} + \delta_{\mu\nu}\delta_{rs} \quad (60)$$

(a_s^* and a_μ are operators of creation).

The two-particle parts need not be listed in detail. They describe processes of transition of two distinct existing particles, double pair production, double pair annihilation, pair production or annihilation accompanied by transition of a particle not belonging to the pair, or destruction of a pair and creation of another pair.

The one-particle parts comprise the following:

Two-step pair production: $a_n^* a_\mu \delta_{rs} + a_\rho a_s^* \delta_{\mu\nu}$, (61)

Two-step pair annihilation: $a_\nu^* a_m \delta_{rs} + a_r a_\sigma^* \delta_{\mu\nu}$, (62)

Scattering by an electron: $a_n^* a_m \delta_{rs} - a_s^* a_r \delta_{\mu\nu}$, (63)

Scattering by a positron: $a_\rho a_\sigma^* \delta_{\mu\nu} - a_\mu a_\nu^* \delta_{rs}$. (64)

The single vacuum term is:

Delbrück scattering: $\delta_{\mu\nu}\delta_{rs}$. (65)

The first term of the expression (63) gives to the probability amplitude for scattering a contribution that corresponds to the picture of the electron passing through the intermediate state $s=r$. In the original Dirac theory of the electron, with negative energy intermediate states allowed, the other term would be $a_n^* a_m \delta_{\rho\sigma}$. The positron-theoretic expression (63) contains instead the term $-a_s^* a_r \delta_{\mu\nu}$, indicating a process in which the intermediate state contains a pair. Since the subscripts are mere summation indices, and since we have free choice as to which factor $A_\kappa(x)$, $A_\lambda(x')$ corresponds to which photon—the sum over both choices being taken at the end—the result of the space-integrations in Eq. (59) is the same, apart from sign, for the two theories. The difference in sign visible in the second term of Eq. (63) can be said, in the language of Dirac's suggestion of "filled-up" states, to come from the exchange of the original electron with an electron from the negative energy states. When the time integrations are performed, a second difference of sign is found: in the "filled-state" picture of the positron theory, the transition from a negative-energy state to the final state of the electron occurs first instead of last, and thus the difference of energies of initial and intermediate states has the opposite sign; this energy difference appears in the denominator because of the integration over dt'. Thus, there are in all two changes of sign that cancel each other, and the result of positron theory is identical with that of one-electron theory.[33]

The second term of the expression (63) contains no reference to the non-identity or identity of the states r and s; it contributes just as well to a process in which the electron remains in the state $r=s$ as to one in which the electron changes from state r to another state s. Thus, this expression gives in the positron theory the same result for the scattering as would be found in a one-electron theory, independently of whether the scattering is coherent or incoherent.

It is clear then that the use of the expression (63) leads to a calculation of the scattering of an electron in which the exclusion principle is ignored in intermediate states. The same is true of the use of Eq. (65) to calculate the Delbrück scattering; the expression gives always the same coherent scattering by creation and destruction of all possible virtual pairs, making no reference to whether any states are actually occupied or not.

On the other hand it is certainly correct also to make any calculation of scattering according to the concrete picture of the Dirac method of variation of parameters, considering just the changes in time of the probability amplitudes of different states of the system; indeed the expression (59), as it stands, simply instructs us to do just this in any given case. From this point of view intermediate states are treated on the same basis as initial or final states of the system, the only difference being that their energy is not the same. Thus, it must also be correct to use the exclusion principle in the intermediate states.

There is no contradiction between the two ways of proceeding because the only thing physically observable is the total coherent scattering. In a case in which an electron is present in a state s_1, the use of the exclusion principle in intermediate states requires us to omit the

[33] The occurrence of the two compensating changes of sign was not mentioned by Dirac (reference 1) or Waller (reference 2), and appears not to have been noted in the literature until as late as 1942: H. I. Bhabha and D. Basu, Proc. Indian Acad. Sci. **15A**, 461 (1942).

contribution to the coherent scattering which would be given by $s=r=s_1$ in the second term of Eq. (63); but it also requires us to omit the contribution from Eq. (65) with $s=r=s_1$. Since in this case $a_{s1}{}^*a_{s1}=1=\delta_{s1s1}$ and the two terms have opposite signs, the method that ignores the exclusion principle in intermediate states differs from the other method only by the insertion of two terms whose sum is zero.

