charge on the ion. The value of $e_{14}$ calculated from this expression[14] is $-23\times10^4$. From relation (7) derived in the present paper we find that for a purely ionic crystal $z=2$, and $e_{14}=-15.8\times10^4$. This at first sight appears to be a discrepancy; but since the second relation in expressions (5) of Born is identical with our expression (7), it only means that the value 6.5 of $k-k_0$, which has been used to calculate $e_{14}$ from expression (3) of Born, would give a value of $z$ which is 23/15.8 (or 1.5) times 2. This is also verified with the help of the first relation in expressions (5) of Born which gives a relation between $z$ and $k-k_0$, and we find $z=3.03$. The value of $k-k_0$ (6.5) needs to be revised.

[14] Herzfeld, *Handbuch der Experimental Physik* (VII), Teil 2, p. 341.

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# A Note on the Ground State in Antiferromagnetics

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The lowest state of an antiferromagnetic substance is investigated by using the Ising model for the spins. By means of a method of Luttinger and Tisza it is found possible to derive earlier results of Anderson very simply.

## I. INTRODUCTION

RECENT neutron diffraction experiments of Shull on antiferromagnetics have been interpreted by Anderson[1] in terms of a model in which spins interact with not only their nearest neighbors, but also with their next-nearest neighbors. Néel[2] had already introduced such a model to explain the large ratios of the "paramagnetic" Curie temperature $\theta$ to the actual transition temperature $T_c$. The basic point of improvement in Anderson's version is the finding of configurations of the spins which give lower energies than the ones which were used by Néel. The existence of these configurations has two effects: (1) the $\theta/T_c$ ratios are increased, and are more in conformity with experiment; (2) the configurations predicted seem to be just those found by Shull in his neutron diffraction experiments. Anderson chose the configurations he did largely on the basis of plausibility and intuition, and the question arises as to whether or not these are really the configurations of lowest energy. It is the purpose of this paper to show in a systematic and rigorous manner that this is indeed the case. The technique used is a generalization of a method developed by Luttinger and Tisza,[3] and applied by them to the case of dipole interactions in crystals. The present problem is simpler, however, and allows of a much more complete solution.

## II. GENERAL METHOD

The model used is essentially that of Ising,[4] in which each quantum mechanical spin is replaced by a scalar "spin" which can take the values $\pm1$. It is assumed that there is an interaction energy $\gamma(\geq0)$ between nearest neighbors (NN) and interaction energy $\alpha(\geq0)$ between next-nearest neighbors (NNN). In this case the energy may be written

$$E=\tfrac{1}{2}[\gamma\sum_j \mu_j \sum_{i=NN\text{ of }j} \mu_i+\alpha\sum_j \mu_j \sum_{i=NNN\text{ of }j} \mu_i], \quad (1)$$

where $\mu_j=\pm1$ (or $\mu_j^2=1$). For a given arrangement of spins (i.e., a given set of values of $\mu_j$) it is very easy to calculate the energy of the array. One can also ask what the arrangement of spins on the lattice points must be so that $E$ is a minimum. Now the expression for $E$ is a quadratic form in the spins $\mu_j$, and were it not for the complexity of the constraining conditions $\mu_j^2=1$, it would be a simple matter to find its minimum. We can replace (just as in reference 3) the "strong conditions" $\mu_j^2=1$ by the "weak" conditions

$$\textstyle\sum_j \mu_j^2=N, \quad (2a)$$

where $N$ is the number of spins present. This equation follows (by addition) from the strong conditions, but is, of course, considerably less stringent. If we minimize $E$ under the condition (2a) and find that our solution also satisfies the strong condition, then we will have solved the problem. The essence of our method is that this proves to be the case. The minimization of Eq. (1) under the condition (2a) is, of course, a standard problem in the theory of quadratic forms. The solution is given by the lowest eigenvalue of the matrix of the quadratic form.[5] Our problem is then to find the smallest eigenvalue of Eq. (1) for the case of simple cubic (S.C.), body centered cubic (B.C.) and face centered cubic

[1] P. W. Anderson, Phys. Rev. **79**, 705 (1950). The reference to Shull's work is also to be found in this paper. A very clear summary of Anderson's work is to be found in J. H. Van Vleck, Report to the Grenoble Conference, Grenoble, 1950.
[2] L. Néel, Ann. physique **3**, 137 (1948).
[3] J. M. Luttinger and L. Tisza, Phys. Rev. **70**, 954 (1946).
[4] G. Ising, Z. Physik **31**, 253 (1925).
[5] Courant-Hilbert, *Methoden der Mathematischen Physik* (Verlag J. Springer, Berlin, 1931), Vol. I, p. 26.

