masses of Ca^{43}, K^{43}, and Sc^{43} given in Table I may be somewhat too high.

III. DISCUSSION OF RESULTS

Figures 1 and 2 show that although there is considerable deviation of the measured masses from the Bohr-Wheeler formula, all deviations with the exception of the value for Ca^{40} lie on relatively smooth curves. The deviations from the curves are not generally larger than 0.5 Mev. There seems to be no abrupt change of mass or of slope of the curves of mass *versus* neutron number near S^{36}, Cl^{37}, A^{38}, or K^{39}, all of which are atoms with 20 neutrons. Similarly, there is no evidence of shell structure at 20 protons from the curves of mass *versus* proton number, except in the case of Ca^{40} which shows a striking deviation from the smooth curves of about 3.5 Mev. This deviation is far greater than the probable error of mass determinations or than the deviation of any other nucleus plotted.

The absence of any change in slope at 20 nucleons makes it rather questionable whether 20 nucleons should be regarded as the closing point of a major shell. The well-known stability of Ca^{40} seems not to be simply connected with the completion of a shell at 20 nucleons. This case indicates that some large deviations from the stability curve may be encountered for other nuclei which are not attributable to neutron or proton shells alone but depend on the combination of neutron and proton numbers. Perhaps the large mass spread of stable Ca isotopes is due to this exceptional stability of Ca^{40} and to a shell at 28 neutrons which makes Ca^{48} stable rather than to a general stability of 20 protons.

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On a Difference Equation Method in Cosmic-Ray Shower Theory*

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The recent results of Snyder and Bhabha-Chakrabarty for the cascade theory of cosmic-ray showers are shown to be derivable from a general approach involving the use of the Laplace and Mellin transforms, and a general and powerful method, due to Snyder, for solving the resulting difference equations. Boundary conditions are introduced in a natural and automatic way, and the accuracy of the solution is limited by the possible ways of evaluating the resulting triple complex integral.

I. INTRODUCTION

Snyder[1] has recently obtained numerical results for the cascade theory of electron-photon showers which appear to be considerably more accurate than those of Bhabha and Chakrabarty.[2] It is the object of this paper to present a general method of solving the shower equations which yield both of the above-mentioned solutions, and which should be applicable to a number of other problems.

II. THEORY

Using Snyder's[1] notation, we write the diffusion equation for $P(E, t)$, the mean energy spectrum of electrons at depth t, and $\gamma(E, t)$, the mean energy spectrum of photons:

$$\frac{\partial P(E,t)}{\partial t}=\lim_{\delta\to 0}\left[\int_{E+\delta}^{\infty}P(E',t)R(E',E'-E)\frac{E'-E}{E'^2}dE' - P(E,t)\int_{\delta}^{E}R(E,E')\frac{E'dE'}{E^2}\right]+\beta\frac{\partial P(E,t)}{\partial E} +2\int_{E}^{\infty}\gamma(E',t)R(E,E')\frac{dE'}{E'}, \quad (1)$$

$$\frac{\partial \gamma(E,t)}{\partial t}=\int_{E}^{\infty}P(E',t)R(E',E)\frac{EdE'}{E'^2} -\gamma(E,t)\int_{0}^{E}R(E',E)\frac{dE'}{E}. \quad (2)$$

In these equations $R(E', E)$ is a function which yields the elementary probabilities per unit path length of the pair-production and bremsstrahlung processes. In the case of high energies, the asymptotic form of R is that of a function homogeneous in E/E'; this is the only case dealt with in the present treatment.

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* This work was started in 1940 (Ph.D. Thesis, University of Michigan, 1941). It was completed at the Brookhaven National Laboratory, under the auspices of the AEC.

[1] H. S. Snyder, Phys. Rev. **76**, 1563 (1949).

[2] H. J. Bhabha and S. K. Chakrabarty, Phys. Rev. **74**, 1352 (1948).

Here β is the ionization or collision loss per unit path length, and is taken to be a constant. Its inclusion is essential. Solutions of the equations for $\beta=0$ are obtained easily, but are fundamentally different from those for *any* finite value of β, since for $\beta=0$ there is no removal of particles from the showers defined by Eqs. (1) and (2). In what follows, we shall measure energy in units of β by introducing $x=E/\beta$; where an initial energy E_0 is used, we write $x_0=E_0/\beta$. The resulting equations differ from (1) and (2) only by the replacement of E's by x's and the suppression of β.

