with the average energy values of 5.72 and 5.68 kev are ±0.06 kev and ±0.04 kev, respectively. The weighted mean of these values is 5.69±0.04 kev.

Slack *et al.*[11] have pointed out that the accurate determination of the average energy release in tritium enables one to make an equally accurate determination of the maximum beta-energy release. They have carried out a computation using the above value for the average energy and obtained a value of 18.6±0.2 kev for the maximum energy, a result which is in close agreement with the results of proportional counter measurements of the maximum energy.[6,7]

The half-life determined in this study is slightly greater than the value of 12.1±0.5 years reported by Novick,[4] and is considerably greater than the value of 10.7±2 years reported as a preliminary value by Goldblatt *et al.*[2] It does, however, fall within the estimated error associated with each of the previous values.

[11] Slack, Owen, and Primakoff, Phys. Rev. **75**, 1448 (1949).

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# Improved Calculations on Cascade Shower Theory*

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The diffusion equations of cascade shower theory are solved by means of a perturbation method. The first approximation is taken to be that solution obtained by Snyder using the so-called completely screened cross sections for the elementary processes of bremsstrahlung and pair production. The correction to this is then calculated using more refined approximations to the Bethe-Heitler cross sections.

The results, which lend themselves to accurate numerical work only in the case of light elements and high incident energies, indicate with respect to Snyder's solution that (1) the shower maximum is decreased in height and slightly shifted to greater depths, (2) there is a decrease in the average number of electrons present at small absorber depths and a correspondingly larger number at large depths. The magnitude of the correction is larger than had been previously assumed. Numerical results are given so that the correction to the solution under the assumption of completely screened cross sections can be readily calculated for all light elements when the incident particle is either a single photon or a single electron.

## I. INTRODUCTION

The latest and most accurate calculations on the cascade theory of showers are those of Snyder[1] who gave solutions of the diffusion equations of the theory of such a nature that accurate numerical results were readily derivable therefrom. The calculations, however, were based on the assumption of so-called "completely screened" or "asymptotic" cross sections. That is, the cross sections used were those valid for high energy particles, and too high a probability was assigned to those elementary processes associated with the lower energy particles in the shower. The purpose of this paper is to calculate the correction to the Snyder solution introduced by the use of more accurate cross sections.

Efforts to employ more accurate cross sections have already been made by Corben[2] and by Chakrabarty,[3] but their calculations were based on none too numerically accurate solutions of the diffusion equations for the completely screened case, and on none too accurate approximations to the appropriate Bethe-Heitler cross sections.[4] The order of magnitude of the effect of this refinement as here calculated is much greater, and the results are in a more general form than is indicated by the work of these authors.

* This paper is based on a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at New York University. This work was performed at Brookhaven National Laboratory under the auspices of the AEC.

** Now at Westinghouse Research Laboratories.

[1] H. S. Snyder, Phys. Rev. **76**, 1563 (1949).

[2] H. C. Corben, Phys. Rev. **60**, 435 (1941).

[3] S. K. Chakrabarty, Proc. Nat. Inst. Sci. Ind. **9**, 323 (1943).

## II. THE DIFFUSION EQUATIONS

The well-known diffusion equations of the cascade shower theory are

$$\frac{\partial P(E,t,E_0)}{\partial t}=\beta\frac{\partial P(E,t,E_0)}{\partial E}+2\int_E^\infty \gamma(E',t,E_0)R(E,E')\frac{dE'}{E'}+\lim_{\delta\to 0}\left[\int_{E+\delta}^\infty P(E',t,E_0)R(E',E'-E)\frac{E'-E}{E'^2}dE'-P(E,t,E_0)\int_\delta^E R(E,E')\frac{E'dE'}{E^2}\right], \quad (1)$$

$$\frac{\partial \gamma(E,t,E_0)}{\partial t}=\int_E^\infty P(E',t,E_0)R(E',E)\frac{EdE'}{E'^2}-\gamma(E,t,E_0)\int_0^E R(E',E)\frac{dE'}{E}. \quad (2)$$

[4] H. A. Bethe and W. Heitler, Proc. Roy. Soc. **159**, 432 (1937).

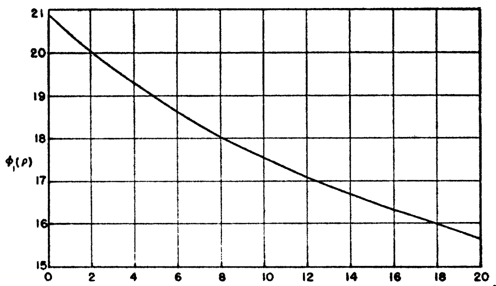


FIG. 1. Cross section modifying function $\phi_1(\rho)$ *vs.* $\rho$.

$P(E, t, E_0)$ is the number of electrons, $\gamma(E, t, E_0)$ the number of photons per unit energy at thickness $t$ in the energy range between $E$ and $E+dE$. $t$ is distance measured in radiation units $\Delta$,

$$\Delta=[(4z^2N/137)(e^2/mc^2)^2\ln(183z^{-\frac{1}{3}})]^{-1}.\dagger$$

$\beta$ is the ionization loss per unit length of $t$. $R(E, E')$ is a factor[4] common to the cross sections for bremsstrahlung and pair production. $E_0$ is the total energy in the incident spectrum of electrons and photons.

In the "complete screening" approximation one takes

$$\begin{aligned}R(E, E')=R_0(E, E')&=1-a(E/E')+a(E/E')^2,\\ a&=\alpha+4/3,\\ \alpha&=9\ln(183z^{-\frac{1}{3}})\sim\text{const.}=0.025,\end{aligned}\qquad(3)$$

in which case the diffusion equations (1) and (2) can be solved by means of a Laplace transform with respect to $t$ and a Mellin transform with respect to $E$.

More accurately, the cross sections are given by

$$R(E, E')=R_0(E, E')\frac{\phi_1(\rho)-\frac{1}{3}4\ln z}{4\ln 183-\frac{1}{3}4\ln z},\qquad(4)$$

where

$$\rho=(100/z^{\frac{1}{3}})|mc^2E'/E(E-E')|\qquad(5)$$

and $\phi_1(\rho)$ is given in Fig. 1.

Equation (4) predicts smaller values of the cross sections for given values of $E$ and $E'$ than does (3), the difference being greater the smaller is $E$.

Since $\phi_1(\rho)$ is known only numerically, it is not feasible to use (4) directly in a solution of the diffusion equations. It is expedient, however, to use the following empirical expression which fits (4) to within two percent over the whole range of Fig. 1, which is the range of importance in the genesis of a cascade shower.

$$R(E, E')=\frac{R_0(E, E')}{1+K|E'/E(E-E')|},\qquad(6)$$

where

$$K=(mc^2/z^{\frac{1}{3}})255/(15.6-\tfrac{1}{3}4\ln z).\qquad(7)$$

In what follows, it is convenient to measure energy in units of $K$ as shown in Table I.

TABLE I. Energy units used *vs.* $Z$.

| $Z$ | 1 | 5 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $u$(Mev) | 8.30 | 5.63 | 4.81 | 4.11 | 3.74 | 3.53 | 3.37 | 3.27 | 3.15 | 3.08 | 3.00 |

## III. THE TRANSFORM METHOD[5-7]

For any admissible function $F(E, t)$ let us define the following:

Laplace transform

$$\mathfrak{L}F(E, s)=\int_0^\infty dt\, e^{-st}F(E, t).\qquad(8)$$

Mellin transform

$$\mathfrak{M}F(p, t)=\int_0^\infty d\left(\frac{E}{\beta}\right)\left(\frac{E}{\beta}\right)^p F(E, t).\qquad(9)$$

Then

$$\mathfrak{M}\mathfrak{L}F(p, s)=\int_0^\infty d\left(\frac{E}{\beta}\right)\left(\frac{E}{\beta}\right)^p\int_0^\infty dt\, e^{-st}F(E, t).\qquad(10)$$

By the Laplace and Mellin inversion theorems

$$F(E, t)=\int_L\frac{ds}{2\pi i}e^{st}\mathfrak{L}F(E, s),\qquad(11)$$

$$F(E, t)=\int_M\frac{dp}{2\pi i}\left(\frac{E}{\beta}\right)^{-p-1}\mathfrak{M}F(p, t),\qquad(12)$$

where $L$ and $M$ are contours parallel to the respective imaginary axes, running from $-i\infty$ to $+i\infty$, and to the right of all singularities in the respective complex planes. Therefore,

$$F(E, t)=\int_L\frac{ds}{2\pi i}e^{st}\int_M\frac{dp}{2\pi i}\left(\frac{E}{\beta}\right)^{-p-1}\mathfrak{M}\mathfrak{L}F(p, s).$$

In this paper we confine ourselves to a consideration of two initial spectra, a single incident electron of energy $E_0$, and a single incident photon of energy $E_0$. In this latter case, the solutions of the diffusion equations will be set off by a dagger. All other incident spectra are but linear combinations of these with appropriate weighting functions. Those solutions obtained by taking $R(E, E')=R_0(E, E')$ (the "complete screening" approximation) will be denoted by a subscript 1, that is, $P_1(E, t, E_0)$ and $\gamma_1(E, t, E_0)$.

