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The Hamiltonian of the General Theory of Relativity with Electromagnetic Field*

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In this paper we have given a specific example of a Hamiltonian of a non-linear field theory, a Hamiltonian density completely free of time derivatives. In accordance with the general theory developed previously, this Hamiltonian is one of the constraints between the canonical variables and, therefore, vanishes everywhere. To obtain this function, we have developed methods that will also permit the construction of Hamiltonian densities in any field theory in which the Lagrangian density is quadratic in the first derivatives. Our Hamiltonian differs from the one obtained by Schild and Pirani in that they use Dirac's method to derive a Hamiltonian that is invariant but contains velocities, so that their canonical field equations cannot be solved with respect to the time derivatives of all canonical variables. In our formalism, the canonical equations contain no time derivatives on the right-hand sides, but the adoption of a particular Hamiltonian is equivalent to the adoption of a particular coordinate condition and gauge condition. However, once we have obtained any one Hamiltonian density, we can readily obtain any other one (and thus go over to arbitrary coordinate and gauge conditions) by combination with the other constraints of the theory in question.

1. INTRODUCTION

In two previous papers,[1,2] it was shown that any set of field equations which can be derived from a variational principle can be cast into the canonical form, with a Hamiltonian which vanishes identically. With the introduction of canonically conjugate variables, the so-called momentum densities, and with the (arbitrary) singling out of some direction at each world point (local "time"-axis), it is possible to reformulate the whole formalism in such a manner that the differential equations are all first-order equations, solved with respect to the "time"-derivatives. On a single "space"-like hypersurface, a small set of equations must be satisfied which do not involve any "time"-derivatives. One of these constraints is the vanishing of the Hamiltonian density. Other constraints are intimately associated with the covariance properties of the theory. If we introduce[2] "parameters," three constraints follow from the invariance of the theory with respect to parameter transformations; if the theory is covariant with respect to general coordinate transformations we shall have four constraints corresponding to coordinate covariance; finally, gauge invariance of the electromagnetic field leads to one constraint of its own. The number of constraints always equals the number of arbitrary functions involved in the transformation group. Once the constraints are satisfied on one hypersurface, the field equations automatically insure that the constraints remain satisfied permanently.

The usefulness of the canonical formalism consists in the relative ease with which the new field equations and their solutions can be discussed. Furthermore, we expect that the quantization of the theories in this form will be a relatively easy and straightforward procedure.

We have given a proof of the existence of the Hamiltonian previously,[2] but did not provide a procedure for its construction. The purpose of this paper is to indicate such a procedure for the large class of theories in which the Lagrangian density is homogeneous and quadratic in the first derivatives. The calculations are then carried through to completion for the best-studied example of a covariant field theory, Einstein's theory of gravitation with an electromagnetic field. The application of the formalism thus obtained to the problem of motion and its quantization will be provided in subsequent papers.

2. THE BASIC FORMS WITH A QUADRATIC LAGRANGIAN

Consider a Lagrangian density which is a homogeneous quadratic function of the first derivatives of the field variables y_A, which, in other words, possesses the form

$$L=\Lambda^{A\rho B\sigma}(y)y_{A,\rho}y_{B,\sigma}. \tag{2.1}$$

For constructing the momentum densities and other pertinent functions, we shall introduce the "parameters" of II, the u^s, t. The modified Lagrangian, JL, will then be homogeneous of the first degree in the "time"-derivatives of all the field variables, including the coordinates. In the special case (2.1), this homogeneous function of the first degree will be a homogeneous quadratic form, divided by a homogeneous linear form. In fact, straightforward calculation shows that JL is given by the expression

$$JL=(l^c\dot{y}_c)^{-1}G^{ab}\dot{y}_a\dot{y}_b, \tag{2.2}$$

where the coefficients G^{ab} and l^a are:

$$\begin{aligned} G^{AB}&=\Lambda^{A\rho B\sigma}Jt_{,\rho}Jt_{,\sigma},\\ G^{A}{}_{\nu}&=\Lambda^{A\rho B\sigma}Jt_{,\rho}y_{B|s}J(u^s{}_{,\sigma}t_{,\nu}-u^s{}_{,\nu}t_{,\sigma}),\\ G_{\mu\nu}&=\Lambda^{A\rho B\sigma}y_{A|r}J(u^r{}_{,\rho}t_{,\mu}-u^r{}_{,\mu}t_{,\rho})y_{B|s}J(u^s{}_{,\sigma}t_{,\nu}-u^s{}_{,\nu}t_{,\sigma}), \end{aligned} \tag{2.3}$$

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[1] P. G. Bergmann, Phys. Rev. **75**, 680 (1949), referred to as I.

[2] P. G. Bergmann and J. H. M. Brunings, Rev. Mod. Phys. **21**, 480 (1949), referred to as II.

and

$$l^A=0, \quad l_\mu=Jt_{,\mu}. \tag{2.4}$$

The significance of the indices $a, b, \cdots$ is the same as in II, p. 485. Now we can form without difficulty the expressions for the $(N+4)$ canonical momentum densities π^a and also for the matrix Λ of II, Eq. (3.13), which is closely associated with the algebraic constraints on the canonical field variables y_a, π^a. We obtain the following expressions for the momentum densities, first in the "lumped" notation (in which the original field variables and the coordinates are treated uniformly),

$$\pi^a=(l^d\dot{y}_d)^{-2}(G^{ab}l^c+G^{ac}l^b-G^{bc}l^a)\dot{y}_b\dot{y}_c, \tag{2.5}$$

and then in the "extended" notation,

$$\begin{aligned}\pi^A&=2J^{-1}(G^{AB}\dot{y}_B+G^A{}_\nu\dot{x}^\nu),\\ \lambda_\mu&=2J^{-1}(G_\mu{}^B\dot{y}_B+G_{\mu\sigma}\dot{x}^\sigma)\\ &\quad -J^{-2}Jt_{,\mu}(G^{AB}\dot{y}_A\dot{y}_B+2G^A{}_\sigma\dot{y}_A\dot{x}^\sigma+G_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma).\end{aligned} \tag{2.6}$$

The canonical momenta are all homogeneous of the zeroth degree in the dotted variables. The components π^A are fractions in which both numerator and denominator are linear homogeneous forms; the λ_μ are fractions of quadratic forms.

