obtain

$$J(t) = (e\hbar/Vam^*)\sin(eEnat/\hbar).$$

For $t \ll \hbar/eEna$ we have

$$J(t) = (ne^2/Vm^*)Et, \quad \partial J(t)/\partial t = (ne^2/Vm^*)E. \quad (15)$$

The quantity multiplying E in (15) is however exactly the quantity $1/\Lambda$ defined by Tisza so as to give Eq. (1). Therefore we obtain finally

$$\partial(\Lambda J)/\partial t = E. \quad (16)$$

This is Eq. (2). The fact that the same Λ occurs in both equations speaks to a certain extent for the internal consistency of the theory.

In conclusion I would like to thank Professor Tisza for allowing me to see his manuscript prior to publication, and for several valuable discussions.

PHYSICAL REVIEW VOLUME 80, NUMBER 4 NOVEMBER 15, 1950

Quantum Theory of the Longitudinal Electromagnetic Field

S. T. MA
University of Chicago, Chicago, Illinois
(Received April 24, 1950)

The special features of the quantum theory of the longitudinal electromagnetic field originated from the supplementary condition are discussed in connection with the vacuum expectation value of the bilinear form in the electromagnetic potentials. The present investigation confirms the view that the unified treatment of the electromagnetic field is valid for gauge-invariant integrals.

I. INTRODUCTION

PRIOR to the recent developments in quantum electrodynamics it was customary to eliminate the longitudinal electromagnetic field and its supplementary condition by a canonical transformation, and to deal with the interaction of the electron waves with the transverse photons, together with the Coulomb interaction of the electron waves that arises from the elimination of the longitudinal field.[1] In order to simplify the calculations, a unified treatment of the longitudinal and transverse components of the electromagnetic field has been generally adopted in recent works on quantum electrodynamics in dealing with the virtual processes.[2] Since the longitudinal field is no longer eliminated from the theory, it has become necessary to make a closer study of its properties.

In the following discussion we shall use the notation of Schwinger's papers with the constants c and $\hbar$ put equal to unity. In this notation the state vector representing the vacuum state of the free electromagnetic field satisfies the supplementary condition:

$$\frac{\partial}{\partial x_\mu} A_\mu(x)\Psi_0 = 0 \quad (1)$$

for the longitudinal field and the condition of vacuum:

$$\mathcal{A}_\mu^{(+)}(x)\Psi_0 = 0 \quad (2)$$

for the transverse field. In Schwinger's formulation of quantum electrodynamics the elimination of virtual photons is achieved by evaluating vacuum expectation values of the form:

$$\langle\Omega\rangle_0 = \Psi_0^\dagger \Omega \Psi_0. \quad (3)$$

In order to do this in a covariant way, Eq. (2) is generalized into:

$$A_\mu^{(+)}(x)\Psi_0 = 0. \quad (4)$$

This generalized vacuum condition leads to the result:

$$\langle A_\mu(x) A_\nu(x')\rangle_0 = i\delta_{\mu\nu} D^{(+)}(x-x'), \quad (5)$$

which forms the basis of the unified treatment of the electromagnetic field. Belinfante[3] has pointed out that Eq. (4) is incompatible with the supplementary condition. As a result of this incompatibility the vacuum expectation value

$$\left\langle A_\mu(x) \frac{\partial}{\partial x_\nu'} A_\nu(x') \right\rangle_0$$

derived from formula (5) does not vanish as it should. Nevertheless, it has been shown by Schwinger that direct calculations of the self-energy of the electron based on the rigorous method and the unified treatment give results which become identical after a gauge transformation. It has also been shown by Hu[4] that the two methods give the same result for the iterated form of the S-matrix. It is not clear from these investigations

[1] W. Heisenberg and W. Pauli, Zeits. f. Physik **59**, 168 (1930); E. Fermi, Rev. Mod. Phys. **4**, 125 (1932).