† As kindly pointed out to me by Professor J. A. Wheeler, in the case of light elements, the $z^2$ appearing in the expression for $\Delta$ should be replaced by $z(z+1)$: J. A. Wheeler and W. E. Lamb, Phys. Rev. **55**, 858 (1939).

[5] L. Landau and G. Rumer, Proc. Roy. Soc. **166**, 277 (1938).
[6] W. T. Scott, Phys. Rev. **80**, 611 (1950).
[7] G. Doetsch, *Theorie und Anwendung der Laplace Transformation* (Verlag. Julius Springer, Berlin, 1937).

In addition to these it will be necessary, for use in the iteration scheme to follow, to have a solution to the equation obtained from (1) by deleting the term

$$2\int_E^\infty \gamma(E', t, E_0)R(E, E')dE'/E'$$

taking $R(E, E')=R_0(E, E')$, and with the initial condition of one incident electron of energy $E_0$. This solution will be denoted by $P_0(E, t, E_0)$.

If now one operates on the diffusion equations with

$$\int_0^\infty d\left(\frac{E}{\beta}\right)\left(\frac{E}{\beta}\right)^p \int_0^\infty dt\, e^{-st}$$

the result is

$$G_0(p, s)\mathfrak{M}\mathfrak{L}P_0(p, s, E_0)+p\mathfrak{M}\mathfrak{L}P_0(p-1, s, E_0) = (1/\beta)(E_0/\beta)^p, \quad (13)$$

$$G_1(p, s)\mathfrak{M}\mathfrak{L}P_1(p, s, E_0)+p\mathfrak{M}\mathfrak{L}P_1(p-1, s, E_0) = (1/\beta)(E_0/\beta)^p, \quad (14)$$

$$\mathfrak{M}\mathfrak{L}\gamma(p, s, E_0)+[C(p)/(s+D)]\mathfrak{M}\mathfrak{L}P_1(p, s, E_0), \quad (15)$$

$$G_1(p, s)\mathfrak{M}\mathfrak{L}P_1^\dagger(p, s, E_0)+p\mathfrak{M}\mathfrak{L}P_1^\dagger(p-1, s, E_0) = \frac{B(p)}{s+D}\frac{1}{\beta}\left(\frac{E_0}{\beta}\right)^p, \quad (16)$$

$$\mathfrak{M}\mathfrak{L}\gamma_1^\dagger(p, s, E_0) = \frac{C(p)}{s+D}\mathfrak{M}\mathfrak{L}P_1^\dagger(p, s, E_0)+\frac{1}{s+D}\left(\frac{E_0}{\beta}\right)^p, \quad (17)$$

where

$$G_0(p, s)=s+A(p), \quad (18)$$

$$G_1(p, s)=s+A(p)-B(p)C(p)/(s+D), \quad (19)$$

and

$$A(p)=a[\psi(p+1)-\psi(1)]+\tfrac{1}{2}-1/(p+1)(p+2), \quad (20)$$

$$B(p)=2\left(\frac{1}{p+1}-\frac{a}{p+2}+\frac{a}{p+3}\right), \quad (21)$$

$$C(p)=\frac{1}{p+2}-\frac{a}{p+1}+\frac{a}{p}, \quad (22)$$

$$D=1-\tfrac{1}{6}a, \quad (23)$$

$$\psi(p)=d\ln\Gamma(p)/dp. \quad (24)$$

Equations (13), (14), and (16) are all linear difference equations of the form

$$G(p, s)g(p, s)+pg(p-1, s)=\varphi(p, s). \quad (25)$$

A formal solution of (25) in the form of a complex integral is

$$g(p, s)=-\int_K \frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2)} \times\varphi(p+\sigma+1, s)Q(p+1, \sigma, s), \quad (26)$$

where $K$ is a contour parallel to the imaginary axis, running from $-i\infty$ to $+i\infty$ in the strip $-1<Re\,\sigma<0$ (26) is valid if the integrand is regular in this strip, goes to 0 sufficiently strongly at $\infty$, and $Q(p+1, \sigma, s)$ satisfies

$$Q(p+1, 0, s)=1, \quad (27)$$

$$G(p, s)Q(p+1, \sigma, s)=Q(p, \sigma+1, s). \quad (28)$$

A formal solution of (27) and (28) is

$$Q(p+1, \sigma, s)=\lim_{N\to\infty} G(N+1, s)^\sigma \prod_{K=1}^{N}\frac{G(p+K, s)}{G(p+\sigma+K, s)}. \quad (29)$$

For those $G(p, s)$ and $\varphi(p, s)$ with which we are concerned, the conditions on the integrand are satisfied, the limit in (29) exists, and (26) is a true solution.

In what follows that solution of (28) in which $G(p, s)=G_0(p, s)$ will be denoted by a subscript 0, that in which $G(p, s)=G_1(p, s)$ by a subscript 1.

One can now write the solutions of the electron diffusion equations as

$$P_0(E, t, E_0)=-\frac{1}{\beta}\int_L\frac{ds}{2\pi i}e^{st}\int_M\frac{dp}{2\pi i}\left(\frac{E}{\beta}\right)^{-p-1}\int_K\frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2)}\left(\frac{E_0}{\beta}\right)^{p+\sigma+1}Q_0(p+1, \sigma, s), \quad (30)$$

$$P_1(E, t, E_0)=-\frac{1}{\beta}\int_L\frac{ds}{2\pi i}e^{st}\int_M\frac{dp}{2\pi i}\left(\frac{E}{\beta}\right)^{-p-1}\int_K\frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2)}\left(\frac{E_0}{\beta}\right)^{p+\sigma+1}Q_1(p+1, \sigma, s), \quad (31)$$

$$P_1^\dagger(E, t, E_0)=-\frac{1}{\beta}\int_L\frac{ds}{2\pi i}e^{st}\int_M\frac{dp}{2\pi i}\left(\frac{E}{\beta}\right)^{-p-1}\int_K\frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2)}\left(\frac{E_0}{\beta}\right)^{p+\sigma+1}\frac{B(p+\sigma+1)}{s+D}Q_1(p+1, \sigma, s). \quad (32)$$

The quantity which can be readily calculated however, is not $P(E, t, E_0)$ but rather $N(0, t, E_0)$, where $N(E, t, E_0)$ is the so-called integral spectrum defined by

$$N(E, t, E_0)=\int_E^\infty dE\, P(E, t, E_0). \quad (33)$$

Note that $N(0, t, E_0)$ is the average number of electrons at thickness $t$ and is given in terms of the Mellin transform as

$$N(0, t, E_0)=\beta\mathfrak{M}P(0, t, E_0). \quad (34)$$

In order to show the equivalence of say (31) with the result of Snyder, it is necessary to invert the Laplace transform, To this end we define

$$\mu(p)=-\tfrac{1}{2}[A(p)+D] +\tfrac{1}{2}\{[A(p)-D]^2+4B(p)C(p)\}^{\frac{1}{2}}, \quad (35)$$

$$\nu=(p)=-\tfrac{1}{2}[A(p)+D]-\tfrac{1}{2}\{[A(p)-D]^2+4B(p)C(p)\}^{\frac{1}{2}}. \quad (36)$$