The method here presented consists in applying both a Laplace transformation in t and a Mellin transformation in E, and then solving the resulting difference equation. Following the usual practice for Laplace transforms, we shall denote by s the transformed variable corresponding to t. The variables referred to by various authors as (λ, μ), (λ_1, λ_2) or (μ, ν) will appear later on after evaluation of the Laplace inversion integral by the calculus of residues. The variable for the Mellin transform we then call p.

Let us, therefore, set

$$g(p, s)=\int_0^\infty e^{-st}dt\int_0^\infty x^p dx P(x, t),$$
$$\theta(p, s)=\int_0^\infty e^{-st}dt\int_0^\infty x^p dx\gamma(x, t). \tag{3}$$

Multiply Eqs. (1) and (2) by $e^{-st}dtx^p dx$ and integrate over x and t. We find

$$\begin{aligned} sg(p, s)-\varphi_P(p)&=-A(p)g(p, s)\\ &\quad+B(p)\theta(p, s)-pg(p-1, s), \\ s\theta(p, s)-\varphi_\gamma(p)&=C(p)g(p, s)-D\theta(p, s). \end{aligned} \tag{4}$$

The functions $A(p)$, $B(p)$, $C(p)$ and the constant D are given by Snyder[1] and by Rossi and Greisen.[3] $\varphi_P(p)$ is the Mellin transform of the initial spectrum of particles, and $\varphi_\gamma(p)$ is that for the photons:

$$\varphi_P(p)=\int_0^\infty P(x, 0)x^p dx, \quad \varphi_\gamma(p)=\int_0^\infty \gamma(x, 0)x^p dx.$$

It is to be noticed, first, that the form of Eqs. (4) depends on the homogeneity property of the asymptotic cross-sections; second, that this method brings the initial conditions in automatically and exactly; third, that the term $-pg(p-1, s)$ results from the collision loss term $\beta\partial P/\partial t$ and is the main source of difficulty with this approach.[4]

Equations (4) can be solved for g:

$$pg(p-1, s)+\left[s+A(p)-\frac{B(p)C(p)}{s+D}\right]g(p, s)=\varphi(p, s), \tag{5}$$

where

$$\varphi(p, s)=\frac{B(p)\varphi_\gamma(p)}{s+D}+\varphi_P(p). \tag{6}$$

$\theta(p, s)$ is given by

$$\theta(p, s)=\frac{C(p)g(p, s)+\varphi_\gamma(p)}{s+D}. \tag{7}$$

In what follows, we shall deal only with $g(p, s)$ inasmuch as exactly similar results for $\theta(p, s)$ can be found by using (7). We shall also suppress the letter s in $\varphi(p, s)$ for brevity. Finally, we write for the square bracket in (5)

$$G(p, s)=s+A(p)-\frac{B(p)C(p)}{s+D}\equiv\frac{[s-\mu(p)][s-\nu(p)]}{s+D}, \tag{8}$$

where μ and ν are functions given by Snyder[1] [his Eq. (17)], by Rossi and Greisen,[3] who call them λ_1 and λ_2, and by Bhabha and Chakrabarty,[2] who call them $-\lambda(p+1)$ and $-\mu(p+1)$. As $|p|\to\infty$, $G(p, s)\sim c_1 \ln p$, where c_1 is a constant.

We thus have to solve the difference equation

$$pg(p-1, s)+G(p, s)g(p, s)=\varphi(p). \tag{9}$$

The condition on $g(p, s)$ which determines the particular one of the infinite manifold of solutions of (9) that yields the desired solutions of Eqs. (1) and (2) is that $g(p, s)$ must be a proper Mellin transform function in p. From (3) we see that, if the initial spectrum of particles contains none above $x_0=E_0/\beta$, then $g(p, s)$ increases as $p\to\infty$ less rapidly than x_0^p. That is, $x_0^{-p}g(p, s)\to 0$ as $p\to+\infty$. Consequently, we can solve (9) by iteration for $g(p, s)$, successively replacing p by $p+1$, $p+2$, $p+3$, $\cdots$, and eliminating the intermediate values of $x_0^{-p-m}g(p+m, s)$. We obtain in this way an infinite series, with a remainder which approaches zero. In the final result, x_0^{-p} can be factored out, and we have a convergent series

