A Geometrical Model for the Unified Theory of Physical Fields*

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IT is possible to give a unified field theory based on the generalized geometry introduced by Finsler.[1] Finsler's geometry, as is well known,[2] is a generalization of Riemann's and can be regarded as a manifold of line elements in the space of which the metric is defined by

$$ds=L(x^i, \dot{x}^i)dt \tag{1}$$

where x^i represents a point of the n-dimensional space and $\dot{x}^i$ will be in our case the derivative of x^i with respect to the arc length of the curve $x^i=x^i(t)$. The space has a general covariant formalism, which also is a direct generalization of Riemann's. As a consequence of (1), the metrical ground tensor, being a symmetric covariant tensor of the second order is

$$g_{ik}=g_{ik}(x^i, \dot{x}^i)=\tfrac{1}{2}\partial^2(L^2)/\partial\dot{x}^i\partial\dot{x}^k \tag{2}$$

and the parallel displacement of a covariant vector ξ_i is defined by

$$d\xi_i=C_i{}^r{}_s\,\xi_r d\dot{x}^s+\Gamma_i{}^r{}_s\,\xi_r dx^s \tag{3}$$

where the coefficients of the affine connection $C_i{}^r{}_s$ and $\Gamma_i{}^r{}_s$ are deducible from $L(x^i, \dot{x}^i)$ as follows

$$\begin{aligned}
&C_i{}^k{}_j=g^{kr}C_{irj},\\
&C_{ikj}=\frac{1}{2}\frac{\partial g_{ik}}{\partial\dot{x}^j}=\frac{1}{4}\frac{\partial^3(L^2)}{\partial\dot{x}^i\partial\dot{x}^k\partial\dot{x}^j},\\
&\Gamma_i{}^k{}_j=g^{kr}\Gamma_{irj},\\
&\Gamma_{ijk}=\frac{1}{2}\left\{\frac{\partial g_{ij}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right\}+C_{ikr}\frac{\partial G^r}{\partial\dot{x}^j}-C_{jkr}\frac{\partial G^r}{\partial\dot{x}^i},\\
&\quad G^r=G_i g^{ir},\\
&\quad G_i=\frac{1}{4}\left\{\frac{\partial^2(L^2)}{\partial\dot{x}^i\partial x^r}\dot{x}^r-\frac{\partial(L^2)}{\partial x^i}\right\},\\
&\Gamma^*{}_i{}^r{}_k=\Gamma_i{}^r{}_k-C_i{}^r{}_s\,\Gamma_k{}^s{}_h\,\dot{x}^h.
\end{aligned} \tag{4}$$

The space is characterized by its curvature and torsion. It is usual to introduce the totally symmetric covariant tensor of the third order

$$A_{ikh}=L(x^i, \dot{x}^i)\cdot C_{ikh} \tag{5}$$

which determines the deviation of Finsler's space from Riemann's, which arises from the original space by discarding the dependence of the g_{ik} on the $\dot{x}^i$ (this is equivalent to $A_{ikh}\equiv 0$). The curvature and torsion are given by the tensors

$$\begin{aligned}
R^k{}_{ihj}=&\frac{\partial\Gamma^*{}_i{}^k{}_h}{\partial x^j}-\frac{\partial\Gamma^*{}_i{}^k{}_h}{\partial\dot{x}^s}\frac{\partial G^s}{\partial\dot{x}^j}-\left\{\frac{\partial\Gamma^*{}_i{}^k{}_j}{\partial x^h}-\frac{\partial\Gamma^*{}_i{}^k{}_j}{\partial\dot{x}^s}\frac{\partial G^s}{\partial\dot{x}^h}\right\}\\
&+\Gamma^*{}_i{}^s{}_h\cdot\Gamma^*{}_s{}^k{}_j-\Gamma^*{}_i{}^s{}_j\cdot\Gamma^*{}_s{}^k{}_h,\\
P_{ijkh}=&\Gamma^*{}_j{}^r{}_k\cdot A_{ihr}-\Gamma^*{}_i{}^r{}_k\cdot A_{jkr}-L\cdot\frac{\partial C_{ijh}}{\partial x^k}+L\cdot\frac{\partial\Gamma_{ijk}}{\partial\dot{x}^h},\\
S_{ijkh}=&A_j{}^r{}_k\cdot A_{ihr}-A_j{}^r{}_h\cdot A_{ikr}.
\end{aligned} \tag{6}$$

We now suppose that, in accordance with Einstein's original idea, the curvature of the space is determined by the distribution of the matter localized in the space and the deviation from Riemann's geometry; as an example, the torsion of the space originates from the electromagnetic, or generally from the meson field.