Then

$$G_1(p,s)=[s-\mu(p)][s-\nu(p)]/(s+D). \quad (37)$$

Moreover, one can write

$$Q(p+1,\sigma,s)=\frac{Q(p+\sigma+j,1-j,s)Q(p+1,\sigma+j,s)}{G(p+\sigma+j,s)},\quad j=1,2,3,\cdots \quad (38)$$

whence

$$Q_1(p+1,\sigma,s)=\frac{(s+D)Q_1(p+\sigma+j,1-j,s)Q_1(p+1,\sigma+j,s)}{[s-\mu(p+\sigma+j)][s-\nu(p+\sigma+j)]} \quad (39)$$

and it is convenient to invert the Laplace transform in terms of the poles of the integrand of the inversion integral. Hence

$$P_1(E,t,E_0)=-\frac{1}{\beta}\int_K\frac{dp}{2\pi i}\int_M\frac{d\sigma}{2\pi i}\left(\frac{E}{\beta}\right)^{-p-1}\frac{\pi}{\sin\pi\sigma}\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2)}\left(\frac{E_0}{\beta}\right)^{p+\sigma+1}\sum_{j=1}^{\infty}\left\{\frac{[\mu(p+\sigma+j)+D]e^{t\mu(p+\sigma+j)}}{\mu(p+\sigma+j)-\nu(p+\sigma+j)}\right.$$
$$\times Q_1(p+\sigma+j,1-j,\mu(p+\sigma+j))Q_1(p+1,\sigma+j,\mu(p+\sigma+j))+\frac{[\nu(p+\sigma+j)+D]e^{t\nu(p+\sigma+j)}}{\nu(p+\sigma+j)-\mu(p+\sigma+j)}$$
$$\left.\times Q_1(p+\sigma+j,1-j,\nu(p+\sigma+j))Q_1(p+1,\sigma+j,\nu(p+\sigma+j))\right\}. \quad (40)$$

While $\mu$ and $\nu$ have simultaneous branch points, note that upon going around a branch point the net effect is an interchange of $\mu$ and $\nu$. This circumstance leaves the integrand of (40) unchanged since it is symmetric in $\mu$ and $\nu$. Therefore, because the integrand is single valued and non-singular over that region of the new $\sigma$-plane over which we shift contours, we may let $\sigma+j$ go into $\sigma$ and then shift all contours to the contour $K+2$. By contour $K+2$ is meant the line obtained by shifting $K$ parallel to itself, 2 units to the right. The result is

$$P_1(E,t,E_0)=\sum_{j=1}^{\infty}\left(-\frac{\beta}{E_0}\right)^{j-1}\int_M\frac{dp}{2\pi i}\int_{K+2}\frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2-j)}\left(\frac{E_0}{\beta}\right)^{p+\sigma}\left(\frac{E}{\beta}\right)^{-p-1}\frac{1}{\mu(p+\sigma)-\nu(p+\sigma)}$$
$$\cdot\{[\mu(p+\sigma)+D]e^{t\mu(p+\sigma)}Q_1(p+\sigma,1-j,\mu(p+\sigma))Q_1(p+1,\sigma,\mu(p+\sigma))$$
$$-[\nu(p+\sigma)+D]e^{t\nu(p+\sigma)}Q_1(p+\sigma,1-j,\nu(p+\sigma))Q_1(p+1,\sigma,\nu(p+\sigma))\}. \quad (41)$$

This is identical with Eq. (38) of Snyder[1] if one makes the following identifications:

$$n=j-1,\quad y=p+\sigma,\quad y+s+1=\sigma,$$
$$K_\mu(y,s)=\Gamma(s+1)Q_1(y+s+1,-s,\mu(y)),$$
$$K_\nu(y,s)=\Gamma(s+1)Q_1(y+s+1,-s,\nu(y)),$$
$$A_n(y)=(-1)^n\frac{\Gamma(y+1)}{\Gamma(y+1-n)}Q_1(y,-n,\mu(y)),$$
$$B_n(y)=(-1)^n\frac{\Gamma(y+1)}{\Gamma(y+1-n)}Q_1(y,-n,\nu(y)).$$

One property of the diffusion equations which must be possessed by any true solution and which, in particular, is possessed by the Snyder solution is

$$\int_0^\infty dtN(0,t,E_0)=\beta\mathfrak{M}\mathfrak{L}P(0,0,E_0)=E_0/\beta. \quad (42)$$

This is a statement of the conservation of energy, namely that all the energy originally in the incident spectrum, in the model used in setting up the diffusion equations, is absorbed by ionization loss.

## IV. ITERATION SCHEME

The Mellin transform method depends for its success on the fact that in the "complete screening" approximation the function $R_0(E,E')$ is homogeneous, depending only on the variable $E/E'$. This is no longer true if one wishes to use Eq. (6). However, since one would not expect the correction to the "complete screening" solution caused by the introduction of (6) to be excessively large, a perturbation method is indicated.

The most convenient perturbation method is arrived at by writing the diffusion equations in the form of a single integral equation. To do this one first solves Eq. (2) in the form

$$\gamma(E,t,E_0)=\gamma(E,0,E_0)\exp[-\mathfrak{D}(E)t]+\int_0^t dx\exp[-(t-x)\mathfrak{D}(E)]\times\int_E^\infty P(E',x,E_0)R(E',E)\frac{EdE'}{E'^2}, \quad (43)$$

where

$$\mathfrak{D}(E)=\int_0^E \frac{dE'}{E}R(E',E) \tag{44}$$

is the gamma-ray absorption coefficient.

The electron diffusion equation (1) can now be written

$$\frac{\partial P(E,t,E_0)}{\partial t}-\beta\frac{\partial P(E,t,E_0)}{\partial E}-\lim_{\delta\to 0}\left[\int_{E+\delta}^{\infty}P(E',r,E_0)R_0(E',E'-E)\frac{E'-E}{E'^2}dE'-P(E,t,E_0)\int_\delta^E R_0(E,E')\frac{E'dE'}{E^2}\right]=S(E,t,E_0), \tag{45}$$

where

$$S(E,t,E_0)=\lim_{\delta\to 0}\left[\int_{E+\delta}^{\infty}P(E',t,E_0)R_1(E',E'-E)\frac{E'-E}{E'^2}-P(E,t,E_0)\int_\delta^E R_1(E,E')\frac{E'dE'}{E^2}\right]+2\int_E^\infty\frac{dE'}{E'}R(E,E')\left\{\gamma(E,0,E_0)\exp[-\mathfrak{D}(E')t]+\int_0^t dx\exp[-(t-x)\mathfrak{D}(E')]\times\int_E^\infty P(E'',x,E_0)R(E'',E')\frac{E'dE''}{E''^2}\right\} \tag{46}$$

and we have written

$$R(E,E')=R_0(E,E')+R_1(E,R'). \tag{47}$$

Interpreting the left-hand side of (45) as governing the propagation of an electron suffering only ionization loss and bremsstrahlung under the assumption of complete screening and the right side as a source function incorporating the effects of pair production and screening, one can write, since $P_0(E,t,E_0)$ is the Green's function:

$$P(E,t,E_0)=\int_0^t dt'\int_0^\infty dE'P_0(E,t-t',E')\times S(E',t',E_0)+\pi(E,t,E_0), \tag{48}$$

where $\pi(E,t,E_0)$ is some multiple of $P_0(E,t,E_0)$ satisfying the desired initial condition on $P(E,t,E_0)$.

The integral equation is now in a form suitable for solution by iteration. The first approximation to the solution of the diffusion equations using (6) can be obtained by using $P_1(E,t,E_0)$ in evaluating $S(E,t,E_0)$ according to (44), and substituting this in the right-hand side of (46), and will be denoted by $P_2(E,t,E_0)$. The next approximation is obtained by so using $P_2(E,t,E_0)$, etc.

For calculational purposes it is more convenient to work with the Laplace-Mellin transform of Eq. (36) rather than directly with the equation itself. This is

$$\mathfrak{M}\mathfrak{L}P(p,s,E_0)=\int_0^\infty dE'\,\mathfrak{M}\mathfrak{L}P_0(p,s,E')\times\mathfrak{L}S(E',s,E_0)+\mathfrak{M}\mathfrak{L}\pi(p,s,E_0), \tag{49}$$

which yields

$$\mathfrak{L}N(0,s,E_0)=\beta\int_0^\infty dE'\,\mathfrak{M}\mathfrak{L}P_0(0,s,E')\times\mathfrak{L}S(E',s,E_0)+\beta\mathfrak{M}\mathfrak{L}\pi(0,s,E_0). \tag{50}$$

## V. THE CORRECTION IN THE CASE OF A SINGLE INCIDENT ELECTRON

Let us calculate the first iteration, based on Eqs. (50) and (6), in the event that the initial condition is a single incident electron. Here $\pi(E,t,E_0)=P_0(E,t,E_0)$.