[2] T. Tati and S. Tomonaga, Prog. Theor. Phys. **3**, 391 (1948); J. Schwinger, Phys. Rev. **74**, 1439 (1948); **75**, 651 (1949); **76**, 790 (1949); R. P. Feynman, Phys. Rev. **74**, 1430 (1948); D. Rivier and E. C. G. Stuecklberg, Phys. Rev. **74**, 218 (1948); F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

[3] F. J. Belinfante, Phys. Rev. **76**, 226 (1949).

[4] N. Hu, Phys. Rev. **76**, 391 (1949); **77**, 150 (1950).

why two apparently incompatible mathematical schemes should lead to equivalent results.

The situation has been clarified by recent investigations of Dyson,[5] Coester and Jauch.[6] These authors have shown that it is possible to justify the unified treatment with no reference to either the traditional separate treatment of the electromagnetic field or the generalized vacuum condition, as long as we are dealing with gauge-invariant integrals. In Coester and Jauch's evaluation of the S-matrix the virtual photons are eliminated by using the commutation relations alone without any special consideration of the vacuum expectation value $\langle A_\mu(x)A_\nu(x')\rangle_0$. The vacuum expectation value of the bilinear form has been investigated by Dyson. Dyson shows that $\langle A_\mu(x)A_\nu(x')\rangle_0$ can be determined, except for an unknown function, from Eq. (2) and the equation:

$$\frac{\partial}{\partial x_\mu}A_\mu^{(+)}(x)\Psi_0=0, \tag{6}$$

which is the positive-frequency part of the supplementary condition. The result thus obtained is of the form:

$$\langle A_\mu(x)A_\nu(x')\rangle_0=i\delta_{\mu\nu}D^{(+)}(x-x')+\frac{1}{2}\frac{\partial^2}{\partial x_\mu\partial x_\nu}\Phi(x-x'), \tag{7}$$

Φ being a function left undetermined by the theory. From the special form in which Φ appears in Eq. (7) it can be seen that it contributes nothing to integrals of the form:

$$\int\int K_{\mu\nu}(x, x')\langle A_\mu(x)A_\nu(x')\rangle_0 d^4xd^4x', \tag{8}$$

if the tensor $K_{\mu\nu}$ satisfies the conditions of gauge invariance:

$$\frac{\partial}{\partial x_\mu}K_{\mu\nu}=0, \quad \frac{\partial}{\partial x_\nu'}K_{\mu\nu}=0, \tag{9}$$

and the integrals extend over the whole space. Since Eq. (7) differs from Eq. (5) only by the term involving the function Φ, it is justified to use formula (5) for the evaluation of gauge-invariant integrals.

It is interesting to find out whether it is possible to take into account the complete supplementary condition, Eq. (1), in the evaluation of the vacuum expectation value in question. It will be seen in the next section that this requires a slight modification in the way the vacuum expectation value is taken, but the result that follows from this modification bears out the view that the unified treatment of the electromagnetic field is valid for gauge-invariant integrals.

In connection with the exclusive use of the positive-frequency part of the supplementary condition, we note that Gupta[7] has recently proposed a modified formulation of quantum electrodynamics which consists in taking Eq. (6) instead of Eq. (1) as the supplementary condition. In the present note we shall be concerned with only the original formulation of the theory based on the supplementary condition (1). A systematic formulation of Gupta's idea is a problem for further investigation.

[5] F. J. Dyson, Phys. Rev. **77**, 420 (1950).

[6] F. Coester and J. M. Jauch, Phys. Rev. **78**, 149 (Section V), 827 (1950).

II. VACUUM EXPECTATION VALUE IN THE QUANTUM THEORY OF THE ELECTROMAGNETIC FIELD

Quantum-mechanical representations for the longitudinal electromagnetic field have been given by Fock and Podolsky,[8] Dirac,[9] and Wentzel.[10] In all of these representations the state vector satisfying Eq. (1) is infinite in magnitude. Since the state vector is not normalized in the conventional sense, the usual concept of expectation value does not exist. If we formally take the state vector to be normalized and apply the conventional rules for evaluating expectation values, contradictions of the following kind will arise.[11] From Eq. (1) and its Hermitian conjugate:

$$\Psi_0{}^\dagger\frac{\partial}{\partial x_\mu}A_\mu(x)=0, \tag{10}$$

we obtain

$$\Psi_0{}^\dagger\left[\frac{\partial}{\partial x_\mu}A_\mu(x),\quad A_\nu(x')\right]\Psi_0=0$$

or

$$\frac{\partial}{\partial x_\nu}D(x-x')\Psi_0{}^\dagger\Psi_0=0,$$

which obviously cannot be true.