$$\begin{aligned} g(p, s)=&\frac{\varphi(p+1)}{p+1}-\frac{\varphi(p+2)G(p+1, s)}{(p+1)(p+2)}+\frac{\varphi(p+3)G(p+1, s)G(p+2, s)}{(p+1)(p+2)(p+3)}-\cdots \\ &+(-1)^m\frac{\varphi(p+m+1)G(p+1, s)G(p+2, s)\cdots G(p+m, s)}{(p+1)(p+2)\cdots(p+m+1)}+\cdots. \end{aligned} \tag{10}$$

This series yields an appropriate Mellin transform in p, which can, in fact, be integrated termwise. But the mth term as a function of s behaves like s^m, and cannot be used in the Laplace inversion integral, which requires that the Laplace transform vanish as s^{-1} as $s\to+\infty$.

[3] B. Rossi and K. Greisen, Rev. Mod. Phys. 13, 240 (1941).

[4] F. L. Friedman, Cosmic Ray Shower Theory, MIT Laboratory for Nuclear Science and Engineering, Tech. Rpt. 31, pp. 43–45.

Thus we must sum the series (10), or obtain another representation for it. We shall first write the general term in such a way that we can replace the integer m by a complex variable σ, so that we can in the well-known way convert the series into a contour integral.

Now, the mth term of (10) contains m factors G in the numerator. Just as the denominator can be written as a ratio of two gamma-functions to suppress the variable number of factors, so we shall use the ratio of two new functions (definable as infinite products) to rewrite the numerator.

For the denominator, we have

$$(p+1)(p+2)\cdots(p+m+1)=\frac{\Gamma(p+m+2)}{\Gamma(p+1)}. \quad (11)$$

with $p\Gamma(p)=\Gamma(p+1)$.

For the numerator, we shall take

$$G(p+1,s)G(p+2,s)\cdots G(p+m,s)=\frac{L(p+m+1,s)}{L(p+1,s)} \quad (12)$$

where $L(p,s)$ must satisfy the recursion relation

$$G(p,s)L(p,s)=L(p+1,s). \quad (13)$$

Snyder[5] has shown how to solve equations of type (13) in a very general way for general values of p in terms of infinite products. His solution for this case can be written as follows:

$$L(p,s)=G(b,s)^{p-a}\lim_{N\to\infty}\prod_{j=0}^{N}\frac{G(a+j,s)}{G(p+j,s)}\times\left\{\frac{G(b+j+1,s)}{G(b+j,s)}\right\}^{p-a}$$

$$=\lim_{N\to\infty}G(b+N+1,s)^{p-a}\prod_{j=0}^{N}\frac{G(a+j,s)}{G(p+j,s)}. \quad (14)$$

The infinite product is convergent for a wide class of functions G. Indeed, if G satisfies a relation of the form

$$\frac{G(p_1+j,s)}{G(p_2+j,s)}=1+O\left(\frac{1}{j}\right)\quad \text{as}\quad j\to\infty \quad (15)$$

for any p_1, p_2 and s, the infinite product will be convergent.[6] Functions which behave like $p^{\pm r}$, $\ln p$, $p^{\pm r}e^{\lambda p}$, etc., will thus yield convergent products.

To see that (14) satisfies (13), write

$$\frac{L(p+1,s)}{L(p,s)}=G(b,s)\lim_{N\to\infty}\prod_{j=0}^{N}\frac{G(p+j,s)}{G(p+j+1,s)}\frac{G(b+j+1,s)}{G(b+j,s)}$$

$$=G(b,s)\lim_{N\to\infty}\left\{\frac{G(p,s)}{G(p+N+1,s)}\frac{G(b+N+1,s)}{G(b,s)}\right\}$$

$$=G(p,s)$$

by Eq. (15). Here a is a constant which merely determines that value of p for which $L=1$. When $p-a$ is an integer, (14) reduces to a finite product. Also, b is a convergence-controlling number which does not affect the value of L, since

