On the Interactions of Birkhoff's Gravitational Field with the Electromagnetic and Pair Fields

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The flat space-time theory of gravitation proposed by G. D. Birkhoff is discussed from the standpoint of a field theory and, in particular, the interaction of Birkhoff's field with other fields is developed. This interaction is shown to take place through the interaction Lagrangian function $h_{\mu\nu}\theta_{\mu\nu}$, and when applied to the electromagnetic and pair fields, it leads to a modification of the Maxwell and Dirac equations. These modified equations determine a change in the index of refraction and a shift of the energy levels of the atom, which account for the bending of light and the red shift in gravitational phenomena. Furthermore, they predict a gravitational correction to the magnetic moment associated with the spin and orbital motion of the electron.

I. INTRODUCTION

At the present time there is a wide interest in relativistic field theories in connection with the interactions between meson, electromagnetic, and pair fields. The effect of the gravitational field on these interactions has been mainly neglected, partly because of the smallness of the gravitational interactions, and partly because of the special position given to the gravitational field in the general theory of relativity.

It seems therefore, worthwhile to examine descriptions of gravitation in flat space-time, from the standpoint of field theory. The theory of gravitation that we propose to discuss from this viewpoint was introduced by G. D. Birkhoff,[1-5] and in it, the gravitational field is described in flat space-time by a symmetric tensor potential $h_{\alpha\beta}$, $\alpha, \beta=0, 1, 2, 3$. The equation of motion for a particle of rest mass m is given by the usual relation of special relativity, $m(d^2x_\alpha/ds^2)=f_\alpha$, where f_α the ponderomotive force for the gravitational field, has the form:

$$f_\alpha=m(\partial h_{\alpha\beta}/\partial x^\gamma-\partial h_{\beta\gamma}/\partial x^\alpha)dx^\beta/ds\, dx^\gamma/ds. \qquad (1)$$

The motion is therefore independent of the mass of the particle.

The field caused by a distribution of matter represented by the energy momentum tensor $T_{\alpha\beta}$ is given by the field equation:[6]

$$\Box h_{\alpha\beta}=4\pi G/c^4 T_{\alpha\beta}, \qquad (2)$$

where $\Box\equiv\nabla^2-1/c^2\,\partial^2/\partial t^2$, and G is the gravitational constant.

It can be shown from (2) and other assumptions that the field outside of an homogeneous sphere of mass M at a distance r from its center, is given by:[2]

$$h_{\alpha\beta}=MG/c^2r\delta_{\alpha\beta}, \quad \text{where} \quad \delta_{\alpha\beta}=\begin{cases}0 & \text{if} \quad \alpha\neq\beta\\ 1 & \text{if} \quad \alpha=\beta\end{cases} \qquad (3)$$

and that this is also the field of a point particle. With the field (3) introduced in the equation of motion (1), one obtains an advance for the perihelion of a planet in the field of a star, which has the same value as in the general theory of relativity.[1-4]

While the bending of light rays and the red shift could be discussed in Birkhoff's theory with the help of the equation of motion (1) and the photon concept,[1-4] it is more proper to view them as due to the action of the gravitational field on the electromagnetic field and on the process of emission of light. For this purpose, the procedure for setting the interaction of Birkhoff's field with other fields, must be developed.

The Lagrangian function[7] from which the field equations (2) could be derived, is given by:

$$L=-\frac{c^4}{8\pi G}g^{\alpha\beta}\frac{\partial h^{\gamma\delta}}{\partial x^\alpha}\frac{\partial h_{\gamma\delta}}{\partial x^\beta}+h^{\gamma\delta}T_{\gamma\delta}\equiv L_B+h^{\gamma\delta}T_{\gamma\delta}, \qquad (4)$$

as $\delta\int Ld^4x=0$ leads to (2). The first part of this Lagrangian function, L_B, corresponds to a free gravitational field, as the variational procedure applied to it leads to $\Box h_{\gamma\delta}=0$. The second part, $h^{\gamma\delta}T_{\gamma\delta}$, in which $T_{\gamma\delta}$ is the energy momentum tensor of the external field, gives rise to the interaction between the gravitational and external fields. This term is somewhat similar to the corresponding interaction Lagrangian function $j_\alpha A^\alpha/c$ between the electromagnetic and matter fields.

When we introduce the Lagrangian function L_B for Birkhoff's free field, we obtain from the general formalism of field theory[7,8] an energy momentum tensor for this field, and this energy momentum tensor of gravitational origin, will affect the field equations (2). It is clear therefore, that an interaction term of the form $h^{\gamma\delta}T_{\gamma\delta}$, in which $T_{\gamma\delta}$ corresponds to the external field,