(F.C.). The S.C. case does not, so far as we know, have a realization in nature, but it forms a simple introduction to the techniques employed here.

### III. THE SIMPLE CUBIC CASE

If we introduce a matrix $E_{ij}$ defined by

$$E=\tfrac{1}{2}\sum_{i,j}E_{ij}\mu_i\mu_j, \tag{3a}$$

then our problem is to find the lowest eigenvalue of the matrix $\|E_{ij}\|$. It proves to be convenient (just as in the theory of fourier series) to allow complex values for the spins $\mu_j$. Any array occurring in nature will, of course, have real $\mu_j$; and these will be built up of superpositions of arrays with complex elements. With this in mind we redefine the weak condition as

$$\sum_j \mu_j^*\mu_j=N, \tag{2b}$$

and the energy as

$$E=\tfrac{1}{2}\sum_{i,j}E_{ij}\mu_i^*\mu_j. \tag{3b}$$

Just as in the case of the normal coordinates of a solid, we have translational symmetry in our energy matrix $\|E_{ij}\|$; and, therefore, the characteristic vectors (or basic arrays, as we shall call them here) must be of the form[6]

$$\mu_{l_1l_2l_3}=e^{i(\kappa_1 l_1+\kappa_2 l_2+\kappa_3 l_3)}=\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})]. \tag{4}$$

Here $l_1$, $l_2$, $l_3$ are integers which define a point in the crystal lattice, $\kappa_i=(2\pi/L)n_i$, the $n_i$ being integers $-\frac{1}{2}L\leq n_i\leq+\frac{1}{2}L$, and $L^3=N$. We have assumed a periodic boundary condition, but it is easy to see that this involves no loss in generality. This result can be verified directly by substitution. Let us call an array of the form (4) $\mathbf{Z}_\kappa$. Then any array may be written (finite fourier series)

$$\mathfrak{u}=\sum_\kappa c_\kappa\mathbf{Z}_\kappa$$

or

$$\mu_{l_1l_2l_3}=\sum_\kappa c_\kappa e^{i(\kappa_1 l_1+\kappa_2 l_2+\kappa_3 l_3)}.$$

Now the nearest neighbors to $l_1$, $l_2$, $l_3$ are given by $(l_1\pm1, l_2, l_3)$, $(l_1, l_2\pm1, l_3)$, $(l_1, l_2, l_3\pm1)$ and the next-nearest neighbors are given by $(l_1, l_2\pm1, l_3\pm1)$, $(l_1\pm1, l_2, l_3\pm1)$, $(l_1\pm1, l_2\pm1, l_3)$. Therefore, the expression (3b) reduces to

$$\begin{aligned}E=\tfrac{1}{2}\sum_{\kappa,\kappa'}\sum_{l_1l_2l_3}&\exp[i(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\cdot\mathbf{l}]c_\kappa c_{\kappa'}^*\\ &\times[\gamma(e^{i\kappa_1'}+e^{-i\kappa_1'}+e^{i\kappa_2'}+e^{-i\kappa_2'}+e^{i\kappa_3'}+e^{-i\kappa_3'})\\ &+\alpha(e^{i(\kappa_1'+\kappa_2')}+e^{-i(\kappa_1'+\kappa_2')}+e^{i(\kappa_1'-\kappa_2')}\\ &+e^{-i(\kappa_1'-\kappa_2')}+e^{i(\kappa_1'+\kappa_3')}+e^{-i(\kappa_1'+\kappa_3')}\\ &+e^{i(\kappa_1'-\kappa_3')}+e^{-i(\kappa_1'-\kappa_3')}+e^{i(\kappa_2'+\kappa_3')}\\ &+e^{-i(\kappa_2'+\kappa_3')}+e^{i(\kappa_2'-\kappa_3')}+e^{-i(\kappa_2'-\kappa_3')})].\end{aligned}$$