These features of the theory of the longitudinal electromagnetic field are typical of the theory of continuous spectrum. Consider for example a pair of canonically conjugate variables q, p satisfying the commutation relation:

$$[q, p]=i. \tag{11}$$

Let $\psi(q')$ be an eigenvector of q belonging to the eigenvalue q'. Then the normalization condition in the theory of discrete spectrum is replaced by the formula:

$$\psi(q')^\dagger\psi(q'')=\delta(q'-q''), \tag{12}$$

and the matrix elements of p are of the form:

$$\psi^\dagger(q')p\psi(q'')=-i\frac{\partial}{\partial q'}\delta(q'-q''). \tag{13}$$

[7] S. N. Gupta, Proc. Phys. Soc. London **63**, 681 (1950); K. Bleuler (to be published).

[8] V. A. Fock and B. Podolsky, Physik. Zeits. Sowjetunion **1**, 801 (1931); L. Rosenfeld, Zeits. f. Physik **76**, 729 (1932); S. T. Ma, Phys. Rev. **75**, 535 (1949); F. J. Belinfante (reference 3).

[9] P. A. M. Dirac, Ann. Inst. M. Poincaré, **9**, 13 (1939).

[10] G. Wentzel, *Quantum Theory of Fields*, Interscience Publishers, New York (1949). The writer is grateful to Professor Wentzel for a discussion of this representation.

[11] This was pointed out to me by C. N. Yang several years ago.

It follows from Eqs. (12) and (13) that:

$$\psi^\dagger(q')pq\psi(0)=0, \tag{14}$$

$$\psi^\dagger(q')qp\psi(0)=i\delta(q'), \tag{15}$$

$$\psi^\dagger(0)pq\psi(q')=-i\delta(q'), \tag{16}$$

$$\psi^\dagger(0)qp\psi(q')=0. \tag{17}$$

These equations are mathematically consistent as long as q' is treated as a variable of integration. On the other hand, expectation values—diagonal matrix elements—calculated from these equations are usually divergent and ambiguous. In the first place, it follows from Eq. (12) that

$$\psi^\dagger(0)\psi(0)=\delta(0),$$

which shows that $|\psi(0)|^2$ is infinite. As can be expected from this divergence, Eqs. (14) to (17) lead to contradictory results if q' is put equal to zero. For example, from Eqs. (14), (17) with $q'=0$ we have

$$\psi^\dagger(0)[q,p]\psi(0)=0$$

or

$$\psi^\dagger(0)\psi(0)=0.$$

In view of these special features of the longitudinal electromagnetic field, and of the fact that only the transverse photons are of physical interest, it seems more consistent to take the vacuum expectation value only for the transverse degrees of freedom. Let χ_0 and Φ_0 be the two factors of Ψ_0 referring, respectively, to the longitudinal and transverse degrees of freedom. These state vectors satisfy the equations:

$$\frac{\partial}{\partial x_\mu}A_\mu(x)\chi_0=0, \tag{18}$$

$$\mathcal{A}_\mu^{(+)}(x)\Phi_0=0, \tag{19}$$

$$\Phi_0^\dagger\mathcal{A}_\mu^{(-)}(x)=0. \tag{20}$$

Instead of the quantities defined by Eq. (3) we consider the quantities $\Phi_0^\dagger\Omega\Phi_0$. The latter quantities are operators when Ω involves the longitudinal variables, and operate on χ_0.