In order to take account of the exclusion principle in the general case with particles originally present in an arbitrary number of states,[34] one must replace[35] the last terms of Eqs. (63) and (64) by the following expressions:

$$-a_s{}^*a_r\delta_{\mu\nu}\rightarrow -a_s{}^*a_r(1-\delta_{rs})\delta_{\mu\nu}(1-a_\mu a_\nu{}^*),$$
$$-a_\mu a_\nu{}^*\delta_{rs}\rightarrow -a_\mu a_\nu{}^*(1-\delta_{\mu\nu})\delta_{rs}(1-a_s{}^*a_r)$$

and also the expression (65) by

$$\delta_{\mu\nu}\delta_{rs}(1-a_s{}^*a_r)(1-a_\mu a_\nu{}^*).$$

The sum of the new expressions is not identical with the sum of those given first, but the discrepancy is precisely accounted for by replacing the first term (two-particle part) in the right-hand member of Eq. (60) as follows:

$$a_s{}^*a_\mu a_\nu{}^*a_r\rightarrow a_s{}^*a_\mu a_\nu{}^*a_r(1-\delta_{\mu\nu})(1-\delta_{rs}).$$

This is just the change required to apply the exclusion principle to the intermediate state of the process in question: the pair (μ, s) is to be produced *provided* there is already present a pair (ν, r), which is thereupon annihilated.

All available evidence indicates that the Delbrück scattering given by Eq. (65), which is the same whether or not actual electrons are present, is very small. Using the Born approximation, Kemmer and Ludwig[4] give for the scattering cross section of a Coulomb field for long wavelengths

$$\sigma\approx\text{const.}\cdot(Z/137)^4(h/mc\lambda)^4(e^2/mc^2)^2,\quad \lambda\gg h/mc. \quad (66)$$

Here the numerical constant factor is probably quite small. Achieser and Pomerantchuk[5] give this result, and also a result for short wavelengths:

$$\sigma\approx\text{const.}\cdot(Z/137)^4(\lambda mc/h)^2\ln(h/mc\lambda)\cdot(e^2/mc^2)^2,\quad \lambda\ll h/mc. \quad (67)$$

From these results it can be surmised that the maximum cross section should occur at $\lambda\approx h/mc$ and that even for the largest values of Z it is very small compared to the Thomson cross section $(8\pi/3)(e^2/mc^2)^2$.

Particularly at long wavelengths one is inclined to doubt the validity of the calculation by the Born approximation, which certainly is inadequate for the bound states. An accurate calculation, however, would be extremely laborious. It is thus of some interest to examine a nonrelativistic formula that approximates the effect of bound states and of ionized states in which the electron's speed is not large. The "true scattering" term $(e^2A^2/2mc^2)\psi$ in the Schrödinger equation gives an effect equivalent to that of the transitions between all negative energy levels and any given positive energy state of not too high energy. To get the Delbrück scattering we must sum over all positive energy states. Then, if $\mathbf{k}$, $\mathbf{k}'$ are the propagation vectors of the incident and scattered radiation, the probability amplitude is in this approximation proportional to

$$\sum_n\int\psi_n{}^*(\mathbf{r})\exp\{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\}\psi_n(\mathbf{r})dv. \quad (68)$$

The sum is taken over the states of the non-relativistic problem of the electron in the given field.

The expression (68) is a divergent one, but can reasonably be given a meaning by modification in two respects. One modification is the rather usual one of inserting a factor $e^{-\beta r}$ in the integrand and taking the limit $\beta\rightarrow 0+$ at the end of the calculation. The other modification is based on the fact that this nonrelativistic approximation is not valid for the states in which the momentum of the electron is large; for such states, indeed, the relativistic Born approximation is more suitable. One can allow only the states with not too large momentum their full importance by replacing the Schrödinger function $\psi_n(\mathbf{r})$ with an average value,

$$\psi_n(\mathbf{r})\rightarrow(3/4\pi b^3)\int_{r'>b}\psi_n(\mathbf{r}+\mathbf{r}')dv' \quad (69)$$

over a sphere whose radius b is of the order of the Compton wavelength. When these modifications are made and the formula

$$\sum_n\psi_n{}^*(\mathbf{r})\psi_n(\mathbf{r}'')=\delta(\mathbf{r}-\mathbf{r}'') \quad (70)$$

is used, the expression (68) becomes

$$(3/4\pi b^3)\lim_{\beta\rightarrow 0+}\int\exp\{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}-\beta r\}dv=0. \quad (71)$$

This result indicates that the contribution to the Delbrück scattering from positive energy states in the nonrelativistic region, including bound states, is very small.