The Laplace transform of $N_2(0,t,E_0)$ is then, after appropriate changes of variable,

$$\mathfrak{L}N(0,s,E_0)=\mathfrak{L}N_0(0,s,E_0)+\int_M\frac{dp}{2\pi i}\int_K\frac{d\sigma}{2\pi i}\times\int\frac{d\sigma'}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\pi}{\sin\pi\sigma'}\frac{1}{\Gamma(\sigma'+2)}\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2)}\times\left(\frac{E_0}{\beta}\right)^{p+\sigma+1}\beta^{p-\sigma'-1}Q_0(1,\sigma',s)Q_1(p+1,\sigma,s)\times\{2\mathfrak{A}(p,\sigma',s)+\mathfrak{B}(p,\sigma')-\mathfrak{C}(p,\sigma')\}, \tag{51}$$

where

$$\mathfrak{A}(p,\sigma',s)=\int_0^\infty d\xi\,\xi^{p-\sigma'-2}\times\int_0^1 dx\frac{x^p(1-ax+ax^2)}{\left[\dfrac{\xi}{1-x}+1\right]\left[s+\mathfrak{D}\left(\dfrac{1}{x\xi}\right)\right]}\times\int_0^1 d\omega\frac{\omega^{p-1}(\omega^2-a\omega+a)}{\dfrac{\omega^2x\xi}{1-\omega}+1}, \tag{52}$$

$$\mathfrak{B}(p,\sigma')=\int_0^\infty d\xi\,\xi^{p-\sigma'-1}\int_0^1 dx\frac{x^2-ax+a}{1+x(\xi-1)}, \tag{53}$$

$$\mathfrak{C}(p,\sigma')=\int_0^\infty d\xi\,\xi^{p-\sigma'-1}\int_0^1 dx\frac{x^p[x^2-(2-a)x+a]}{1+\xi(1-x)}. \tag{54}$$

Note that this result has been obtained by reversing the order of integration from the stepwise inversion of the transforms. This implies, that, while the $\sigma$ and $\sigma'$ contours remain as before, the $p$ contour in any term in (51) be parallel to the imaginary axis and in a strip determined by the conditions for convergence on the integrals (52), (53), and (54). But the regions of convergence are as follows:

$$(52) \text{ converges if } -1<Re(p-\sigma'-2)<0 \quad 0<Re\,p; \tag{55}$$

$$(53) \text{ converges if } -1<Re(p-\sigma'-1)<0; \tag{56}$$

$$(54) \text{ converges if } -1<Re(p-\sigma'-1)<0 \quad -1<Re\,p, \tag{57}$$

which conditions follow from a consideration of the several integrands at the limits of integration. The requirements are then that the $p$ contour lie in the strip $-1<Re(p-\sigma'-2)<0$ for the term containing $\mathfrak{A}(p,\sigma',s)$ and in the strip $-1<Re(p-\sigma'-1)<0$ for the other terms.

The next step in the evaluation of the first iteration is an evaluation of the $p$ integral in (51) in terms of the poles of its integrand. Integration by parts of (52) shows that the analytic continuation of the function there defined by the real integral has poles at $p-\sigma'=0, \pm1, \pm2, \cdots; p=0, -1, -2, \cdots$. Moreover, the strength of the first-order pole at $p=\sigma'+1$ is

$$B(\sigma'+1)C(\sigma'+1)/2(s+D) = \tfrac{1}{2}[G_0(\sigma'+1,s)-G_1(\sigma'+1,s)] \tag{58}$$

while the strength of the first-order pole at $p=0$ is

$$h(\sigma',s)=\frac{a}{\sigma'+1}\int_0^\infty d\xi\,\xi^{-\sigma'-1}\frac{\partial}{\partial\xi}\int_0^1 \times\frac{1-ax+ax^2}{\left[\dfrac{\xi}{1+x}+1\right]\left[s+\mathfrak{D}\left(\dfrac{1}{x\xi}\right)\right]}. \tag{59}$$

Therefore, if we deform the path of integration of the $g(p,\sigma',s)$ term to coincide with that of the other terms in the integrand of (51) the contributions of the intervening poles will be

$$I_1=\int_K\frac{d\sigma}{2\pi i}\int_K\frac{d\sigma'}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\pi}{\sin\pi\sigma'}\left(\frac{E_0}{\beta}\right)^{\sigma+\sigma'+2} \times\frac{1}{\Gamma(\sigma+\sigma'+3)}Q_0(1,\sigma',s)Q_1(\sigma'+2,\sigma,s) \times[G_0(\sigma'+1,s)-G_1(\sigma'+1,s)] \tag{60}$$

from the pole $p=\sigma'+1$, and

$$I_2=\int_K\frac{d\sigma}{2\pi i}\int\frac{d\sigma'}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\pi}{\sin\pi\sigma'}\frac{1}{\Gamma(\sigma'+2)}\frac{1}{\Gamma(\sigma+2)} \times\left(\frac{E_0}{\beta}\right)^{\sigma+1}\beta^{-\sigma'-1}Q_0(1,\sigma',s)Q_1(1,\sigma,s)2h(\sigma',s)$$

from the pole at $p=0$. One can write

$$I_2=-\mathfrak{L}N_1(0,s,E_0)\int_K\frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{1}{\Gamma(\sigma+2)} \times\beta^{-\sigma-1}2h(\sigma,s)Q_0(1,\sigma,s), \tag{61}$$

since

$$\mathfrak{L}N_1(0,s,E_0)=-\int_K\frac{d\sigma}{2\pi i}\frac{1}{\Gamma(\sigma+2)}\left(\frac{E_0}{\beta}\right)^{\sigma+1}Q_1(1,\sigma,s). \tag{62}$$

In order to simplify $I_1$ we note that, as a result of (29),

$$G_0(\sigma'+1,s)Q_0(1,\sigma',s)=Q_0(1,\sigma'+1,s),$$
$$G_1(\sigma'+1,s)Q_1(\sigma'+2,\sigma,s)=Q_1(\sigma'+1,\sigma+1,s),$$

and therefore

$$I_1=\int_K\frac{d\sigma}{2\pi i}\int_K\frac{d\sigma'}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\pi}{\sin\pi\sigma'}\frac{1}{\Gamma(\sigma+\sigma'+3)} \times\left(\frac{E_0}{\beta}\right)^{\sigma+\sigma'+2}[Q_0(1,\sigma'+1,s)Q_1(\sigma'+2,\sigma,s) -Q_0(1,\sigma',s)Q_1(\sigma'+1,\sigma+1,s)]. \tag{63}$$

In the first term of (63) let $\sigma'$ go into $\sigma'-1$, and then shift the contour back to the contour $K$. The pole at $\sigma'=0$ contributes. In the second term let $\sigma$ go into $\sigma-1$ and shift the contour back to $K$. The pole at $\sigma=0$ contributes. The result is:

$$\begin{aligned}I_1=&-\int_K\frac{d\sigma}{2\pi i}\int_K\frac{d\sigma'}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\pi}{\sin\pi\sigma'}\frac{1}{\Gamma(\sigma+\sigma'+2)} \times\left(\frac{E_0}{\beta}\right)^{\sigma+\sigma'+1}Q_0(1,\sigma',s)Q_1(\sigma'+1,\sigma,s)\\ &-\int_K\frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{1}{\Gamma(\sigma+2)}\left(\frac{E_0}{\beta}\right)^{\sigma+1} \times Q_0(1,0,s)Q_1(1,\sigma,s)\\ &+\int_K\frac{d\sigma}{2\pi i}\int\frac{d\sigma'}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\pi}{\sin\pi\sigma'}\frac{1}{\Gamma(\sigma+\sigma'+2)} \times\left(\frac{E_0}{\beta}\right)^{\sigma+\sigma'+1}Q_0(1,\sigma',s)Q_1(\sigma'+1,\sigma,s)\\ &+\int_K\frac{d\sigma'}{2\pi i}\frac{\pi}{\sin\pi\sigma'}\frac{1}{\Gamma(\sigma'+2)}\left(\frac{E_0}{\beta}\right)^{\sigma'+1} \times Q_0(1,\sigma',s)Q_1(\sigma'+1,0,s)\\ =&\,\mathfrak{L}N_1(0,s,E_0)-\mathfrak{L}N_0(0,s,E_0)\end{aligned} \tag{64}$$

since

$$\mathfrak{L}N_0(0,s,E_0)=-\int_K d\sigma\frac{\pi}{\sin\pi\sigma}\frac{1}{\Gamma(\sigma+2)}\left(\frac{E_0}{\beta}\right)^{\sigma+1}Q_0(1,\sigma,s)$$

and in view of (27), and the cancellation of the double integrals.