It follows from Eqs. (19) and (20) that:

$$\Phi_0^\dagger\mathcal{A}_\mu(x)\Phi_0=0, \tag{21}$$

$$\Phi_0^\dagger\mathcal{A}_\mu(x)\mathcal{A}_\nu(x')\Phi_0=i\delta_{\mu\nu}D^{(+)}(x-x')+P_{\mu\nu}(x-x'), \tag{22}$$

where

$$P_{\mu\nu}(x)=-i\left\{\frac{\partial^2}{\partial x_\mu\partial x_\nu}+\left(n_\mu\frac{\partial}{\partial x_\nu}+n_\nu\frac{\partial}{\partial x_\mu}\right)n_\lambda\frac{\partial}{\partial x_\lambda}\right\}\mathfrak{D}^{(+)}(x). \tag{23}$$

Resolving $A_\mu(x)$ into the longitudinal and transverse components according to the general formula:

$$A_\mu(x)=n_\mu n_\lambda\frac{\partial}{\partial x_\lambda}[\Lambda(x)-L(x)]-\frac{\partial}{\partial x_\mu}L(x)+\mathcal{A}_\mu(x), \tag{24}$$

where we write L instead of Schwinger's Λ', we have:

$$\Phi_0^\dagger A_\mu(x)A_\nu(x')\Phi_0 =i\delta_{\mu\nu}D^{(+)}(x-x')+P_{\mu\nu}(x-x')+Q_{\mu\nu}(x-x') +R_{\mu\nu}(x-x')+\frac{\partial^2}{\partial x_\mu\partial x_\nu'}L(x)L(x'), \tag{25}$$

with:

$$Q_{\mu\nu}(x-x')=n_\mu n_\lambda\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\nu'}[L(x),L(x')] =-in_\mu n_\lambda\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\nu'}\mathfrak{D}(x-x'), \tag{26}$$

$$R_{\mu\nu}(x-x')=\left\{n_\mu n_\lambda\frac{\partial}{\partial x_\lambda}[\Lambda(x)-L(x)] -\frac{\partial}{\partial x_\mu}L(x)\right\}n_\nu n_\kappa\frac{\partial}{\partial x_\kappa'}[\Lambda(x')-L(x')] -\frac{\partial}{\partial x_\nu'}L(x')n_\mu n_\lambda[\partial/\partial x_\lambda][\Lambda(x)-L(x)]. \tag{27}$$

The supplementary condition (18) can be written in the form:

$$[\Lambda(x)-L(x)]\chi_0=0. \tag{28}$$

Hence:

$$\Phi_0^\dagger A_\mu(x)A_\nu(x')\Phi_0\chi_0 =\left\{i\delta_{\mu\nu}D^{(+)}(x-x')+P_{\mu\nu}(x-x')+Q_{\mu\nu}(x-x') +\frac{\partial^2}{\partial x_\mu\partial x_\nu'}L(x)L(x')\right\}\chi_0. \tag{29}$$

The right-hand side of Eq. (29) may be replaced by:

$$i\delta_{\mu\nu}D^{(+)}(x-x')\chi_0 \tag{30}$$

in the integrand of a gauge-invariant integral, which shows that the state vector χ_0 is transformed into the same state vector multiplied by the factor $i\delta_{\mu\nu}D^{(+)}(x-x')$. Thus, formula (29), like formula (7), provides a justification for applying formula (5) to gauge-invariant integrals.

The integral (8) is gauge-invariant if (i) the tensor $K_{\mu\nu}$ satisfies the conditions of gauge invariance (9), and (ii) the integral extends over the whole space. Expressed or implied, the conditions of gauge invariance have been used in all the recent investigations on the equivalence of the two methods of treating the longitudinal electromagnetic field. As the investigations of Wentzel and others show,[12] the evaluation of the induction tensor in the theory of vacuum polarization involves a great deal of ambiguity, so that gauge invariance of quantum electrodynamics is by no means a well-established fact at the present stage of the theory. However, gauge invariance is a basic requirement of a satisfactory formulation of quantum electrodynamics, and it is reasonable to take Eqs. (9) as an assumption or to ensure their validity by means of the regularization of Pauli and Villars.