However,

$$\sum_{l_1l_2l_3}\exp[i(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\cdot\mathbf{l}]=N\delta_{\kappa\kappa'};$$

and, therefore,

$$\begin{aligned}E=N\sum_\kappa[&\gamma(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)+2\alpha(\cos\kappa_1\cos\kappa_2\\ &+\cos\kappa_2\cos\kappa_3+\cos\kappa_3\cos\kappa_1)]|c_\kappa|^2.\end{aligned} \tag{5}$$

This shows that the energy is now diagonal, and that the eigenvalues are

$$\begin{aligned}E(\kappa)=N\gamma[&(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)\\ &+2r(\cos\kappa_1\cos\kappa_2+\cos\kappa_2\cos\kappa_3+\cos\kappa_3\cos\kappa_1)],\end{aligned}$$

where $r=\alpha/\gamma$, the ratio of next-nearest to nearest neighbor interaction. Writing the energy in units of $N\gamma$ as $\epsilon(\kappa)$ we have, finally,

$$\begin{aligned}\epsilon(\kappa)=&\cos\kappa_1+\cos\kappa_2+\cos\kappa_3\\ &+2r(\cos\kappa_1\cos\kappa_2+\cos\kappa_2\cos\kappa_3+\cos\kappa_3\cos\kappa_1).\end{aligned} \tag{6}$$

We now must seek the minimum value of Eq. (6) as a function of $\kappa$. It can easily be shown that for $r<\frac{1}{4}$ this minimum is given by $\cos\kappa_1=\cos\kappa_2=\cos\kappa_3=-1$, while for $r>\frac{1}{4}$ the lowest state is given by two of the cosines being $-1$ and the other being $+1$ (for example, $\cos\kappa_1=\cos\kappa_2=-1$, $\cos\kappa_3=+1$). For $r<\frac{1}{4}$, the configuration is given by

$$\mu_{l_1l_2l_3}=(-)^{l_1+l_2+l_3},\quad \mu^2{}_{l_1l_2l_3}=1,$$

so that the strong condition is satisfied. The configuration consists in each spin being surrounded by six antiparallel neighbors. For $r>\frac{1}{4}$, the configuration is given by (for example)

$$\mu_{l_1l_2l_3}=(-)^{l_1+l_2},\quad \mu^2{}_{l_1l_2l_3}=1,$$

so that the strong condition is once again satisfied. The configuration consists of rows of spins along the $z$ axis, each of which is surrounded by four antiparallel rows of spins.

### IV. THE BODY CENTERED CASE

In treating the B.C. case it is convenient to introduce the concept of the field $F_l$ at a lattice point $\mathbf{l}$,

$$F_l\equiv-[\gamma\sum_{NN\text{ to }l}\mu_{l'}+\alpha\sum_{NNN\text{ to }l}\mu_{l'}].$$

With this definition the energy (3b) becomes

$$E=-\tfrac{1}{2}\sum_l\mu_l^*F_l.$$

Since $F_l$ is a linear function of the $\mu_{l'}$'s, we can introduce vector notation (in $N$ space) and write

$$\mathbf{F}=\begin{pmatrix}F_1\\F_2\\\cdot\\\cdot\\\cdot\\F_N\end{pmatrix}=\begin{pmatrix}\mathfrak{F}_{11}\cdots\mathfrak{F}_{1N}\\\cdot\quad\cdot\\\cdot\quad\cdot\\\cdot\quad\cdot\\\mathfrak{F}_{N1}\cdots\mathfrak{F}_{NN}\end{pmatrix}\begin{pmatrix}\mu_1\\\mu_2\\\cdot\\\cdot\\\cdot\\\mu_N\end{pmatrix}=\mathfrak{F}\mathfrak{u}.$$

[6] F. Seitz, *Modern Theory of Solids* (McGraw-Hill Book Company, Inc., New York, 1940), p. 125.

The characteristic vectors of the energy will then be the characteristic vectors of the matrix $\mathfrak{F}$, i.e.,

$$\mathfrak{F}\mathfrak{y}=f\mathfrak{y}. \tag{7}$$

That is, the characteristic vectors (or basic arrays, as we have called them) have the property that the field at any point in the lattice is proportional to the spin at that point.