One can, therefore, write the Laplace transform of the correction to Snyder's solution in the form:

$$\mathfrak{L}N_2(0, s, E_0)-\mathfrak{L}N_1(0, s, E_0)$$

$$=-\mathfrak{L}N_1(0, s, E_0)\int_K \frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{1}{\Gamma(\sigma+2)}$$

$$\times Q_0(1, \sigma, s)\beta^{-\sigma-1}2h(\sigma, s)+\int_K \frac{d\sigma}{2\pi i}\int_K \frac{d\sigma'}{2\pi i}$$

$$\times\int \frac{dp}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{\pi}{\sin\pi\sigma'}\frac{1}{\Gamma(\sigma'+2)}\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2)}$$

$$\times\left(\frac{E_0}{\beta}\right)^{p+\sigma+1}\beta^{p-\sigma'-1}Q_0(1, \sigma', s)Q_1(p+1, \sigma, s)$$

$$\times\{2\mathfrak{A}(p, \sigma', s)+\mathfrak{B}(p, \sigma')-\mathfrak{C}(p, \sigma')\}. \quad (65)$$

The $p$ contour for the triple complex integral in (65) is a line parallel to the imaginary axis in the strip $0<Re(p-\sigma'-1)<0$; $-1<Re\,p<0$. The next singularity to the right of the contour lies at $p=\sigma'$.

The order of magnitude of the contribution of this term can be estimated crudely by considering $p-\sigma'$ to be very nearly zero, and noting that the integral is very roughly $\mathfrak{L}N_1(0, s, E_0)(1/E_0)$. (In the same fashion the single complex integral term is roughly of order $1/\beta$.) But in all cases of interest in shower theory $E_0$ is greater than $10\beta$. Hence we neglect the contribution of this term.

Therefore, to this order, in virtue of the convolution theorem for Laplace transform one can write:

$$N_2(0, t, E_0)-N_1(0, t, E_0)$$

$$=\int_0^t dxN_1(0, t-x, E_0)F(x), \quad (66)$$

where

$$\mathfrak{L}F(s)=-\int_K \frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{1}{\Gamma(\sigma+2)}$$

$$\times Q_0(1, \sigma, s)\beta^{-\sigma-1}2h(\sigma, s). \quad (67)$$

To this order the first iteration (66) satisfies the boundary conditions since the correction as calculated in (66) is obviously zero for $t=0$.

It is convenient to evaluate (67) in terms of the poles of its integrand to the right of the $\sigma$-contour. The first of these is of the third order and occurs at $\sigma=0$, a $1/\sigma$ contribution coming from the term $\pi/\sin\pi$, a $b/\sigma^2$ contribution from the term $h(\sigma,s)$. The pole strength, as determined by integration by parts of (59), is

$$b=-a\left[\frac{1}{s+D}-\frac{D}{(s+D)^2}\right]=\frac{-as}{(s+D)^2}. \quad (68)$$

Equation (67) can now be written as

$$\mathfrak{L}F(s)=-a\left[\frac{1}{s+D}-\frac{D}{(s+D)^2}\right]\frac{\partial^2}{\partial\sigma^2}\left\{\beta^{-\sigma-1}\frac{Q_0(1, \sigma, s)}{\Gamma(\sigma+2)}\right\}$$

$$-\int_{K+1}\frac{d\sigma}{2\pi i}\frac{\pi}{\sin\pi\sigma}\frac{1}{\Gamma(\sigma+2)}\beta^{-\sigma-1}Q_0(1, \sigma, s)2h(\sigma, s). \quad (69)$$

Since the location of the next pole is at $\sigma=1$, the integral in (69) can be estimated to be of order $1/\beta^2$. It is not feasible to evaluate the contribution of the next pole since it is of fourth order and its residue is too complicated for ready numerical calculation. Moreover, it is not even clear that this residue, by itself, is a proper Laplace transform. Therefore, to order $1/\beta$, we can write the correction to the completely screened solution in the form

$$N_2(0, t, E_0)-N_1(0, t, E_0)=a\int_0^t dxN_1(0, t-x, E_0)$$

$$\times\left\{\frac{\ln^2\beta}{\beta}W_1(x)-\frac{2\ln\beta}{\beta}W_2(x)+\frac{1}{\beta}W_3(x)\right\}, \quad (70)$$

where

$$\mathfrak{L}W_1(s)=\left.\frac{Q_0(1, \sigma, s)}{\Gamma(\sigma+2)}\right|_{\sigma=0}\frac{-as}{(s+D)^2}$$

$$=-as/(s+D)^2, \quad (71)$$

$$\mathfrak{L}W_2(s)=\left.\frac{\partial}{\partial\sigma}\frac{Q_0(1, \sigma, s)}{\Gamma(\sigma+2)}\right|_{\sigma=0}\frac{-as}{(s+D)^2}$$

$$=\frac{-as}{(s+D)^2}\left\{\lim_{N\to\infty}\left(\ln[s+A(N+1)]\right.\right.$$

$$\left.\left.-\sum_{k=1}^N\frac{A'(k)}{s+A(k)}\right)-\psi(1)\right\}, \quad (72)$$

$$\mathfrak{L}W_3(s)=\left.\frac{\partial^2}{\partial\sigma^2}\frac{Q_0(1, \sigma, s)}{\Gamma(\sigma+2)}\right|_{\sigma=0}\frac{-as}{(s+D)^2}$$

$$=\frac{-as}{(s+D)^2}\left\{\left[\lim_{N\to\infty}\left(\ln[s+A(N+1)]\right.\right.\right.$$

$$\left.-\sum_{k=1}^N\frac{A'(k)}{s+A(k)}\right)\Bigg]^2+\sum_{k=1}^\infty\left[\frac{A'(k)^2}{[s+A(k)]^2}\right.$$

$$\left.\left.-\frac{A''(k)}{s+A(k)}\right]+\psi(1)^2-\psi'(1)\right\}$$

$$-2\psi(1)\mathfrak{L}W_1(s). \quad (73)$$

Here primes indicate differentiations. $\partial Q_0(1, \sigma, s)/\partial\sigma$ and $\partial^2Q_0(1, \sigma, s)/\partial\sigma^2$ were calculated by taking logarith-

TABLE II. Tables of the $W(t)$'s for use in numerically effecting the convolutions indicated in Eqs. (70) and (70′).

| $t$ | $W_1(t)$ | $W_2(t)$ | $W_3(t)$ | $t$ | $W_1(t)$ | $W_2(t)$ | $W_3(t)$ |
|---|---|---|---|---|---|---|---|
| 0 | −1.000 | $-\infty$ | $-\infty$ | 5.8 | 0.039 | −0.065 | 0.121 |
| 0.2 | −0.724 | −0.302 | 2.051 | 6.0 | 0.034 | −0.060 | 0.119 |
| 0.4 | −0.507 | +0.299 | 1.590 | 6.2 | 0.030 | −0.055 | 0.114 |
| 0.6 | −0.337 | 0.492 | 0.756 | 6.4 | 0.027 | −0.051 | 0.110 |
| 0.8 | −0.205 | 0.521 | 0.215 | 6.6 | 0.024 | −0.047 | 0.106 |
| 1.0 | −0.104 | 0.498 | −0.154 | 6.8 | 0.021 | −0.043 | 0.101 |
| 1.2 | −0.028 | 0.431 | −0.393 | 7.0 | 0.019 | −0.039 | 0.097 |
| 1.4 | +0.028 | 0.356 | −0.519 | 7.2 | 0.017 | −0.035 | 0.091 |
| 1.6 | 0.069 | 0.279 | −0.584 | 7.4 | 0.015 | −0.032 | 0.086 |
| 1.8 | 0.097 | 0.207 | −0.590 | 7.6 | 0.013 | −0.028 | 0.080 |
| 2.0 | 0.116 | 0.142 | −0.547 | 7.8 | 0.012 | −0.026 | 0.075 |
| 2.2 | 0.128 | 0.087 | −0.500 | 8.0 | 0.010 | −0.023 | 0.072 |
| 2.4 | 0.134 | 0.040 | −0.416 | 8.2 | 0.0094 | −0.021 | 0.064 |
| 2.6 | 0.135 | 0.002 | −0.339 | 8.4 | 0.0083 | −0.018 | 0.059 |
| 2.8 | 0.134 | −0.028 | −0.267 | 8.6 | 0.0072 | −0.016 | 0.055 |
| 3.0 | 0.130 | −0.052 | −0.202 | 8.8 | 0.0064 | −0.014 | 0.050 |
| 3.2 | 0.124 | −0.069 | −0.145 | 9.0 | 0.0056 | −0.013 | 0.047 |
| 3.4 | 0.117 | −0.080 | −0.098 | 9.2 | 0.0050 | −0.011 | 0.043 |
| 3.6 | 0.110 | −0.091 | −0.041 | 9.4 | 0.0040 | −0.010 | 0.040 |
| 3.8 | 0.103 | −0.096 | −0.004 | 9.6 | 0.0038 | −0.0089 | 0.037 |
| 4.0 | 0.095 | −0.098 | 0.026 | 9.8 | 0.0034 | −0.0080 | 0.034 |
| 4.2 | 0.087 | −0.099 | 0.052 | 10.0 | 0.0029 | −0.0071 | 0.031 |
| 4.4 | 0.080 | −0.097 | 0.074 | 11 | 0.0029 | −0.0041 | 0.031 |
| 4.6 | 0.073 | −0.094 | 0.092 | 12 | 0.0015 | −0.0023 | 0.020 |
| 4.8 | 0.066 | −0.091 | 0.104 | 13 | 0.0008 | −0.0014 | 0.011 |
| 5.0 | 0.059 | −0.086 | 0.112 | 14 | 0.0004 | −0.0007 | 0.0057 |
| 5.2 | 0.054 | −0.080 | 0.118 | 15 | 0.0002 | −0.0004 | 0.0030 |
| 5.4 | 0.048 | −0.075 | 0.121 | 16 | | −0.0002 | 0.0015 |
| 5.6 | 0.044 | −0.070 | 0.122 | 17 | | −0.0001 | 0.0001 |

mic derivatives of the infinite product representation for $Q_0(1, \sigma, s)$.