The partial derivatives of the algebraic constraints with respect to the momentum densities are all null "vectors" of the matrix Λ. The components of Λ are

$$\Lambda^{ab}=2(l^i\dot{y}_i)^{-3}\times(G^{ab}l^cl^d-G^{ac}l^bl^d-G^{bd}l^al^c+G^{cd}l^al^b)\dot{y}_c\dot{y}_d \tag{2.7}$$

in the "lumped" notation and

$$\begin{aligned}\Lambda^{AB}&=2J^{-1}\Lambda^{A\rho B\sigma}Jt_{,\rho}Jt_{,\sigma},\\ \Lambda^A{}_\nu&=2J^{-1}G^A{}_\nu-2J^{-2}Jt_{,\nu}\Lambda^{A\rho B\sigma}Jt_{,\rho}\\ &\quad\times[Jt_{,\sigma}\dot{y}_B+y_{B|s}J(u^s{}_{,\sigma}t_{,\mu}-u^s{}_{,\mu}t_{,\sigma})\dot{x}^\mu], \qquad (2.8)\\ \Lambda_{\mu\nu}&=2J^{-1}G_{\mu\nu}\\ &\quad-2J^{-2}[Jt_{,\mu}(G^A{}_\nu\dot{y}_A+G_{\rho\nu}\dot{x}^\rho)+Jt_{,\nu}(G_\mu{}^B\dot{y}_B+G_{\mu\sigma}\dot{x}^\sigma)]\\ &\quad+2J^{-3}Jt_{,\mu}Jt_{,\nu}J^2L\end{aligned}$$

in the "extended" notation.

In what follows, we shall denote the algebraic constraints by identifying symbols: the three "parameter" constraints

$$0=y_{a|s}\pi^a, \tag{2.9}$$

by g_s; the four "coordinate" constraints

$$0=F_{A\mu}{}^{B\nu}y_B\pi^AJt_{,\nu}-K_\mu(y_c,y_{c|s},x^\rho{}_{|s}), \tag{2.10}$$

by g_μ; and in the presence of an electromagnetic field, the "gauge" constraint

$$0=Jt_{,\rho}\psi^\rho, \tag{2.11}$$

by ψ. The remaining constraint will serve as the Hamiltonian density and will, therefore, be denoted by H. This last and most important constraint is connected with the homogeneity of the Lagrangian. Inasmuch as the canonical momenta are homogeneous of the zeroth degree in the "time"-derivatives, including the $\dot{x}^\rho$, they must satisfy at least one algebraic identity, and we shall find that this identity is algebraically independent of the constraints (2.9) and (2.10). Our task is to discover this identity.

The algebraic constraints between the canonical variables must hold for any combinations of the field variables consistent with the expression (2.5) for the momentum densities. We shall, therefore, for any combination of parameter values and field variables, construct a symbolic "vector space" in which the time derivatives $\dot{y}_a$ are the coordinates. The functions π^a, (2.5), are then specific zeroth-degree homogeneous functions of the coordinates in that vector space. In any transformation of the $\dot{y}_a$ into new $\dot{y}_a{}'$ with nonvanishing Jacobian, the "coordinates" of our vector space will undergo a linear transformation, and the momentum densities will transform contragrediently to them. We shall refer to the $\dot{y}_a$ as "coordinates," and we shall call quantities with the same transformation law "contravariant vectors." By the same token, the π^a form a covariant vector, the G^{ab} a covariant symmetric tensor, etc. In attempting to find an algebraic relationship between the momenta (2.5) and the y_a which will serve as our Hamiltonian density, we shall look for combinations which are invariant with respect to the "coordinate transformations" in this symbolic vector space. We are thus led to examine the typical vector-algebraic formations available.

If we look over the "building blocks" that might possibly be used in setting up our desired relation, we find that there are given to us a covariant vector, l^a, and a covariant symmetric tensor G^{ab}, apart from the coordinates $\dot{y}_a$ themselves. But since our relationship is to be a constraint in the y_a and π^a only, satisfied identically in Eqs. (2.5), the coordinates must enter only by way of the functions π^a. In addition, we have the invariant subspaces of the null vectors of the tensors Λ^{ab} and G^{ab}. We know that the π-derivatives of the various algebraic constraints are null vectors of the tensor Λ^{ab}. Some of them are null vectors of G^{ab} as well, as we shall show now.

Take first the derivatives of g_s. We have

$$\begin{aligned}G^{ab}(\partial g_s/\partial\pi^b)&=G^{ab}y_{b|s},\\ G^{Ab}y_{b|s}&=G^{AB}y_{B|s}+G^A{}_\nu x^\nu{}_{|s}\\ &=\Lambda^{A\rho B\sigma}Jt_{,\rho}[Jt_{,\sigma}y_{B|s}+y_{B|r}J(u^r{}_{,\sigma}t_{,\nu}-u^r{}_{,\nu}t_{,\sigma})x^\nu{}_{|s}]\\ &=0, \qquad (2.12)\\ G_\rho{}^by_{b|s}&=G_\rho{}^By_{B|s}+G_{\rho\sigma}x^\sigma{}_{|s}=0.\end{aligned}$$

In other words, the "vector" with the components $y_{a|s}$ is a null vector of G^{ab}. Next we shall form the product of G^{ab} by the π-derivaties of g_μ. In this case, the result is not zero, but proportional to $Jt_{,\rho}$:

$$\partial g_\mu/\partial\pi^B = F_{B\mu}{}^{C\tau} y_C Jt_{,\tau}, \quad \partial g_\mu/\partial\lambda_\rho = 0$$
$$G^{AB}(\partial g_\mu/\partial\pi^B) = \Lambda^{A\rho B\sigma} F_{B\mu}{}^{C\tau} y_C Jt_{,\tau} Jt_{,\rho} Jt_{,\sigma} = 0, \tag{2.13}$$
$$G_\rho{}^B(\partial g_\mu/\partial\pi^B) = Jt_{,\rho}(\Lambda^{A\nu B\sigma} y_{A,\nu} F_{B\mu}{}^{C\tau} Jt_{,\sigma} Jt_{,\tau} y_C).$$

Thus, while the four vectors $(\partial g_\mu/\partial\pi^B)$ are not themselves null vectors of G^{ab}, there exist three independent linear combinations that are.