For integrals which do not satisfy the requirement (ii), further investigation is necessary before the validity and limitation of the unified treatment can be established. For example, in Schwinger's calculations of the self-energy of the electron, the results obtained by the two methods of treating the longitudinal electromagnetic field become identical only after a gauge transformation. On the other hand, the requirement (ii) is satisfied by the iterated form of the S-matrix, so that the equivalence of the two methods of evaluating the S-matrix follows immediately from the above considerations.

[12] G. Wentzel, Phys. Rev. **74**, 1070 (1948); W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949).

PHYSICAL REVIEW VOLUME 80, NUMBER 4 NOVEMBER 15, 1950

Hall Coefficient and Resistivity of Thin Films of Antimony Prepared by Distillation*

W. F. LEVERTON† AND A. J. DEKKER
Department of Physics, University of British Columbia, Vancouver, Canada
(Received June 30, 1950)

An a.c. method is described for measuring the Hall coefficient. An apparatus for the preparation of very pure evaporated metal films is also described. This apparatus eliminates such sources of contamination as hot filaments.

The effect of annealing on both Hall coefficient and resistivity of evaporated films of antimony is examined. A tentative explanation is presented for the observed increase in the Hall coefficient and decrease in the resistivity. It is based on the assumption of partial recombination of electrons and holes.

The Hall coefficient of unannealed evaporated films is 0.215_9 c.g.s.m., and of annealed films is 0.24_14 c.g.s.m. with an accuracy of one percent.

The resistivity of our annealed films is 1.28 times that of bulk antimony.

I. INTRODUCTION

If a conductor carrying a current is placed in a magnetic field at right angles to the current, a potential difference develops across the conductor in a direction perpendicular to both magnetic field and current flow. This potential difference is known as the Hall voltage

$$V=(RIH/t)\times 10^{-9},$$

where I is the current in amperes, H is the magnetic field in gauss, t is the thickness of the conductor in cm in the direction of H, and R is the Hall coefficient of the conducting material in c.g.s. magnetic units.

Examination of the literature reveals that most observers have used direct current in their determinations of the Hall coefficient. Since such a determination measures the sum of the Hall and Ettingshausen voltages, the results must be corrected by measuring the latter separately. Since the voltages are very small, the Hall voltage determined in this manner may be in serious error where the Ettingshausen voltage is large compared with the Hall voltage. The Ettingshausen voltage may be eliminated by using Hall contacts of the same material as the sample; however, this is practicable only in a limited number of cases.

In the measurements described in this paper an alternating current was used in the determination of Hall coefficients. Since the temperature gradient which leads to the Ettingshausen effect requires a time of the order of seconds to become established,[1] use of alternating current completely eliminates this effect from the observed transverse voltage. A further advantage of the a.c. method is that it is much easier to amplify very small alternating voltages than to amplify direct voltages of the same magnitude.

Despite these advantages, few observers[1–3] have used a.c. methods. Smith[1] found the Hall coefficient of bismuth to be constant within experimental error for frequencies up to 120,000 cycles per second. Wood[2] found similar results for tellurium up to 10,000 cycles per second. Busch and Labhart[3] worked only at line frequency (50 cycles). Some early observers[4–6] used a.c. of the same frequency in both the Hall sample and in the

* Assisted by the Defence Research Board of Canada.
† Holder of National Research Council of Canada Studentship, 1948–49, 1949–50.

[1] A. Smith, Phys. Rev. **35**, 81 (1912).
[2] L. A. Wood, Phys. Rev. **41**, 231 (1932).
[3] G. Busch and H. Labhart, Helv. Phys. Acta **19**, 463 (1946).
[4] T. Des Coudres, Physik. Zeits. **2**, 586 (1901).
[5] Von Traubenberg, Ann. d. Physik **17**, 78 (1905).
[6] H. Zahn, Ann. d. Physik **36**, 553 (1911).