The Laplace inverses of (71), (72), and (73) are readily obtainable in terms of single and double series of the Dirichlet type, but it is more convenient for the derivation of a table of values to adopt the following scheme.

Let us define

$$\mathfrak{L}U_1(s)=\lim_{N\to\infty}\left(\ln[s+A(N+1)]-\sum_{k=1}^{N}\frac{A'(k)}{s+A(k)}\right)\frac{1}{s+D}, \tag{74}$$

$$\mathfrak{L}U_2(s)=U_1(s)/(s+D), \tag{75}$$

$$\mathfrak{L}U_3(s)=[\mathfrak{L}U_1(s)]^2, \tag{76}$$

$$\mathfrak{L}U_4(s)=(s+D)\mathfrak{L}U_3(s), \tag{77}$$

$$\mathfrak{L}U_5(s)=\sum_{k=1}^{\infty}\left[\left(\frac{A'(k)}{s+A(k)}\right)^2-\frac{A''(k)}{s+A(k)}\right], \tag{78}$$

$$\mathfrak{L}U_6(s)=\mathfrak{L}U_5(s)/(s+D), \tag{79}$$

$$\mathfrak{L}U_7(s)=\mathfrak{L}U_6(s)/(s+D). \tag{80}$$

The $W$'s can be defined in terms of these, as is done later.

Then

$$U_1(t)=e^{-Dt}\lim_{N\to\infty}\left\{\ln[A(N+1)-D]-\sum_{k=1}^{N}\frac{A'(k)}{A(k)-D}\right\}+\sum_{k=1}^{\infty}\frac{A'(k)}{A(k)-D}e^{-A(k)t}. \tag{81}$$

Expression (81) converges for all $t>0$ and diverges like $-\ln t$ as $t$ approaches zero. The series in (81) can be summed numerically by means of the Euler summation formula,[8] which also permits one to develop an approximation formula for $U_1(t)$ valid near $t=0$. Also one can write

$$U_5(t)=\sum_{k=1}^{\infty}\{tA'(k)^2-A''(k)\}e^{-A(k)t}. \tag{82}$$

Equation (82) too is evaluable numerically by means of the Euler summation formula, the series converging for $t\geq 0$.

The other quantities, both $U$'s and $W$'s, can then be computed numerically, in stepwise fashion, according to the prescription

$$\begin{aligned}
U_2(t)&=\int_0^t dxU_1(t-x)e^{-Dx},\\
U_3(t)&=\int_0^t dxU_1(x)U_1(t-x),\\
U_4(t)&=U_3'(t)+DU_3(t),\\
U_6(t)&=\int_0^t dxU_5(t-x)e^{-Dx},\\
U_7(t)&=\int_0^t dxU_6(t-x)e^{-Dx},\\
W_1(t)&=e^{-Dt}(Dt-1),\\
W_2(t)&=-\psi(1)W_1(t)-U_1(t)+DU_2(t),\\
W_3(t)&=-U_4(t)+DU_3(t)-U_6(t)+DU_7(t)\\
&\quad+2\psi(1)[U_1(t)-DU_2(t)]\\
&\quad+[\psi(1)^2-\psi'(1)]W_1(t).
\end{aligned} \tag{83}$$

The $W$'s are tabulated in Table II and suitable approximation formulas for $t\sim 0$ are given in Table III.

It is now possible to evaluate the correction to $N_1(0, t, E_0)$ to the order of accuracy represented by (70), by carrying out numerically the indicated integrations, using the values of $N_1(0, t, E_0)$ given by Snyder.[1]

It is interesting to note that to this approximation

TABLE III. Approximation formulas for use in effecting part of the convolutions indicated in Eqs. (70) and (70′).

| | |
|---|---|
| $W_2(t)\sim -\ln t$ | $[-1+1.550t-0.60t^2]+1.007-0.881t+0.17t^2$ |
| $W_3(t)\sim \ln^2 t$ | $[-1+1.549t-0.30t^2]-\ln t[2.015-1.757t+0.63t^2]$ |
| | $+1.276-21844t+1.85t^2$ |
| | $0<t\leq 0.2$ |

[8] K. Knopp, *Theory and Application of Infinite Series* (Blackie and Son, Ltd., London, 1928).

condition (42) is satisfied by the corrected solution because

$$\int_0^\infty dtN_2(0,t,E_0)$$

$$=\int_0^\infty dtN_1(0,t,E_0)+\int_0^\infty dt\,a\int_0^t dxN_1(0,t-x,E_0)$$

$$\times\left\{\frac{\ln^2\beta}{\beta}W_1(x)-\frac{2\ln\beta}{\beta}W_2(x)+\frac{1}{\beta}W_3(x)\right\}$$

$$=\mathfrak{L}N_1(0,0,E_0)+aN_1(0,0,E_0)$$

$$\times\left\{\frac{\ln^2\beta}{\beta}\mathfrak{L}W_1(0)-\frac{2\ln\beta}{\beta}\mathfrak{L}W_2(0)+\frac{1}{\beta}\mathfrak{L}W_3(0)\right\}$$

$$=E_0/\beta$$

since

$$\mathfrak{L}N_1(0,0,E_0)=E_0/\beta$$

while the Laplace transforms of all the $W(t)$'s contain as a factor $s/(s+D)^2$ and hence all the $\mathfrak{L}W(0)$'s are zero.

If one examines the case in which the incident spectrum is a single photon of energy $E_0$, then to order $1/E_0$ the results come out in the same form. The result there is

$$\mathfrak{L}N_2^\dagger(0,s,E_0)-\mathfrak{L}N_1^\dagger(0,s,E_0)=\mathfrak{L}N_1^\dagger(0,s,E_0)\mathfrak{L}F(s)\quad(66')$$

and again to order $1/\beta$

$$N_2^\dagger(0,t,E_0)-N_1^\dagger(0,t,E_0)$$

$$=a\int_0^t dxN_1^\dagger(0,t-x,E_0)$$

$$\times\left\{\frac{\ln^2\beta}{\beta}W_1(x)-\frac{2\ln\beta}{\beta}W_2(x)+\frac{1}{\beta}W_3(x)\right\}.\quad(70')$$

Since in the "complete screening" approximation (subscript 1) the solution of (1) and (2) for any initial spectrum is but a linear combination of the solutions for a single incident electron and single incident photon, we see that to order $1/E_0$ in the first iteration the correction introduced by screening is always in the form of a convolution over the "complete screening" solution.

## VI. DISCUSSION AND CONCLUSIONS

The introduction of the more accurate cross sections into the diffusion equations means that the probabilities

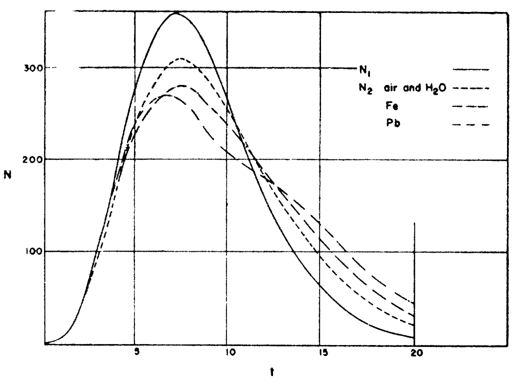


FIG. 2. $N(t)$ the average number of electrons induced by a single incident electron of energy $E_0$ *vs.* absorber thickness $t$. $\beta$=ionization loss per unit radiation length; Snyder solution $N_1(t)$; corrected solution $N_2(t)\cdot\ln E_0/\beta=8$.

for the various elementary processes (bremsstrahlung and pair production) are decreased, the decrease being larger the smaller is the energy $E_0$ of the parent particle. This implies that $N(0,t,E_0)$, the average number of electrons at thickness $t$, must be smaller initially than that calculated under the assumption of complete screening. But, the area under $N(0,t,E_0)$ is $E_0/\beta$ regardless of the cross sections used. Therefore, there must be a corresponding increase in $N(0,t,E_0)$ for large values of $t$. These qualitative results are verified by those of the numerical calculations which can be considered to be accurate.