The dot product of the "vector" l by any of the seven known null vectors of Λ^{ab} vanishes, as can be proven by a brief computation. Finally, the dot product of these null vectors by the "vector" π^a leads back to the constraints already known. Thus, we require an additional "tensor" to produce the Hamiltonian, and such a tensor would naturally be the inverse of G^{ab}, were it not for the fact that G^{ab} is a singular matrix and therefore possesses no inverse.

3. THE QUASI-UNIVERSE

We can construct a contravariant symmetric tensor by finding the solution of the following conditions:

$$G^{ab}E_{bc}G^{cd} = 0, \quad E_{ab}G^{bc}E_{cd} = 0. \tag{3.1}$$

These conditions possess a solution, even though G^{ab} is singular, but the solution is not uniquely determined. [Only for a regular matrix G, Eqs. (3.1) will determine uniquely the ordinary inverse G^{-1}.] For a singular matrix G, like the one we have to deal with in our present problem, we shall call E the "quasi-inverse" of G. The significance of E can be ascertained most easily in a "special" coordinate system, in which the null vectors of G are parallel to coordinate axes.[3] In such a special coordinate system, G takes the form

$$G = \begin{pmatrix} g, & 0 \\ 0, & 0 \end{pmatrix}. \tag{3.2}$$

The most general solution of Eqs. (1.14) in this special coordinate system is

$$E = \begin{pmatrix} g^{-1}, & h \\ h^T, & h^T g h \end{pmatrix}. \tag{3.3}$$

Here g^{-1} is the inverse of the matrix g, and the rectangular matrix h is completely arbitrary. The superscript ^{T} denotes the transpose. If we set h equal to zero, we get as a solution a matrix which commutes with G and which we obtain by replacing each non-zero eigenvalue of G by its reciprocal value, while retaining the zero eigenvalues unchanged. (Naturally, if there are no zero eigenvalues, i.e. if G is regular, then the quasi-inverse of this form goes over into the inverse.) All possible solutions of (3.1) can be transformed into each other by means of suitably chosen coordinate transformations. In the special coordinate system [in which G has the form (3.2)], the transformation matrix leading from

$$E_0 = \begin{pmatrix} g^{-1}, & 0 \\ 0, & 0 \end{pmatrix} \tag{3.4}$$

to (3.3) has the form

$$S = \begin{pmatrix} 1, & 0 \\ h^T g, & 1 \end{pmatrix}, \quad y' = Sy, \quad E' = SE_0S^T = E. \tag{3.5}$$

The transformation law for G is

$$G' = (S^T)^{-1} G S^{-1}, \quad \text{where} \quad S^{-1} = \begin{pmatrix} 1, & 0 \\ -h^T g, & 1 \end{pmatrix}, \tag{3.6}$$

and if S, (3.5), is applied to G, (3.2), the latter goes over into itself.

For use in the following section, we shall prove the covariant relationship

$$GEl = l. \tag{3.7}$$

Being normal to all null vectors of Λ and, therefore, *a fortiori*, to all null vectors of G, the covariant vector l must, in a "special" coordinate system, possess the form

$$l = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}. \tag{3.8}$$

This form is invariant with respect to coordinate transformations (3.5), since the transformation law for l is

$$l' = (S^T)^{-1} l. \tag{3.9}$$

If we now compute the left-hand side of Eq. (3.7) in a system in which E has the form (3.4), we have the result (3.7) immediately, and since the expression $(GEl - l)$ is a covariant vector, it will vanish in every coordinate system if it vanishes in one.

4. THE HAMILTONIAN DENSITY

If we possess the form E [any one solution of (3.1)], we are able to form additional invariants. Applying matrix notation to (2.5), we can write for the vector π:

$$\pi = \frac{2}{J} G\dot{y} - \frac{1}{J^2}(\dot{y}^T G \dot{y}) l, \quad J = l^T \dot{y}, \tag{4.1}$$

and if we multiply this expression by E to form a contravariant vector, we get

$$E\pi = \frac{2}{J} EG\dot{y} - \frac{1}{J^2}(\dot{y}^T G \dot{y}) El. \tag{4.2}$$

[3] Of course, G does not transform as a matrix, but as a symmetric tensor; but just as in matrix calculus, the existence of null vectors precludes the formation of a "contravariant metric tensor," which would be the precise analog to the inverse in matrix calculus.

Now we can form the scalar $\pi^T E\pi$,

$$\pi^T E\pi = (l^T El)(\dot{y}^T G\dot{y}/J^2)^2 \tag{4.3}$$

and the scalar $l^T E\pi$,

$$l^T E\pi = 2-(l^T El)(\dot{y}^T G\dot{y}/J^2). \tag{4.4}$$

Between these two quantities, we can eliminate the $\dot{y}_a$ completely, and we obtain the algebraic relationship

$$(l^T El)(\pi^T E\pi)-[2-(l^T E\pi)]^2 \equiv 0. \tag{4.5}$$

The left-hand side of this last constraint is suitable as a Hamiltonian density. In theories which possess coordinate covariance in addition to the trivial parameter invariance, the Hamiltonian simplifies even more. In Section 7, we shall show that the scalar $(l^T El)$ vanishes, and therefore, because of Eq. (4.3), the Hamiltonian density reduces to

$$H = \tfrac{1}{4}\pi^T E\pi = 0. \tag{4.6}$$

5. THE FIRST TRANSFORMATION

With the establishment of Eqs. (4.5) and (4.6), the construction of a Hamiltonian density has been reduced to an algebraic problem, namely the determination of the "tensor" E. Instead of merely reporting the result, which can be verified, of course, by substitution into Eqs. (3.1), we shall go through the complete calculations, because they show how the same work may be carried out with a different theory. The guiding idea in these calculations is the continued transformation of G until it is brought as closely as possible into the form (3.2). To find the inverse of the regular matrix g is relatively easy.