To realize this assumption we give the connection between the structure of the Finsler space and the electromagnetic (meson) field. We characterize the field by its symmetrized stress-energy tensory S_{ik}, and from this we deduce the following total symmetric covariant tensor of the third order

$$A_{ikh}=-\tfrac{1}{2}\cdot\{S_{ik|h}+S_{kh|i}+S_{hi|k}\} \tag{7}$$

(where $S_{ik|h}$ is the covariant derivative of S_{ik}) which we shall regard as the torsion tensor of Finsler's geometry.

Assumption (7) has also a physical meaning ,and it is easy to see that it can be considered as a generalization of the Lorentz force. To justify this remark we deduce from (7), by contraction of the indices k and h, the following covariant vector

$$A_i=A_i{}^k{}_k=g^{kr}\cdot A_{irk}=-\partial S_i{}^k/\partial x^k, \tag{8}$$

which is just the density of the Lorentz force of the field.

Mathematically (7) gives in the case of the four-dimensional space-time continuum 20 independent partial differential equations for the determination of the dependence of L (or g_{ik}) on the $\dot{x}^i$.

For the determination of the dependence on the x^i we have also Einstein's original equations

$$R_{ik}-\tfrac{1}{2}g_{ik}R=\kappa T_{ik} \tag{9}$$

but now R_{ik} and R are deduced from (6) by the well-known method. T_{ik} is as usual the energy-momentum tensor of the matter. Equation (9) gives also the further 10 partial differential equations for the determination of the dependence of the g_{ik} on the x^i.

Finally, we must pay attention to the fact that the proposed connection between the field theories and the geometrical basis is not only a formal one. We have mentioned that Finsler's space can be regarded as a manifold of line elements defined along curves of the space. If we regard such a curve as the world-line of a particle (electron, nucleon, etc.) by which the field is originated, the line elements have also a direct physical meaning since the unit vector

$$l^i=\dot{x}^i/L \tag{10}$$

deduced from $\dot{x}^i$, can be regarded as the velocity of the particle along the world-line. It is well known that the field of such a moving particle has a strong inhomogeneity. Now this inhomogeneity is faithfully expressed by Finsler's geometry. In cases in which the inhomogeneity would be dissolved (e.g., the static case) every component of A_{ikh} vanishes, as can be verified by elementary calculations, taking account of some of the methods of Finsler's geometry and the usual relations of homogeneous functions. Thus we are not led by our method away from the Riemann space. This theory is also interesting especially in the non-static case, which will be discussed in detail in the near future.

* Extract of the author's habilitation lecture.

[1] P. Finsler, Über Kurven und Flächen in allgemeinen Räumen, Diss., Göttingen, 1918.

[2] E. Cartan, Les espaces de Finsler, Act. scient. et ind. 79 (Hermann et Cie, Paris, 1934); O. Varga, Monat. f. Math. und Phys. Leipzig und Wien **50**, 165 (1941); Comm. Math. Helv. **19**, 367 (1947).

The Diamagnetic Correction for Protons in Water and Mineral Oil

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THE proton nuclear resonance shifts between H_2 gas and water, mineral oil, and the standard sample used in the determination of the proton gyromagnetic ratio[1] have been measured as previously suggested.[1,2] The shifts were measured in a large electromagnet having a $4\frac{1}{2}$-inch gap with pole faces 24 inches across and regulated[3] by means of nuclear resonance. The spatial field variation over a region somewhat larger than that occupied by the sample was of the order of 10 p.p.m. and produced a line width for the distilled water sample of 0.043 gauss.

The three samples were contained in identical spherical Pyrex glass vessels, 1.4-cm i.d. and 2.0-cm o.d. The hydrogen sample was tank hydrogen at a pressure of 40 atmospheres. The distilled water sample was free of dissolved oxygen and the oil sample was Petrolatum U.S.P.-Light.

The resonance shifts between samples were measured relative to a fixed sample and resonance detector probe by means of calibrated Helmholtz coils. Because it is known that slight unpredictable changes in field distribution with time might occur, a