One would expect that the screening correction should depend on the energy of the incident particle in such a way that if this latter is high then the screening correction is low, and vice versa. This property is possessed by both (66) and (70). In both of these the dependence on $E_0/\beta$ is given through $N_1(0,t,E_0)$, the average number of electrons at thickness $t$ under the assumption of complete screening. The larger $E_0/\beta$, the larger is the value of $t$ for which the maximum in $N_1(0,t,E_0)$ occurs. Since in both cases the function with which $N_1(0,t,E_0)$ is convoluted exhibits exponential decrease for sufficiently large values of $t$, (66) and (70) yield proportionally smaller corrections, the larger is $E_0/\beta$.

The dependence on material is introduced essentially through the $\beta$ (ionization loss per unit length) dependence of the term with which $N_1(0,t,E_0)$ is convoluted. This term, and hence the correction, is an increasing function of $1/\beta$ and, therefore, of $Z$.

The estimate which led to (66) as a good approximation to the correction as given by the first iteration is crude but plausible. Physically, if we note the origins of the terms $\mathfrak{A}(p,\sigma',s)$, $\mathfrak{B}(p,\sigma')$, and $\mathfrak{C}(p,\sigma')$ in (57), the estimate says that the correction to the Snyder (complete screening) solution is largely due to the correction of the cross section for pair production alone.

For reasonably high incident energies, the correction is only about 10 percent at the maximum of $N_1(0,t,E_0)$.

TABLE IV. Representative values of various parameters.

| | Air | $H_2O$ | Al | Fe | Pb |
|---|---|---|---|---|---|
| $\beta$ (Mev) | 103 | 115 | 55.6 | 25.9 | 7.00 |
| $\Delta$ (cm) | 34.2 | 43.4 | 9.80 | 1.84 | 0.525 |
| $K$ (Mev) | 5.15 | 5.70 | 4.55 | 3.85 | 3.05 |
| $\beta/K$ | 20.0 | 20.0 | 12.2 | 6.72 | 2.27 |

TABLE V. Part 1. $N_1(t)$ = Snyder solution in the case of a single incident electron, $N_2(t)$ = the corresponding corrected solution. $\epsilon = \ln E_0/\beta$, $N_1{*}W = \int_0^t dx N_1(t-x)W(x)$, $N_2(t) = N_1(t) + 1.358[N_1{*}W_1(\ln^2\beta/\beta) - N_1{*}W_2(2\ln\beta/\beta) + N_1{*}W_3(1/\beta)]$.

| $t$ | $N_1$ | $N_1{*}W_1$ | $N_1{*}W_2$ | $N_1{*}W_3$ | $N_1$ | $N_1{*}W_1$ | $N_1{*}W_2$ | $N_1{*}W_3$ | $N_1$ | $N_1{*}W_1$ | $N_1{*}W_2$ | $N_1{*}W_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\epsilon=3$ | | | | $\epsilon=4$ | | | | $\epsilon=5$ | | |
| 1 | 3.75 | −1.36 | −0.44 | 1.08 | 4.87 | −1.57 | −0.69 | 0.95 | 5.68 | −1.54 | −0.87 | 0.87 |
| 2 | 4.55 | −2.00 | 0.86 | 0.92 | 8.86 | −3.55 | 0.55 | 2.44 | 14.46 | −5.38 | −0.25 | 3.06 |
| 3 | 3.85 | −1.47 | 1.58 | −0.96 | 10.13 | −4.00 | 2.43 | 0.03 | 21.17 | −8.27 | 2.91 | 2.20 |
| 4 | 2.81 | −0.44 | 1.55 | −2.37 | 9.07 | −2.84 | 3.63 | −3.08 | 23.60 | −8.55 | 6.28 | −2.13 |
| 5 | 1.94 | 0.36 | 0.76 | −2.23 | 7.09 | −0.88 | 3.08 | −4.93 | 22.01 | −5.33 | 7.70 | −7.67 |
| 6 | 1.22 | 0.80 | 0.03 | −1.29 | 5.23 | 0.71 | 1.54 | −4.51 | 18.40 | −2.26 | 6.43 | −11.30 |
| 7 | 0.68 | 0.97 | −0.48 | −0.30 | 3.52 | 1.72 | 0.10 | −2.70 | 13.97 | −1.28 | 3.60 | −10.50 |
| 8 | 0.48 | 0.88 | −0.67 | 0.55 | 2.39 | 2.08 | −1.03 | −0.60 | 9.96 | −3.64 | 0.48 | −6.50 |
| 9 | | | | | 0.90 | 1.75 | −1.56 | 1.15 | 4.72 | 4 57 | −3.46 | 1.59 |
| 12 | | | | | 0.31 | 1.06 | −1.32 | 2.59 | 1.99 | 3.55 | −0.39 | 6.55 |
| 14 | | | | | | | | | 0.79 | 2.05 | −0.20 | 5.28 |
| 16 | | | | | | | | | 0.30 | 1.10 | −0.08 | 3.88 |
| 18 | | | | | | | | | 0.12 | 0.55 | −0.05 | 2.21 |
| | | $\epsilon=6$ | | | | $\epsilon=7$ | | | | $\epsilon=8$ | | |
| 1 | 6.33 | −1.89 | −0.99 | 0.88 | 6.7 | −1.57 | −1.22 | 0.01 | 8.5 | −1.89 | −1.60 | −0.38 |
| 2 | 26.7 | −6.98 | −1.38 | 3.10 | 33.1 | −10.64 | −3.70 | 5.48 | 44.3 | −13.12 | −5.67 | 9.08 |
| 3 | 41.0 | −15.26 | 1.51 | 5.07 | 68.7 | −25.26 | 1.43 | 11.46 | 115.8 | −39.96 | −2.99 | 27.61 |
| 4 | 53.42 | −20.22 | 9.79 | 2.19 | 106.3 | −38.96 | 12.79 | 6.68 | 201.5 | −74.53 | 15.59 | 31.64 |
| 5 | 57.33 | −18.85 | 16.28 | −9.40 | 135.0 | −46.89 | 26.71 | −6.63 | 287.5 | −102.71 | 44.70 | 11.02 |
| 6 | 54.20 | −12.79 | 17.56 | −20.08 | 142.1 | −42.46 | 38.89 | −31.19 | 335.4 | −107.34 | 77.12 | −38.33 |
| 7 | 46.42 | −4.72 | 14.55 | −23.82 | 135.2 | −28.13 | 39.92 | −48.92 | 357.5 | −93.99 | 90.89 | −84.57 |
| 8 | 37.25 | +2.69 | 7.86 | −20.95 | 119.2 | −10.65 | 31.18 | −54.26 | 346.0 | −64.42 | 89.60 | −115.81 |
| 10 | 20.16 | 10.68 | −3.88 | −3.44 | 75.6 | −18.22 | 3.30 | −28.66 | 261.3 | 6.27 | 40.83 | −118.63 |
| 12 | 9.67 | 10.70 | −9.94 | 10.77 | 41.1 | 27.43 | −17.34 | 12.04 | 155.2 | 58.8 | −18.32 | 11.86 |
| 14 | 4.0 | 7.45 | −8.39 | 15.45 | 20.2 | 23.72 | −22.30 | 34.06 | 85.5 | 67.4 | −53.22 | 76.72 |
| 16 | 1.67 | 4.35 | −5.62 | 12.56 | 9.1 | 16.11 | −18.06 | 36.01 | 42.2 | 51.5 | −52.63 | 98.60 |
| 18 | 0.67 | 2.14 | −3.21 | 8.10 | 3.9 | 9.00 | −11.84 | 26.73 | 18.9 | 32.9 | −41.13 | 84.34 |
| 20 | 0.10 | 1.02 | −1.58 | 4.41 | 1.6 | 4.49 | −6.62 | 17.06 | 8.0 | 18.8 | −29.62 | 59.56 |

Hence one might expect that the first iteration is a sufficient approximation. In view of the complicated nature of the functions involved, a more accurate determination of the errors is not feasible.

It is possible to obtain numerical results from (70), but it is prohibitively laborious to calculate a more accurate approximation to (66). However, one can expect that (70) is a good approximation to (66) if $\beta$ is large. This is true for light elements. Table IV gives values of the various parameters to be used in the calculations.