The first of this series of transformations isolates the three parameter constraints g_s. Inasmuch as the corresponding three null vectors of the matrix G have the form

$$\partial g_s/\partial\pi^a = y_{a|s} = (y_{A|s},\ x^\rho{}_{|s}), \tag{5.1}$$

we shall introduce a new "coordinate system" in the linear vector space of the $\dot{y}_a$ in which these three vectors become coordinate axes. The null vectors of the covariant tensor G are, of course, themselves contravariant vectors. If the three null vectors (5.1) are to become parallel to three particular coordinate axes, then the transformed matrix G (which we shall denote by G') will have only zeros in the corresponding three rows and columns. Naturally, these requirements do not determine the transformation matrix uniquely, though it is clear that the transformation matrix for covariant vectors must contain the three null vectors (5.1) as matrix rows or columns. Calling that matrix A, so that

$$G' = AGA^T, \quad A = (S^T)^{-1} \tag{5.2}$$

[see Eq. (3.6)], and separating for convenience the rows and columns with indices A, B, $\cdots$ from those with indices ρ, σ, $\cdots$, we find that a convenient transformation matrix is

$$A = \begin{pmatrix} \delta^A{}_B, & 0 \\ y_{B|r}, & x^\sigma{}_{|r} \\ 0, & v^\sigma \end{pmatrix}, \tag{5.3}$$

where v^σ are four quantities which, for the time being, shall remain undetermined. The determinant of A equals $v^\sigma Jt_{,\sigma}$ and we must, therefore, require that

$$v = v^\sigma Jt_{,\sigma} \neq 0. \tag{5.4}$$

Applying the transformation A, we find that

$$G' = \begin{pmatrix} G^{AB} & 0, & G^A{}_\sigma v^\sigma \\ 0, & 0, & 0 \\ v^\rho G_\rho{}^B, & 0, & G_{\rho\sigma}v^\rho v^\sigma \end{pmatrix}, \tag{5.5}$$

$$l' = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \tag{5.6}$$

and

$$\pi' = \begin{pmatrix} \pi^A \\ 0 \\ v^\rho\lambda_\rho \end{pmatrix}. \tag{5.7}$$

Clearly, the quasi-inverse of the matrix (5.5) can be chosen so that three of its rows and columns consist entirely of zeros, thus:

$$E' = \begin{pmatrix} E_{AB} & 0 & E_A \\ 0 & 0 & 0 \\ E_B & 0 & E_0 \end{pmatrix}. \tag{5.8}$$

If we make that choice, we have from now on to deal only with covariants in an $(N+1)$-dimensional space, instead of an $(N+4)$-dimensional space. In this reduced space, our covariants will have the forms:

$$\bar{G} = \begin{pmatrix} G^{AB}, & G^A{}_\sigma v^\sigma \\ v^\rho G_\rho{}^B, & G_{\rho\sigma}v^\rho v^\sigma \end{pmatrix}, \tag{5.9}$$

$$\bar{l} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \bar{\pi} = \begin{pmatrix} 0 \\ v^\rho\lambda_\rho \end{pmatrix} \tag{5.10}$$

and

$$\bar{E} = \begin{pmatrix} E_{AB}, & E_A \\ E_B, & E_0 \end{pmatrix}. \tag{5.11}$$

6. THE SECOND AND THIRD TRANSFORMATIONS

We shall write the four cooordinate constraints (2.10) in the abbreviated form

$$g_\mu = u_{\mu A}\pi^A - K_\mu, \quad u_{\mu A} = F_{A\mu}{}^{B\nu} y_B Jt_{,\nu}. \tag{6.1}$$

In the $(N+1)$-dimensional space and in the coordinate system denoted by bars ($\bar{G}$, etc.), the gradients of these four constraints take the form

$$\bar{u}_\mu = \begin{pmatrix} u_{\mu A} \\ 0 \end{pmatrix}. \tag{6.2}$$

Because of Eqs. (2.13), we have further

$$\bar{G}\bar{u}_\mu = \begin{pmatrix} 0 \\ U_\mu \end{pmatrix}, \qquad U_\mu = v y_{A,\rho} \Lambda^{A\rho B\sigma} J t_{,\sigma} u_{\mu B} = v^\rho G_\rho{}^B u_{\mu B}. \tag{6.3}$$

Our task is now to further transform the coordinates so that additional rows and columns of the $\bar{G}$-matrix will be filled with zeros. To this end, we shall introduce the transformation matrix D with the components

$$D = \begin{pmatrix} D^{A'}{}_A, & 0 \\ u_\mu{}^A, & 0 \\ 0, & 1 \end{pmatrix}, \tag{6.4}$$

to be used for the transformation of covariant vectors and tensors. The indices $A', B', \cdots$, are to run from 1 to $N-4$, and the coefficients $D^{A'}{}_A$ are to be chosen according to convenience, with the only proviso that the determinant of the matrix D (and that means the N-rowed sub-determinant in the upper left-hand corner) shall not vanish. The resulting covariants shall be denoted by double primes. In the following expressions, each column and row is broken down into three portions, of which the first has $(N-4)$, the second 4, and the third 1 component. For G'' we get

$$G'' = D\bar{G}D^T = \begin{pmatrix} D^{A'}{}_A G^{AB} D^{B'}{}_B, & 0, & D^{A'}{}_A G^A{}_\sigma v^\sigma \\ 0, & 0, & U_\mu \\ v^\rho G_\rho{}^B D^{B'}{}_B, & U_\nu, & G_{\rho\sigma} v^\rho v^\sigma \end{pmatrix}, \tag{6.5}$$

the vector $\bar{l}$ remains unchanged,

$$l'' = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \tag{6.6}$$

and for the vector π'', finally, we have the two alternative expressions

$$\pi'' = \begin{pmatrix} D^{A'}{}_A \pi^A \\ u_{\mu A} \pi^A \\ v^\rho \lambda_\rho \end{pmatrix} = \begin{pmatrix} D^{A'}{}_A \pi^A \\ K_\mu \\ v^\rho \lambda_\rho \end{pmatrix}. \tag{6.7}$$

Inasmuch as the vectors $\bar{u}_{\mu A}$ are not null vectors of $\bar{G}$, the second transformation does not make any complete row or column of G'' vanish, as the first transformation did. Of course, it would be an easy matter to find three independent linear combinations of the four numbers U_μ that vanish, and thus, by a further transformation, to produce three further completely empty rows and columns. Presumably, the resulting $(N-2)$-rowed matrix would be non-singular and could be used as the submatrix g of Eqs. (3.2) and (3.4). But such a procedure would destroy the symmetry between the four coordinate directions of physical space, the four directions characterized by Greek indices. That is why we shall adopt a different procedure to find the quasi-inverse.