$$\lim_{N\to\infty}\left[\frac{G(b_1+N+1,s)}{G(b_2+N+1,s)}\right]^{p-a}=1$$

(excepting the case of reference 6). We may, in fact, write the factor $G(b+N+1)^{p-a}$ as a product of several factors with different values of b and different exponents, provided the sum of the exponents is $p-a$. These values of b and the corresponding exponents can be so chosen that the product in (14) yields exact values *for any* N at specified values of p. Then for neighboring values of p the convergence will be especially rapid, and (14) will be useful for numerical work.[7]

For $G(p,s)=p$, L becomes $\Gamma(p)$; values calculated in the manner aforementioned yield extremely rapid convergence.[8]

A useful asymptotic expression for $L(p,s)$ may be obtained by writing $\ln G(p,s)=\ln L(p+1,s)-\ln L(p,s)\simeq\partial/\partial p[\ln L(p,s)]$ and using $G(p,s)\simeq c_1\ln p$ yielding on integration

$$L(p,s)\simeq(c_1\ln p)^p \quad (16)$$

as $|p|\to\infty$.[9]

We may now write the mth term of the series (10) in a form which does not involve a variable number of factors, and which allows m to be replaced by σ. To make the writing simpler, we shall first write

$$\frac{L(p+m+1,s)}{L(p+1,s)}=Q(p+1,m,s)$$

$$=G(b,s)^m\prod_{j=0}^{\infty}\frac{G(p+j+1,s)}{G(p+j+m+1,s)}\left\{\frac{G(b+j+1,s)}{G(b+j,s)}\right\}^m. \quad (17)$$

$Q(p+1,m,s)$ obeys the recursion relations

$$G(p,s)Q(p+1,m,s)=Q(p,m+1,s) \quad (18a)$$

and

$$G(p+m,s)Q(p,m,s)=Q(p,m+1,s). \quad (18b)$$

We also have

$$Q(p,0,s)=1. \quad (19)$$

The series (10) for $g(p,s)$ now becomes

$$g(p,s)=\sum_{m=0}^{\infty}(-1)^m\varphi(p+m+1)\frac{\Gamma(p+1)}{\Gamma(p+m+2)}\times Q(p+1,m,s), \quad (20)$$

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[5] H. S. Snyder, Phys. Rev. **75**, 906 (1949).

[6] If $G=e^{\lambda p}$ where λ is any function of s, the convergence is trivial. If a factor $e^{\lambda p}$ is included in G, (15) will be altered by a factor $e^{\lambda(p_1-p_2)}$ but the convergence will be unaffected.

[7] See Eq. (25) of reference 1, and the discussion following.

[8] H. S. Snyder (private communication).

[9] I wish to thank Mr. I. Bernstein for informing me of this asymptotic expression.

and we can write $g(p, s)$ as a contour integral as follows

$$g(p,s)=\frac{1}{2\pi i}\int_{C_\sigma}d\sigma\frac{\pi}{\sin\pi\sigma}\frac{\phi(p+\sigma+1)\Gamma(p+1)}{\Gamma(p+\sigma+2)}\times Q(p+1,\sigma,s). \quad (21)$$

where C_σ is a contour from $+\infty$ around the origin in the positive sense, and back to $+\infty$. It is readily shown that the contour can be shifted to C_σ', a contour parallel to the imaginary axis, cutting the real axis between -1 and 0. The negative of the integral must be taken if C_σ' is described in the usual positive sense.

Since $Q(p+1\ \sigma, s)$ behaves like s^σ for large $|s|$, we see that the integral over σ will in general behave like s^δ where $-1<\delta<0$. We can now write the inverse Laplace and Mellin transforms, since we have the correct behavior in s and p:

$$P(x,t)=\frac{-1}{(2\pi i)^3}\int_{C_s}e^{st}ds\int_{C_p}x^{-p-1}dp\int_{C_\sigma'}d\sigma\frac{\pi}{\sin\pi\sigma}\times\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2)}Q(p+1,\sigma,s)\varphi(p+\sigma+1) \quad (22)$$

where C_s and C_p are contours parallel to the imaginary axis in the s-plane and p-plane, respectively, each taken to the right of all singularities. A similar formula can easily be obtained for $\gamma(E, t)$.