In view of this discussion we see that the numerical calculations made are applicable only to light elements and moderate or high initial energies. Tables of the convolutions necessary for the calculation in the case of those initial conditions considered in this paper are given in Table V. In these the indices 0 and $E_0$ are suppressed. They are based on the values for the completely screened solution given by Snyder.[1]

Representative results are given in Figs. 2 and 3.

Figure 4 indicates how well the approximation to the cross sections here used, that is Eq. (6), agrees with the

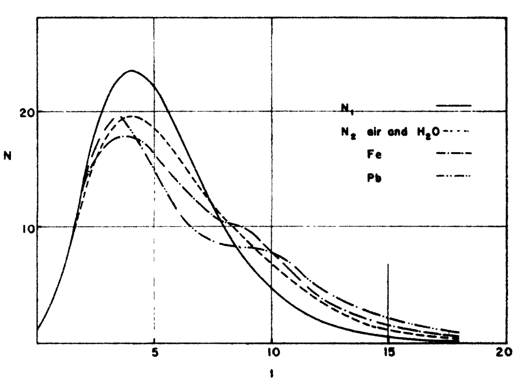


FIG. 3. $N(t)$ the average number of electrons induced by a single incident electron of energy $E_0$ vs. absorber thickness $t$. $\beta$ = ionization loss per unit radiation length. Snyder solution $N_1(t)$; corrected solution $N_2(t)\cdot\ln E_0/\beta = 5$.

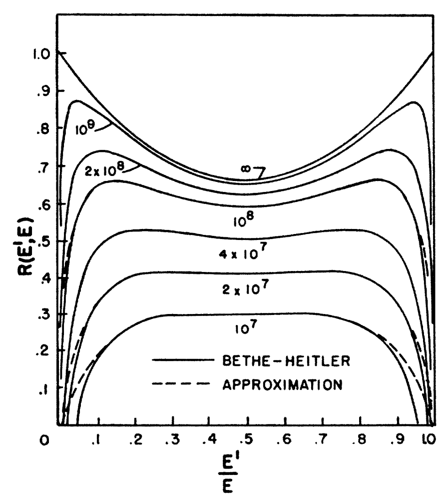


FIG. 4. $R(E, E')$ the differential probability of pair production per unit radiation length vs. fractional energy transferred. Labels on solid curves indicate $E$, the energy in electron volts of the parent particle.

TABLE V. Part 2. $N_1^\dagger(t)$ = Snyder solution in the case of a single incident photon, $N_2^\dagger(t)$ = the corresponding corrected solution. $\epsilon = \ln E_0/\beta$, $N_1^\dagger * W = \int_0^t dx N_1(t-x)W(x)$, $N_2^\dagger(t) = N_1^\dagger(t) + 1.358[N_1^\dagger * W_1(\ln^2\beta/\beta) - N_1^\dagger * W_2(2\ln\beta/\beta) + N_1^\dagger * W_3(1/\beta)]$.

| $t$ | $N_1\dagger$ | $N_1\dagger * W_1$ | $N_1\dagger * W_2$ | $N_1\dagger * W_3$ | $N_1\dagger$ | $N_1\dagger * W_1$ | $N_1\dagger * W_2$ | $N_1\dagger * W_3$ | $N_1\dagger$ | $N_1\dagger * W_1$ | $N_1\dagger * W_2$ | $N_1\dagger * W_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\epsilon=3$ | | | | $\epsilon=4$ | | | | | $\epsilon=5$ | |
| 1 | 1.88 | −0.57 | −0.33 | 0.30 | 2.27 | −0.58 | −0.42 | 0.26 | 3.1 | −0.44 | −0.56 | 0.39 |
| 2 | 3.59 | −1.43 | 0.12 | 0.89 | 5.97 | −2.15 | −0.22 | 1.19 | 8.85 | −3.13 | −0.39 | 1.71 |
| 3 | 3.86 | −1.58 | 1.06 | 0.09 | 8.51 | −3.39 | 1.20 | 1.13 | 16.19 | −6.03 | 1.06 | 2.15 |
| 4 | 3.34 | −1.04 | 1.45 | −1.26 | 9.02 | −3.27 | 2.67 | −1.05 | 20.40 | −7.60 | 4.04 | 0.16 |
| 5 | 2.49 | 0.25 | 1.17 | −1.99 | 8.04 | −2.06 | 3.07 | −3.34 | 21.60 | −6.86 | 6.21 | −4.19 |
| 6 | 1.75 | 0.38 | 0.52 | −1.56 | 6.51 | −0.55 | 2.34 | −4.21 | 19.96 | −4.40 | 6.53 | −8.67 |
| 7 | 1.12 | 0.71 | −0.07 | −0.78 | 4.85 | 0.73 | 1.14 | −3.53 | 16.93 | −1.33 | 5.01 | −9.11 |
| 8 | 0.80 | 0.79 | −0.45 | 0.04 | 3.48 | 1.51 | −0.06 | −1.99 | 13.20 | 1.44 | 2.55 | −7.75 |
| 10 | 0.29 | 0.63 | −0.61 | 0.91 | 1.60 | 1.78 | −1.33 | 1.07 | 7.40 | 4.03 | −1.83 | −0.61 |
| 12 | | | | | 0.65 | 1.29 | −1.40 | 2.29 | 3.28 | 4.01 | −3.36 | 4.24 |
| 14 | | | | | | | | | 1.32 | 2.77 | −3.11 | 5.67 |
| | | | | | | | | | 0.52 | 1.50 | −2.05 | 4.66 |
| | | | | | | | | | 0.19 | 0.76 | −1.13 | 2.92 |
| | | $\epsilon=6$ | | | | $\epsilon=7$ | | | | | $\epsilon=8$ | |
| 1 | 3.2 | −0.82 | −0.61 | 0.20 | 3.6 | −0.90 | −0.74 | 0.15 | 8.5 | −2.58 | −1.55 | 1.34 |
| 2 | 13.33 | −4.18 | −1.54 | 1.78 | 13.9 | −4.57 | −1.28 | 2.28 | 24.4 | −8.17 | −1.29 | 3.51 |
| 3 | 27.2 | −10.19 | 0.65 | 4.93 | 46.0 | −14.79 | −2.86 | 5.12 | 71.0 | −23.54 | −2.34 | 8.16 |
| 4 | 42.5 | −15.58 | 5.14 | 2.68 | 78.9 | −29.78 | 5.56 | 12.04 | 140.2 | −49.71 | 5.46 | 16.24 |
| 5 | 51.2 | −17.96 | 11.29 | −2.99 | 109.0 | −39.60 | 18.30 | 1.25 | 218.2 | −78.53 | 25.79 | 12.82 |
| 6 | 53.5 | −15.57 | 15.01 | −12.71 | 130.1 | −41.79 | 29.26 | −16.48 | 285.2 | −95.01 | 52.13 | −14.38 |
| 7 | 49.9 | −9.75 | 14.88 | −18.90 | 133.9 | −35.52 | 36.41 | −33.21 | 326.0 | −97.00 | 74.22 | −51.47 |
| 8 | 43.0 | −2.90 | 11.25 | −20.46 | 126.5 | −22.54 | 34.49 | −46.19 | 342.1 | −81.22 | 83.75 | −75.63 |
| 10 | 26.5 | 7.07 | 0.01 | −9.59 | 93.5 | 6.59 | 13.38 | −37.46 | 291.4 | −14.15 | 59.71 | −108.9 |
| 12 | 13.8 | 10.57 | −7.25 | 5.88 | 54.5 | 23.86 | −9.18 | −3.99 | 201.2 | 39.38 | 4.52 | −49.54 |
| 14 | 6.8 | 7.97 | −8.42 | 13.51 | 29.4 | 24.75 | −20.75 | 25.48 | 113.5 | 64.34 | −37.99 | 28.52 |
| 16 | 3.5 | 4.80 | −6.32 | 13.20 | 13.5 | 18.58 | −19.17 | 34.54 | 61.2 | 57.29 | −53.39 | 81.11 |
| 18 | 1.9 | 2.80 | −4.14 | 9.32 | 5.8 | 11.79 | −14.18 | 29.99 | 29.9 | 39.92 | −44.06 | 85.43 |
| 20 | 0.5 | 1.75 | −2.09 | 5.42 | 2.6 | 6.50 | −9.18 | 20.93 | 9.9 | 25.64 | −29.94 | 66.55 |

corresponding Bethe-Heitler values in the case of pair production.

From these, qualitatively, the effect of the correction on $N_1(0, t, E_0)$ is an initial decrease, a decrease in the maximum and slight shift of the maximum to greater depths, and a consequent increase for large $t$.

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