We shall show that we can define a third transformation matrix T which removes from the matrix G'', Eq. (6.5), the first and third portions in the last row and the last column, which, in other words, reduces G'' to the form

$$G''' = TG''T^T = \begin{pmatrix} G^{A'B'}, & 0, & 0 \\ 0, & 0, & U_\mu \\ 0, & U_\nu, & 0 \end{pmatrix}, \qquad G^{A'B'} = D^{A'}{}_A D^{B'}{}_B G^{AB}. \tag{6.8}$$

We shall find the quasi-inverse of G''' below. The matrix T has the following form:

$$T = \begin{pmatrix} \delta^{A'}{}_{B'}, & 0, & 0 \\ 0, & \delta_\mu{}^\nu, & 0 \\ -G_{B'C'} D^{C'}{}_C G^C{}_\sigma v^\sigma, & \beta^\nu, & 1 \end{pmatrix}. \tag{6.9}$$

$G_{B'C'}$ is the inverse of the (non-singular) matrix $G^{A'B'}$,

$$G_{B'C'} G^{A'B'} = \delta^{A'}{}_{C'}, \tag{6.10}$$

and β^ν are four quantities subject only to one requirement, that the dot product of β^ν by U_ν has a specified value, namely

$$U_\mu \beta^\mu = \tfrac{1}{2} v^\rho v^\sigma (G_\rho{}^A G_\sigma{}^B D^{A'}{}_A D^{B'}{}_B G_{A'B'} - G_{\rho\sigma}). \tag{6.11}$$

This last condition follows from straightforward computation of the transformation that leads from G'' to G''' in accordance with Eqs. (6.8) and (6.9).

Once we have reduced G''' to the form (6.9), we can obtain its quasi-inverse directly. The matrix E''' has the components

$$E''' = \begin{pmatrix} G_{A'B'}, & 0, & 0 \\ 0, & 0, & \sigma^\nu \\ 0, & \sigma^\mu, & 0 \end{pmatrix}, \tag{6.12}$$

where σ^μ are four quantities subject to the only requirement that

$$\sigma^\mu U_\mu = 1. \tag{6.13}$$

That the matrix (6.12) is really the required quasi-inverse, or, at least, that it is one possible form of the quasi-inverse, can be verified by direct substitution into the defining equations (3.1).

Before we can obtain the expression for the Hamiltonian density, we must apply the third transformation to our two vectors l'' and π''. Again, l'' does not change at all,

$$l''' = l'' = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \tag{6.14}$$

and for π''' we get

$$\pi''' = \begin{pmatrix} D^{A'}{}_A \pi^A \\ K_\mu \\ v^\rho \lambda_\rho + \beta^\rho K_\rho - v^\rho G_\rho{}^A D^{A'}{}_A G_{A'B'} D^{B'}{}_B \pi^B \end{pmatrix}. \tag{6.15}$$

7. THE HAMILTONIAN

The first and obvious result of our expressions (6.12), (6.14), and (6.15) is that the scalar (l^TEl) vanishes. To obtain the Hamiltonian density, we have, therefore, to evaluate the expression (4.6). Instead of transforming E''' back into E (by applying the three transformations A, D, and T in reverse), we shall determine the Hamiltonian by working out the expression $(\pi^{T'''}E'''\pi''')$. By substituting the appropriate expressions, we get first

$$H=\tfrac{1}{4}G_{A'B'}D^{A'}{}_AD^{B'}{}_B\pi^A\pi^B + \tfrac{1}{2}K_\mu\sigma^\mu(v^\rho\lambda_\rho+\beta^\rho K_\rho - G_{A'B'}D^{A'}{}_AD^{B'}{}_BG^A{}_\rho v^\rho\pi^B). \quad (7.1)$$

This expression could be used as the Hamiltonian density. Its principal drawback lies in the presence of the four-vectors σ^μ and β^μ which are arbitrary except for one cumbersome normalization restriction each, Eqs. (6.11) and (6.13), respectively. We shall now show that both of these formations can be eliminated. To this end, we must determine the relationship between U_μ and K_μ. We have already obtained an expression for U_μ in Eq. (6.3). If we now use the constraint (2.10) to get an explicit expression for K_μ as well, we find:

$$K_\mu=u_{\mu A}\pi^A=2J^{-1}u_{\mu A}(G^{AB}\dot{y}_B+G^A{}_\sigma\dot{x}^\sigma) = 2u_{\mu A}\Lambda^{A\rho B\sigma}Jt_{,\rho}y_{B,\sigma}=2v^{-1}U_\mu. \quad (7.2)$$

This result enables us to use directly the conditions (6.11) and (6.13) to eliminate σ^μ and β^μ. We have

$$K_\mu\sigma^\mu=2v^{-1}U_\mu\sigma^\mu=2v^{-1} \quad (7.3)$$

and

$$K_\mu\beta^\mu=2v^{-1}U_\mu\beta^\mu = v^{-1}v^\rho v^\sigma(G_\rho{}^AG_\sigma{}^BD^{A'}{}_AD^{B'}{}_BG_{A'B'}-G_{\rho\sigma}). \quad (7.4)$$

These two expressions, substituted into the Hamiltonian density (7.1) yield the expression

$$H=v^{-2}v^\rho v^\sigma \times[\tfrac{1}{4}G_{A'B'}D^{A'}{}_AD^{B'}{}_B(Jt_{,\rho}\pi^A-2G^A{}_\rho)(Jt_{,\sigma}\pi^B-2G^B{}_\sigma) + \tfrac{1}{2}(\lambda_\rho Jt_{,\sigma}+\lambda_\sigma Jt_{,\rho})-G_{\rho\sigma}]. \quad (7.5)$$

In this expression for the Hamiltonian density, the four-vector v^ρ remains undetermined, except for the inequality (5.4). However, a short computation shows that the choice of this vector merely affects the manner in which the parameter constraints (2.9) enter into the Hamiltonian. In II, it was pointed out that the choice of Hamiltonian is not unique, but subject to an algebraic combination with the other constraints. The choice of the rectangular matrix $D^{A'}{}_A$ has no effect on the eventual form of H, except to permit the addition of linear and quadratic combinations of the coordinate constraints (2.10). Thus, the Hamiltonian (7.5) possesses exactly the degree of arbitrariness required by the general theory.