We thus have achieved a solution of our problem, Eqs. (1) and (2), in the form of triple complex integrals. The evaluation of these integrals is naturally not simple. One can evaluate one integral as a series of residues, reduce a second by restricting our attention to finding an integral spectrum for the total number of particles (or for low energies, expanding by residues again in powers of x) and calculate the last integral by the method of steepest descents. It is also possible to use a double saddle-point method for two integrals simultaneously.[2]

We shall first show how Snyder's result can be obtained. We shall restrict ourselves to electron-initiated showers, with

$$P(x,0)=\delta(x-x_0);\quad \gamma(x,0)=0, \quad (23)$$

or $\phi(p, s)=x_0^p$.

The first step is to evaluate the s integral, interchanging the order of integration. Let us write $m=\sigma$ in (17), set $b=0$ and use (8).

$$Q(p+1,\sigma,s)=\lim_{N\to\infty}\frac{[s-\mu(N+1)]^\sigma[s-\nu(N+1)]^\sigma}{[s+D]^\sigma}\times\prod_{j=0}^{N}\frac{[s-\mu(p+j+1)][s-\nu(p+j+1)]}{[s-\mu(p+\sigma+j+1)][s-\nu(p+\sigma+j+1)]}. \quad (24)$$

Poles occur for

$$\left.\begin{array}{l} s=\mu(p+n+\sigma+1) \\ \text{and}\quad s=\nu(p+n+\sigma+1)\end{array}\right\} n=0,1,2,\cdots. \quad (25)$$

As $p\to\infty$, $\mu(p)\to -D$, $\nu(p)\to-\infty$ so the point $s=-D$ is the limit point of one series of poles, and need not be treated itself. The residue of Q at $s=\mu(p+n+\sigma+1)$ is then

$$\lim_{N\to\infty}\left\{\frac{[\mu(p+n+\sigma+1)-\mu(N+1)]^{\sigma+n+1}[\mu(p+n+\sigma+1)-\nu(N+1)]^{\sigma+n+1}}{[\mu(p+n+\sigma+1)+D]^\sigma}\right\}$$
$$\cdot\left\{\prod_{j=0}^{N}\frac{[\mu(p+n+\sigma+1)-\mu(p+j+1)][\mu(p+n+\sigma+1)-\nu(p+j+1)]}{[\mu(p+n+\sigma+1)-\mu(p+\sigma+n+j+2)][\mu(p+n+\sigma+1)-\nu(p+\sigma+n+j+2)]}\right\}$$
$$\cdot\left\{\frac{\prod_{k=0}^{n}[\mu(p+n+\sigma+1)-\mu(p+N+\sigma+k+2)][\mu(p+n+\sigma+1)-\nu(p+N+\sigma+k+2)]}{[\mu(p+n+\sigma+1)-\mu(N+1)]^{n+1}[\mu(p+n+\sigma+1)-\nu(N+1)]^{n+1}}\right\}$$
$$\cdot\left\{\frac{1}{\mu(p+n+\sigma+1)-\nu(p+n+\sigma+1)}\prod_{l=0}^{n-1}\frac{1}{[\mu(p+n+\sigma+1)-\mu(p+l+\sigma+1)][\mu(p+n+\sigma+1)-\nu(p+l+\sigma+1)]}\right\} \quad (26)$$

where the third brace { } contains factors that cancel with those added in the second brace to maintain symmetry. The factors dividing this third product are also inserted in the first numerator; they allow the expression in the third brace to approach 1 as $N\rightarrow\infty$. Thus we may write, using (24) to shorten the writing,