Now let us call the array on the lattice points **A**, that on the body centers **B**. We shall use the notation

$$\mathfrak{y}=\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}$$

to characterize the array completely. From our previous argument on translational symmetry it follows once again that

$$A_l=\rho_1\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})],\quad B_l=\rho_2\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})].$$

Such an array will have for the field at a lattice point $\mathbf{l}$

$$F_l(\text{l.p.})=-\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})][\rho_2\gamma(1+e^{-i\kappa_1})(1+e^{-i\kappa_2})(1+e^{-i\kappa_3})$$
$$+\rho_1\alpha(e^{i\kappa_1}+e^{-i\kappa_1}+e^{i\kappa_2}+e^{-i\kappa_2}+e^{i\kappa_3}+e^{-i\kappa_3})],$$

while the field at the body center $\mathbf{l}$ will be

$$F_l(\text{B.C.})=-\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})][\rho_1\gamma(1+e^{i\kappa_1})(1+e^{i\kappa_2})(1+e^{i\kappa_3})$$
$$+\rho_2\alpha(e^{i\kappa_1}+e^{-i\kappa_1}+e^{i\kappa_2}+e^{-i\kappa_2}+e^{i\kappa_3}+e^{-i\kappa_3})].$$

The characteristic value equation reads

$$F_l(\text{l.p.})=f\rho_1\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})],$$
$$F_l(\text{B.C.})=f\rho_2\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})],$$

or

$$\begin{aligned}&\rho_1\lambda+\rho_2\nu=f\rho_1,\quad \rho_1\nu^*+\rho_2\lambda=f\rho_2,\\ &\lambda\equiv-2\alpha(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3),\\ &\nu\equiv-\gamma(1+e^{-i\kappa_1})(1+e^{-i\kappa_2})(1+e^{-i\kappa_3}).\end{aligned} \tag{8}$$

These equations can be solved directly. It is possible, however, to write down the solution at once, as the following considerations show. Let $\mathbf{T}$ be the operation (translation) which moves the lattice points up to their nearest adjoining body centers. Then

$$\mathbf{T}\begin{bmatrix}\rho_1\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})]\\ \rho_2\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})]\end{bmatrix}=\begin{bmatrix}\rho_2e^{-i(\kappa_1+\kappa_2+\kappa_3)}\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})]\\ \rho_1\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})]\end{bmatrix}. \tag{9}$$

However, this translation is merely a renaming of lattice points and body centers; and, therefore, the new array must also be (apart from a factor) the same basic array. Therefore,

$$\mathbf{T}\begin{bmatrix}\rho_1\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})]\\ \rho_2\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})]\end{bmatrix}=p\begin{bmatrix}\rho_1\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})]\\ \rho_2\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})]\end{bmatrix}, \tag{10}$$

where $p$ is a scalar factor. Using Eqs. (9) and (10) we then obtain

$$\rho_2e^{-i(\kappa_1+\kappa_2+\kappa_3)}=p\rho_1,\quad \rho_1=p\rho_2$$

or

$$p^2=e^{-i(\kappa_1+\kappa_2+\kappa_3)},\quad p=\pm e^{-\frac{1}{2}i(\kappa_1+\kappa_2+\kappa_3)}.$$

If we take $\rho_1=1$, we then obtain

$$\rho_2=\pm e^{\frac{1}{2}i(\kappa_1+\kappa_2+\kappa_3)},$$

which give us the two required solutions of Eq. (8). That these are really solutions can be verified easily by direct substitution in Eq. (8). The corresponding characteristic values are

$$f_\pm=-[2\alpha(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)\pm8\gamma\cos\tfrac{1}{2}\kappa_1\cos\tfrac{1}{2}\kappa_2\cos\tfrac{1}{2}\kappa_3]. \tag{11}$$

Since $-\pi\leq\kappa_i\leq\pi$, $\cos(\frac{1}{2}\kappa_i)\geq0$; and, therefore,

$$f_-\geq f_+,\quad E_\pm=-\tfrac{1}{2}Nf_\pm,\quad E_+\geq E_-.$$

In looking for the lowest state we need only consider $E_-$.