The canonical field equations take the form

$$\dot{y}_a=\partial H/\partial\pi^a, \quad \dot{\pi}^a=-\delta H/\delta y_a. \quad (7.6)$$

In particular, we find that the derivatives of the coordinates x^ρ with respect to the parameter t are determined by the expressions

$$\dot{x}^\rho=\partial H/\partial\lambda_\rho=v^{-1}v^\rho=(Jt_{,\sigma}v^\sigma)^{-1}v^\rho. \quad (7.7)$$

The choice of the four-vector v^ρ will, therefore, be largely governed by the desired relationship between the coordinates and the parameter.

Finally, we shall show in passing that the relationship (4.4), with vanishing last term, does not lead to a suitable Hamiltonian relationship. By substituting into the scalar $(l^TE\pi)$ the expressions (6.12), (6.14), and (6.15), observing in the process the relationship (7.3), we find that the resulting constraint is a linear combination of the coordinate constraints (2.10) alone and, therefore, not suitable as a Hamiltonian density.

In Section 8, we shall work out the general expression (7.5) for the particular Lagrangian that characterizes the general theory of relativity with electromagnetic terms.

8. THE HAMILTONIAN OF GENERAL RELATIVITY WITH ELECTROMAGNETIC FIELD

The Lagrangian density of the general theory of relativity has the following structure, if the electromagnetic field is included:[4]

$$L=L_{\text{grav}}+L_{\text{el}}, \quad (8.1)$$

where

$$L_{\text{grav}}=[(-g)^{\frac{1}{2}}/16\pi\kappa]g^{\mu\nu} \times\left(\begin{Bmatrix}\sigma\\ \rho\sigma\end{Bmatrix}\begin{Bmatrix}\rho\\ \mu\nu\end{Bmatrix}-\begin{Bmatrix}\sigma\\ \mu\rho\end{Bmatrix}\begin{Bmatrix}\rho\\ \nu\sigma\end{Bmatrix}\right) \quad (8.2)$$

and

$$L_{\text{el}}=-[(-g)^{\frac{1}{2}}/16\pi]\phi^{\mu\nu}\phi_{\mu\nu}. \quad (8.3)$$

$\phi_{\mu\nu}$ stands for

$$\phi_{\mu\nu}=\phi_{\mu,\nu}-\phi_{\nu,\mu} \quad (8.4)$$

and

$$\phi^{\mu\nu}=g^{\mu\rho}g^{\nu\sigma}\phi_{\rho\sigma}. \quad (8.5)$$

The ϕ_μ are the four electromagnetic potentials. This Lagrangian density contains the field variables and their first derivatives only, but is not a scalar density. Therefore, it belongs to the general class of Lagrangians considered in I. It is also homogeneous quadratic in the first derivatives of the field variables.

We shall now rewrite L to show the coefficients $\Lambda^{A\rho B\sigma}$. Renaming dummy indices and factoring we get:

$$L=\Lambda^{(\alpha\beta)\rho,(\gamma\delta)\sigma}g_{\alpha\beta,\rho}g_{\gamma\delta,\sigma}+Y^{\mu\rho,\nu\sigma}\phi_{\mu,\rho}\phi_{\nu,\sigma},$$

[4] See, for instance, P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., New York, 1942), p. 193 ff, Eqs. (12.56) and (12.65).

$$\begin{aligned}\Lambda^{(\alpha\beta)\rho,(\gamma\delta)\sigma} &= [(-g)^{\frac{1}{2}}/128\pi\kappa] \\ &\times\{[g^{\alpha\beta}(g^{\gamma\rho}g^{\delta\sigma}+g^{\delta\rho}g^{\gamma\sigma})+g^{\gamma\delta}(g^{\alpha\sigma}g^{\beta\sigma}+g^{\alpha\sigma}g^{\beta\rho})] \\ &+g^{\rho\sigma}(g^{\alpha\gamma}g^{\beta\delta}+g^{\beta\gamma}g^{\alpha\delta}-2g^{\alpha\beta}g^{\gamma\delta}) \\ &-[g^{\alpha\sigma}(g^{\beta\gamma}g^{\delta\rho}+g^{\beta\delta}g^{\gamma\rho})+g^{\beta\sigma}(g^{\alpha\gamma}g^{\delta\rho}+g^{\alpha\delta}g^{\gamma\rho})]\}, \\ Y^{\mu\rho,\nu\sigma} &= [(-g)^{\frac{1}{2}}/8\pi](g^{\mu\sigma}g^{\nu\rho}-g^{\mu\nu}g^{\rho\sigma}).\end{aligned} \tag{8.6}$$

The coefficients Λ and Y are symmetric in $(A\rho)$ and $(B\sigma)$,

$$\Lambda^{(\alpha\beta)\rho,(\gamma\delta)\sigma}=\Lambda^{(\gamma\delta)\sigma(\alpha\beta)\rho},\quad Y^{\mu\rho,\nu\sigma}=Y^{\nu\sigma,\mu\rho}. \tag{8.7}$$

Λ is, besides, symmetric in each bracketed index pair

$$\Lambda^{(\alpha\beta)\rho,(\gamma\delta)\sigma}=\Lambda^{(\beta\alpha)\rho,(\gamma\delta)\sigma}. \tag{8.8}$$

In order to substitute into Eq. (7.5), we must determine the null vectors corresponding to the various constraints and, besides, we must make a choice for the reducing matrix $D^{A'}{}_A$. There is also a slight complication in that we have, in addition to the parameter and coordinate constraints, the gauge constraint (2.11).