The calculations of scattering given by Waller[2] correspond in the language of the positron theory to the complete neglect of Delbrück scattering and the use of the expression (63) to calculate the scattering by an electron. The name "scattering by an electron" is thus given to the whole effect occasioned by the electron's presence, without regard to the details of a picture of successive transitions.

On purely mathematical grounds there is nothing to

[34] In the many-particle case the field $A_\mu{}^{(e)}$ could be suitably chosen to take approximate account of the interaction of the particles. Arnous (reference 7) considers interactions rather from the point of view of the Born approximation.

[35] Such modifications could also be made in other terms to show the effect of using the exclusion principle in intermediate states; but we here concern ourselves only with the terms relating to the pair intermediate states for scattering, about which questions have been raised. These terms all come from the single primitive term that is the left-hand member of Eq. (60).

choose between the procedure of applying the exclusion principle in intermediate states and that of ignoring it for such states; both yield the same total numerical results. The former procedure appears as the natural one when an individual scattering problem is approached by the traditional method of variation of parameters. The latter procedure, on the other hand, corresponds to a simple algorithm for the systematic analysis of the S-matrix, and to a direct and simple statement as to the cause of the scattering.

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The Mobility of Electrons in Diamond

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The mobility of electrons in diamond has been measured by means of the Hall effect. In agreement with the theory of Seitz, the mobility is inversely proportional to the three-halves power of the absolute temperature. The measured room temperature value of the mobility is 900 cm²/volt-sec as compared with the theoretical estimate of 156 cm²/volt-sec.

I. INTRODUCTION

A THEORY of the scattering of electrons by lattice vibrations in nonpolar crystals has been given by Seitz,[1] which predicts that, for a nondegenerate electron gas, the electronic mobility and mean free path vary as $T^{-\frac{3}{2}}$ and T^{-1}, respectively. The experimental data on silicon[2] and germanium[3] are in excellent agreement with the theoretical prediction of the absolute magnitude of the mobility and its temperature dependence.

Diamond is a typical nonpolar crystal and, because of its successful use as a crystal counter, the problem of the electronic mobility in it is of considerable interest. Seitz has estimated the room temperature mobility of diamond to be approximately 156 cm²/volt-sec. The measurements of the Hall effect of Lenz[4] yield a room temperature mobility of about 200 cm²/volt-sec. On the basis of admittedly scanty evidence, Lenz concluded, however, that the mobility was probably independent of temperature.

Some time ago[5] we reported a value of 900 cm²/volt-sec for the room temperature electronic mobility in diamond and a temperature variation in agreement with Seitz. The purpose of the present paper is to present in detail the Hall effect measurements on which these conclusions were based.

The theory and technique of measurement of the Hall effect are well known.[6,7] The electronic mobility, μ, may be defined in terms of the electronic conductivity, σ, by the equation,

$$\sigma = ne\mu, \tag{1}$$

where n is the concentration of free electrons and e is the charge of an electron. The elementary theory[8] of a nondegenerate electron gas subjected to mutually perpendicular electric and magnetic fields results in the following equation for the mobility of the electrons:

$$\mu = (8/3\pi)(c/H)(E_h/E_a), \tag{2}$$

where E_h is the Hall field developed in the crystal when the applied electric and magnetic fields are E_a and H, respectively. The measured Hall and applied potentials are related to their respective fields by the equations $V_h = E_h t$ and $V_a = E_a L$. The length of the specimen measured parallel to the applied field is L, and t is the width of the specimen between the Hall electrodes.

The mobility and mean free path, l, are simply related by the equation,

$$\mu = 4el/3(2\pi mkT)^{\frac{1}{2}}, \tag{3}$$

where m is the electronic mass and k is Boltzmann's constant.

II. EXPERIMENTAL

The intrinsic conductivity of diamond at room temperature and below is too small to permit the measurement of the Hall effect. Upon absorption of ultraviolet and visible radiation, diamond exhibits a photo-conductivity which has been the subject of numerous investigations.[9–11] Photocurrents of the order of 10^{-10}

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