$$E_-=N\gamma[r(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)-4\cos\tfrac{1}{2}\kappa_1\cos\tfrac{1}{2}\kappa_2\cos\tfrac{1}{2}\kappa_3].$$

Defining for convenience $\epsilon=E_-/N\gamma$

$$\epsilon=r(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)-4\cos\tfrac{1}{2}\kappa_1\cos\tfrac{1}{2}\kappa_2\cos\tfrac{1}{2}\kappa_3, \tag{12}$$

we have only to find the minimum value of $\epsilon$. Once again, it is a matter of elementary calculus to show that for $r<\frac{2}{3}$ the minimum is given by $\kappa_i=0$ and the corresponding energy is $\epsilon=3r-4$. This means that we have a uniform arrangement of spins on the lattice points and a uniform (but oppositely directed) arrangement on the body centers. The strong conditions are clearly satisfied, since $\mu^2(\text{l.p.})=1$, $\mu^2(\text{B.C.})=+1$. For $r>\frac{2}{3}$ the situation is a little more complicated. The lowest arrangement is given by $\kappa_i=\pm\pi$, and the corresponding energy is $\epsilon=-3r$. However, at first glance the characteristic vectors seem to be complex. For example

$$\mu_l^-=\begin{bmatrix}1\\ i\end{bmatrix}(-)^{l_1+l_2+l_3},\quad \kappa_i=\pi.$$

The situation is easily remedied by noticing that in this case $E_-=E_+(\cos\frac{1}{2}\pi=0)$, so that there is a degeneracy. When degeneracy is present, we can take a linear combination of the two characteristic vectors; and it is once more a characteristic vector with the same characteristic value. Now,

$$\mu_l^+=\begin{bmatrix}1\\ -i\end{bmatrix}(-)^{l_1+l_2+l_3},$$

so that by building

$$\frac{\mu_l^++\mu_l^-}{2}\pm\frac{\mu_l^--\mu_l^+}{2i}=\begin{bmatrix}1\\ \pm1\end{bmatrix}(-)^{l_1+l_2+l_3},$$

we get basic arrays which are real and which satisfy the strong conditions. Because of this degeneracy, we see that the spin array at the lattice points is completely uncorrelated with the array at the body centers, a result which has already been noticed by Anderson.

## V. THE FACE CENTERED CASE

We shall not give the details for this case, but merely indicate the methods and results. If we split the array into its four components and write

$$\mathfrak{y}=\begin{bmatrix}\mathfrak{y}\ (\text{l.p.})\\ \mathfrak{y}\ (xz\ \text{face})\\ \mathfrak{y}\ (xy\ \text{face})\\ \mathfrak{y}\ (yz\ \text{face})\end{bmatrix},$$

then the translation group gives us

$$\mu_l=\begin{bmatrix}\rho_1\\ \rho_2\\ \rho_3\\ \rho_4\end{bmatrix}\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})].$$

Further, just as in the body centered case, we can apply translation operators which translate the various component lattices into each other. The conditions derived from the requirements of invariance under these translations are enough to determine the form of the characteristic vectors completely. There are four types

$$\mu_l^{(1)}=\begin{bmatrix}1\\ e^{\frac{1}{2}i(\kappa_1+\kappa_3)}\\ e^{\frac{1}{2}i(\kappa_1+\kappa_2)}\\ e^{\frac{1}{2}i(\kappa_2+\kappa_3)}\end{bmatrix}\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})],$$

$$\mu_l^{(2)}=\begin{bmatrix}1\\ e^{\frac{1}{2}i(\kappa_1+\kappa_3)}\\ -e^{\frac{1}{2}i(\kappa_1+\kappa_2)}\\ -e^{\frac{1}{2}i(\kappa_2+\kappa_3)}\end{bmatrix}\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})],$$

$$\mu_l^{(3)}=\begin{bmatrix}1\\ -e^{\frac{1}{2}i(\kappa_1+\kappa_3)}\\ e^{\frac{1}{2}i(\kappa_1+\kappa_2)}\\ -e^{\frac{1}{2}i(\kappa_2+\kappa_3)}\end{bmatrix}\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})],$$

$$\mu_l^{(4)}=\begin{bmatrix}1\\ -e^{\frac{1}{2}i(\kappa_1+\kappa_3)}\\ -e^{\frac{1}{2}i(\kappa_1+\kappa_2)}\\ e^{\frac{1}{2}i(\kappa_2+\kappa_3)}\end{bmatrix}\exp[i(\boldsymbol{\kappa}\cdot\mathbf{l})].$$