Altogether, we have 14 variables, of which 10 are gravitational and 4 electromagnetic potentials. The matrix G^{AB} with 14 rows and columns actually consists of a 10×10 and a 4×4 matrix, with the rectangular off-diagonal spaces filled entirely with zeros. As a first step, we shall determine the actual expression for G^{AB},

$$\begin{aligned}G^{AB} &= \begin{pmatrix} G^{(\alpha\beta)(\gamma\delta)} & 0 \\ 0 & G^{\mu\nu}\end{pmatrix}, \\ G^{(\alpha\beta)(\gamma\delta)} &= [(-g)^{\frac{1}{2}}/128\pi\kappa] \\ &\times[X(g^{\alpha\gamma}g^{\beta\delta}+g^{\beta\gamma}g^{\alpha\delta}-2g^{\alpha\beta}g^{\gamma\delta}) \\ &+2(g^{\alpha\beta}X^\gamma X^\delta+g^{\gamma\delta}X^\alpha X^\beta) \\ &-(g^{\alpha\gamma}X^\beta X^\delta+g^{\alpha\delta}X^\beta X^\gamma+g^{\beta\gamma}X^\alpha X^\delta+g^{\beta\delta}X^\alpha X^\gamma)], \\ G^{\mu\nu} &= [(-g)^{\frac{1}{2}}/8\pi](X^\mu X^\nu-Xg^{\mu\nu}), \\ X^\alpha &= Jt_{,\beta}g^{\alpha\beta},\quad X=Jt_{,\alpha}Jt_{,\beta}g^{\alpha\beta}.\end{aligned} \tag{8.9}$$

The first of these two submatrices has the four null vectors

$$u_{\mu(\gamma\delta)}=Jt_{,\gamma}g_{\mu\delta}+Jt_{,\delta}g_{\mu\gamma}, \tag{8.10}$$

and the second submatrix has the single null vector $Jt_{,\nu}$. These expressions are the null vectors obtainable from the general theory. It is, however, very easy to verify their being null vectors by straightforward computation.

Accordingly, we require two separate matrices suitable for the role designated in the preceding sections by the symbol $D^{A'}{}_A$. One 10×6 matrix must reduce the gravitational submatrix to a non-singular 6×6 matrix, while another 4×3 matrix will reduce the electromagnetic coefficients to a non-singular 3×3 matrix. These reducing matrices must have the further property that they are linearly independent of the null vectors indicated above. In an effort to maintain the symmetry between the four (physical) coordinate directions, we chose matrices which, in effect, project four-vectors and four-tensors (in physical space-time) into the three-dimensional space of the parameters u^s. And since the matrix elements of D must be independent of t-derivatives, we set:

$$\begin{aligned}D_{m\mu} &= g_{\mu\rho}x^\rho{}_{|m}, \\ D_{(mn)\mu\nu} &= \tfrac{1}{2}g_{\mu\rho}g_{\nu\sigma}(x^\rho{}_{|m}x^\sigma{}_{|n}+x^\rho{}_{|n}x^\sigma{}_{|m}).\end{aligned} \tag{8.11}$$

The resulting expression for $G^{A'B'}$ is

$$\begin{aligned}G^{A'B'} &= \begin{pmatrix} G_{(ab)(cd)} & 0 \\ 0 & G_{mn}\end{pmatrix}, \\ G_{(ab)(cd)} &= [(-g)^{\frac{1}{2}}X/128\pi\kappa] \\ &\times(g_{ac}g_{bd}+g_{ad}g_{bc}-2g_{ab}g_{cd}), \\ G_{mn} &= -[(-g)^{\frac{1}{2}}X/8\pi]g_{mn}, \\ g_{mn} &\equiv g_{\mu\nu}x^\mu{}_{|m}x^\nu{}_{|n}.\end{aligned} \tag{8.12}$$

We must now find the inverse of this non-singular matrix. We shall first introduce the expressions

$$D^m{}_\mu=X^{-1}X^\rho J(u^m{}_{,\mu}t_{,\rho}-u^m{}_{,\rho}t_{,\mu}) \tag{8.13}$$

with the property

$$D^m{}_\mu x^\mu{}_{|n}=\delta^m{}_n \tag{8.14}$$

and

$$g^{mn}=g^{\mu\nu}D^m{}_\mu D^n{}_\nu,\quad g^{mn}g_{ns}=\delta^m{}_s. \tag{8.15}$$

The inverse matrix must be built up from the g^{mn}.

The inverse electromagnetic matrix can be found by inspection. The determination of the gravitational matrix is only slightly more laborious. There are only two possible combinations of g^{mn} which satisfy all the requirements of symmetry (namely that the matrix $G^{(ab)(cd)}$ be symmetric within each bracketed pair of indices and that it be symmetric with respect to an interchange between the two index pairs as wholes). All that needs to be done is to determine the numerical coefficients of these two possible combinations. The final expression for the inverse is then

$$\begin{aligned}G_{A'B'} &= \begin{pmatrix} G^{(ab)(cd)}, & 0 \\ 0, & G^{mn}\end{pmatrix}, \\ G^{(ab)(cd)} &= [32\pi\kappa/(-g)^{\frac{1}{2}}X](g^{ac}g^{bd}+g^{ad}g^{bc}-g^{ab}g^{cd}), \\ G^{mn} &= -[8\pi/(-g)^{\frac{1}{2}}X]g^{mn}.\end{aligned} \tag{8.16}$$

This inverse matrix now must be multiplied by $D^{A'}{}_A D^{B'}{}_B$. The result of this operation is

$$\begin{aligned}D^{A'}{}_A D^{B'}{}_B G_{A'B'} \equiv G_{AB} &= \begin{pmatrix} G_{(\alpha\beta)(\gamma\delta)}, & 0 \\ 0, & G_{\mu\nu}\end{pmatrix}, \\ G_{(\alpha\beta)(\gamma\delta)} &= [32\pi\kappa/(-g)^{\frac{1}{2}}X](\gamma_{\alpha\gamma}\gamma_{\beta\delta}+\gamma_{\alpha\delta}\gamma_{\beta\gamma}-\gamma_{\alpha\beta}\gamma_{\gamma\delta}), \\ G_{\mu\nu} &= -[8\pi/(-g)^{\frac{1}{2}}X]\gamma_{\mu\nu}, \\ \gamma_{\mu\nu} &\equiv g_{\mu\nu}-Jt_{,\mu}Jt_{,\nu}.\end{aligned} \tag{8.17}$$

The matrix G_{AB} that we have thus defined is, of course, a quasi-inverse of the matrix G^{AB}, Eq. (8.9). In the expression for $G_{\mu\nu}$, we can replace $\gamma_{\mu\nu}$ by $g_{\mu\nu}$, because $Jt_{,\nu}$ is a null vector of that matrix. But the same replacement can be made in the matrix $G_{(\alpha\beta)(\gamma\delta)}$, because the replacement is nothing but the addition of a linear combination of the null vectors (8.10). Thus, the first term in the bracket of Eq. (7.5) has been determined.