$$P(x,t)=\frac{-1}{(2\pi i)^2}\int_{C_p}\left(\frac{x_0}{x}\right)^{p+1}\Gamma(p+1)dp\int_{C_\sigma'}\frac{\pi d\sigma}{\sin\pi\sigma}\frac{x_0^\sigma}{\Gamma(p+\sigma+2)}\frac{1}{\mu(p+n+\sigma+1)-\nu(p+n+\sigma+1)}$$
$$\sum_{n=0}^{\infty}\left\{\frac{e^{\mu(p+n+\sigma+1)t}[\mu(p+n+\sigma+1)+D]^{n+1}Q(p+1,\sigma+n+1,\mu(p+n+\sigma+1))}{\prod_{l=0}^{n-1}[\mu(p+n+\sigma+1)-\mu(p+l+\sigma+1)][\mu(p+n+\sigma+1)-\nu(p+l+\sigma+1)]}\right.$$
$$\left.-\frac{e^{\nu(p+n+\sigma+1)t}[\nu(p+n+\sigma+1)+D]^{n+1}Q(p+1,\sigma+n+1,\nu(p+\sigma+n+1))}{\prod_{l=0}^{n-1}[\nu(p+n+\sigma+1)-\mu(p+l+\sigma+1)][\nu(p+n+\sigma+1)-\nu(p+l+\sigma+1)]}\right\}\quad(27)$$

Equation (27) is identical with Snyder's[1] Eq. (38) as can be seen if we write

$$\sigma+n+1=-s\qquad\text{(Snyder's } s\text{, not ours!)}$$
$$p+\sigma+n+1=y;\quad dp=dy.\quad(28)$$

We then set

$$Q(p+1,\sigma+n+1,\mu(p+n+\sigma+1))=Q(y+s-1,-s,\mu(y))=\frac{K_\mu(y,s)}{\Gamma(s+1)},\quad(29)$$

and

$$Q(p+1,\sigma+n+1,\nu(p+n+\sigma+1))=K_\nu(y,s)/\Gamma(s+1).$$

We use the formulas for Snyder's $A_n(y)$ and $B_n(y)$ given in the note at the end of his paper. Finally, we can use Snyder's arguments to justify displacing the contour for the y integration n places to the left, to bring it from having a real part $n+\epsilon$ to having a real part ϵ, $1>\epsilon>0$.

To arrive at the Bhabha-Chakrabarty result,[2] we shall write in (22), $p+\sigma+2=\lambda$, $dp=d\lambda$ (λ is identical with Bhabha-Chakrabarty's s), and keep the λ-integration path fixed by shifting the p contour C_p to the right as C_σ' is shifted to the left. We evaluate the σ integral first, at the poles $\sigma=-m-1$, $m=0,1,2,\cdots$. The resulting series is divergent, but if we take only a finite number of poles, leaving a remainder, and then carry out the inverse Laplace transform on each term, the resulting series of p integrals is convergent. Thus we write, after taking residues in σ,

$$P(x,t)=\sum_{m=0}^{\infty}\frac{x^{-m}}{2\pi i x_0}\int dx\left(\frac{x_0}{x}\right)^{\lambda}\frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}(-1)^m\frac{1}{2\pi i}\times\int_{C_s}e^{st}dsQ(\lambda+m,-m-1,s).\quad(30)$$

Using (18a) and (19) repeatedly, we see that

$$Q(\lambda+m,-m-1,s)=\prod_{k=0}^{m}\frac{1}{G(\lambda+k-1,s)},\quad(31)$$

and that the functions

$$\psi_m(\lambda,t)=\frac{1}{2\pi i}\int_{C_s}e^{st}dsQ(\lambda+m,-m-1,s)\quad(32)$$

may be found by the theorem of convolution integrals (faltungs-integrals) from

$$\psi_0(\lambda,t)=\frac{1}{2\pi i}\int e^{st}ds\frac{s+D}{[s-\mu(\lambda-1)][s-\nu(\lambda-1)]}.\quad(33)$$

Equation (33) readily yields $B-C$'s formula (8),[2] and their convolution integrals (11) and (13) are readily deduced from Eqs. (31), (32) and (33). We thus arrive at their expression (14) for the electron spectrum, and may treat it as they did.

The methods of this paper are clearly applicable to a rather general class of differential-integral equations, and in particular should be useful for further work in cascade theory. In fact, Bernstein has shown[10] that corrections to Snyder's results can be made by methods related to those of this paper, using more accurate but inhomogeneous cross sections for the elementary processes.

[10] I. Bernstein, Ph.D. Thesis, New York University (1950).