The corresponding energies are

$$E_1=N\gamma[r(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)+2(\cos\tfrac{1}{2}\kappa_1\cos\tfrac{1}{2}\kappa_2+\cos\tfrac{1}{2}\kappa_2\cos\tfrac{1}{2}\kappa_3+\cos\tfrac{1}{2}\kappa_3\cos\tfrac{1}{2}\kappa_1)],$$
$$E_2=N\gamma[r(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)+2(-\cos\tfrac{1}{2}\kappa_1\cos\tfrac{1}{2}\kappa_2-\cos\tfrac{1}{2}\kappa_2\cos\tfrac{1}{2}\kappa_3+\cos\tfrac{1}{2}\kappa_3\cos\tfrac{1}{2}\kappa_1)],$$
$$E_3=N\gamma[r(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)+2(\cos\tfrac{1}{2}\kappa_1\cos\tfrac{1}{2}\kappa_2-\cos\tfrac{1}{2}\kappa_2\cos\tfrac{1}{2}\kappa_3-\cos\tfrac{1}{2}\kappa_3\cos\tfrac{1}{2}\kappa_1)],$$
$$E_4=N\gamma[r(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)+2(-\cos\tfrac{1}{2}\kappa_1\cos\tfrac{1}{2}\kappa_2+\cos\tfrac{1}{2}\kappa_2\cos\tfrac{1}{2}\kappa_3-\cos\tfrac{1}{2}\kappa_3\cos\tfrac{1}{2}\kappa_1)].$$

Since $\cos\kappa_i/2\geq 0$, it is clear that $E_2$, $E_3$, $E_4\leq E_1$. Therefore, we can ignore $E_1$. By symmetry, it makes no difference which of the three remaining $E$'s we choose. Let us take $E_2$, and define

$$\epsilon=E_2/N\gamma=r(\cos\kappa_1+\cos\kappa_2+\cos\kappa_3)+2(-\cos\tfrac{1}{2}\kappa_1\cos\tfrac{1}{2}\kappa_2-\cos\tfrac{1}{2}\kappa_2\cos\tfrac{1}{2}\kappa_3+\cos\tfrac{1}{2}\kappa_3\cos\tfrac{1}{2}\kappa_1).$$

The finding of the minimum of this expression is as always fairly straightforward. We find for $r>\frac{1}{2}$ that the configuration with $\kappa_i=\pm\pi$ is that of minimum energy. We can choose as our array, any of the arrays

$$\mu_l=\begin{bmatrix}1\\ \pm 1\\ \pm 1\\ \pm 1\end{bmatrix}(-)^{l_1+l_2+l_3}$$

because of the degeneracy in the energy values. Clearly, these satisfy the strong conditions, and show that (for $r>\frac{1}{2}$) there is no correlation between the order in the different faces. As in the body centered case, there are slight complications for $r<\frac{1}{2}$. The complex arrays which minimize the energy are $\kappa_1=\kappa_2=0$, $\kappa_3=\pm\pi$ (or similar arrays with 1, 2, 3 permuted). This gives basic arrays of the type

$$\mu_l^{(2)}=\begin{bmatrix}1\\ \pm i\\ -1\\ \mp i\end{bmatrix}(-)^{l_3},\quad \mu_l^{(4)}=\begin{bmatrix}1\\ \mp i\\ -1\\ \pm i\end{bmatrix}(-)_l{}^{l_3}.$$

By combining these properly it is easy to see that we can construct basic arrays of the form

$$\begin{bmatrix}1\\ 1\\ -1\\ -1\end{bmatrix}(-)^{l_3}\quad\text{and}\quad\begin{bmatrix}1\\ -1\\ -1\\ 1\end{bmatrix}(-)^{l_3},$$

which satisfy the strong conditions. As Anderson has already noted, the existence of this degeneracy shows that the spins are correlated only in planes. That is, the lattice points and $xy$ face are correlated, as are the $yz$ and $zx$ faces, but they are not correlated with each other.