The second term can be copied without change. In the last term, we must substitute the correct expression for $G_{\rho\sigma}$. The expressions for $G^{A}{}_{\rho}$, $G_{\rho\sigma}$ are lengthy, but are obtained by routine calculations from the defining equations (2.3). We shall write them down, too, introducing as an abbreviating notation the differential operator

$$y_{A|s}J(u^{s}{}_{,\rho}t_{,\sigma}-u^{s}{}_{,\sigma}t_{,\rho})\equiv y_{A[\rho\sigma]}. \qquad (8.18)$$

This differential operator satisfies the product rule of differentiation. With its help, we obtain the following expressions:

$$G^{(\alpha\beta)}{}_{\sigma}=[(-g)^{\frac{1}{2}}/128\pi\kappa]\{(\log g)_{[\tau\sigma]}(g^{\alpha\tau}X^{\beta}+g^{\beta\tau}X^{\alpha}) \\ -2g^{\alpha\beta}g^{\rho\tau}{}_{[\tau\sigma]}Jt_{,\rho}-2X^{\tau}[g^{\alpha\beta}{}_{[\tau\sigma]}+(\log g)_{[\tau\sigma]}g^{\alpha\beta}] \\ +2Jt_{,\rho}(g^{\alpha\tau}g^{\beta\rho}{}_{[\tau\sigma]}+g^{\beta\tau}g^{\alpha\rho}{}_{[\tau\sigma]})\}, \qquad (8.19)$$

$$G^{\mu}{}_{\sigma}=[(-g)^{\frac{1}{2}}/8\pi](g^{\mu\tau}X^{\rho}-g^{\mu\rho}X^{\tau})\phi_{\rho[\tau\sigma]}$$

and

$$G_{\rho\sigma}=[(-g)^{\frac{1}{2}}/128\pi\kappa] \\ \times\{g^{\iota\kappa}[2g^{\alpha\beta}{}_{[\iota\rho]}g_{\alpha\beta[\kappa\sigma]}-(\log g)_{[\iota\rho]}(\log g)_{[\kappa\sigma]}] \\ -2[g^{\iota\kappa}{}_{[\kappa\sigma]}(\log g)_{[\iota\rho]}+g^{\iota\kappa}{}_{[\iota\rho]}(\log g)_{[\kappa\sigma]}] \\ -4g_{\alpha\beta}(g^{\alpha\iota}{}_{[\kappa\sigma]}g^{\beta\kappa}{}_{[\iota\rho]})\} \\ +[(-g)^{\frac{1}{2}}/8\pi](g^{\mu\kappa}g^{\nu\iota}-g^{\mu\nu}g^{\iota\kappa})\phi_{\mu[\iota\rho]}\phi_{\nu[\kappa\sigma]}. \qquad (8.20)$$

With these substitutions, the final expression for the Hamiltonian density becomes

$$H=v^{-2}v^{\rho}v^{\sigma}\{\tfrac{1}{2}(\lambda_{\rho}Jt_{,\sigma}+\lambda_{\sigma}Jt_{,\rho})-G_{\rho\sigma} \\ +[8\pi\kappa/(-g)^{\frac{1}{2}}X](g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\beta}g_{\gamma\delta}) \\ \times(\pi^{\alpha\beta}Jt_{,\rho}-2G^{(\alpha\beta)}{}_{\rho})(\pi^{\gamma\delta}Jt_{,\sigma}-2G^{(\gamma\delta)}{}_{\sigma}) \\ -[2\pi/(-g)^{\frac{1}{2}}X]g_{\mu\nu}(\psi^{\mu}Jt_{,\rho}-2G^{\mu}{}_{\rho}) \\ \times(\psi^{\nu}Jt_{,\sigma}-2G^{\nu}{}_{\sigma})\}. \qquad (8.21)$$

9. CONCLUSION

The Hamiltonian density which we have obtained is a rather formidable expression, but it is a quantity composed exclusively of the canonical variables and their "spatial" derivatives. The canonical differential equations are of the first differential order and solved with respect to them. Thus, the continuation of a solution of the field equations in the t-direction can be accomplished by a series of iterated integrations. Naturally, the Hamiltonian (8.21) with (8.19) and (8.20) is not the most general expression imaginable that can be used for the Hamiltonian. We can multiply it by an arbitrary (but non-zero) function of the dynamical variables and the parameters (and such a factor will affect the relationship between the parameter t and the coordinates), and we can add arbitrary linear combinations of the coordinate constraints g_{μ}, Eq. (2.10). The addition of parameter constraints g_{s} will have no other effect than would the adoption of particular expressions for the arbitrary v^{ρ}. Thus, having obtained one expression for H, we can easily find all other possible expressions, and we can thus use the equivalent of any coordinate condition and parameter condition desired. As for the gauge constraint (2.11), adding ψ with any factor to the Hamiltonian amounts to the adoption of a particular gauge condition.

The new formalism available will be used to give a new derivation of the equations of motion, which was first achieved by means of an approximation method by Einstein, Infeld, and Hoffmann. We shall obtain the instantaneous acceleration of singularities rigorously by assuming a certain field at a time t_0 in agreement with all the algebraic constraints (including the vanishing of the Hamiltonian density) and then continuing into the future (or past). Later we expect to quantize the theory and to examine problems involving radiative processes.