[1] G. D. Birkhoff, Proc. Nat. Acad. Sci. **29**, 231 (1943).
[2] G. D. Birkhoff, Proc. Nat. Acad. Sci. **30**, 324 (1944).
[3] Barajas, Birkhoff, Graef, and Vallarta, Phys. Rev. **66**, 138 (1944).
[4] A. Barajas, Proc. Nat. Acad. Sci. **30**, 54 (1944).
[5] G. D. Birkhoff, Bol. Soc. Mat. Mexicana **1** (No. 4, 5) 1 (1944).
[6] In Birkhoff's papers (references 1, 2) the field equations take the form $\Box h_{\alpha\beta}=8\pi G/c^4T_{\alpha\beta}$ where $h_{\alpha\beta}$ is dimensionless and $T_{\alpha\beta}$ has dimensions of energy density. The difference of a factor of 2 from the above expression (2), stems from the special form taken by Birkhoff for the tensor of a perfect fluid. When the usual form of this tensor is introduced, the factor 8π must be changed to 4π (reference 10, p. 71).
[7] W. Pauli, Rev. Mod. Phys. **13**, 203 (1941).
[8] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949) Chap. 1.

will only be valid for weak gravitational fields ($h_{\gamma\delta} \ll 1$), to which we shall restrict ourselves in this article.

The Lagrangian function from which the interaction of Birkhoff's field with any other field can be derived is then given by:

$$L = L_B + h^{\alpha\beta}\theta_{\alpha\beta} + L', \tag{5}$$

where L' is the Lagrangian function of the other field and $\theta_{\alpha\beta}$ its corresponding symmetric energy-momentum tensor.

The modified field equations will be derived from the usual variational principle:

$$\delta \int L d^4x = 0. \tag{6}$$

We will apply this analysis to the interaction of Birkhoff's field with the electromagnetic and pair fields, for which we need the following notation:

$x^\alpha = (ct, \mathbf{r})$ the indices $\alpha, \beta, \gamma, \delta$ take the values 0, 1, 2, 3 with repeated indices summed from 0 to 3.
$x^\mu = (\mathbf{r}, ict)$ the indices μ, ν, ρ, σ take the values 1, 2, 3, 4 with repeated indices summed from 1 to 4.
$g^{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$.
$g_{\mu\nu} = 0$ if $\mu \neq \nu$ and $-g_{11} = -g_{22} = -g_{33} = g_{44} = 1$.
$\delta_{\mu\nu} = 0$ if $\mu \neq \nu$, $\delta_{\mu\nu} = 1$ if $\mu = \nu$.
$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\delta_{\mu\nu} dx^\mu dx^\nu$.
$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$.
$A_\mu = (\mathbf{A}, i\phi)$ electromagnetic potentials.
$E_{\mu\nu} = \partial A_\mu/\partial x^\nu - \partial A_\nu/\partial x^\mu$ electromagnetic field strengths.
$\mathbf{E} = -i(E_{14}, E_{24}, E_{34})$, $\mathbf{B} = (E_{32}, E_{13}, E_{21})$.
ϵ = dielectric constant.
μ = magnetic permeability.
n = index of refraction.
ψ = four component wave function of Dirac's equation.
ψ^+ = the adjoint Dirac wave function.
m, e = mass and charge of the electron.
$\gamma_\mu, \alpha_i, \beta, \sigma_i$ = Dirac matrices defined as in Pauli's book.[9]
j_μ = 4-vector current density.
$L_B = c^4/8\pi G \cdot \partial h_{\rho\sigma}/\partial x^\mu\, \partial h_{\rho\sigma}/\partial x^\mu$ the Lagrange function of the Birkhoff field.
$\delta L/\delta q = -\partial/\partial x^\mu[\partial L/\partial(\partial q/\partial x^\mu)] + \partial L/\partial q)$ the variational derivative of L with respect to the variable q.
$T_{\mu\nu}$ = the energy momentum tensor defined from the Lagrangian of the field as in Wentzel's book.[8]
$\theta_{\mu\nu}$ = the symmetric energy momentum tensor.
$P_\mu = \hbar/i\, \partial/\partial x^\mu$.
M = mass of the body originating the gravitational field.
G = gravitational constant.
$\hbar$ = Planck constant.
c = velocity of light.
r, θ, φ = spherical coordinates.
$u = 1/r$.
$M' \equiv MG/c^2$; $f \equiv MG/c^2 r$.
ν = frequency, $\lambda = c/\nu$ wave-length.

We will use from now on the coordinates x^μ, so that we must transform the tensors defined with respect to the coordinates x^α to the coordinate system x^μ by the usual rules. For example $h_{\alpha\beta}$ goes into $h_{\mu\nu}$ whose components are:

$$h_{\mu\nu} = \begin{pmatrix} h_{ij} & -ih_{i0} \\ -ih_{0j} & -h_{00} \end{pmatrix}.$$

[9] W. Pauli, Handb. der Physik. 2 Aufl., Band 24, (Springer, Berlin, 1933) p. 219–220.

In the system of coordinates x^μ the gravitational field (3) becomes then:

$$h_{\mu\nu} = -MG/c^2 r g_{\mu\nu} \equiv -f g_{\mu\nu}. \tag{7}$$

The metric tensor of the coordinate system x^μ is $-\delta_{\mu\nu}$ and this imples $A^\mu = -A_\mu$, $h^{\mu\nu} = h_{\mu\nu}$. The interaction part of the Lagrangian function becomes $h_{\mu\nu}\theta_{\mu\nu}$.

II. INTERACTION WITH THE ELECTROMAGNETIC FIELD

The Lagrangian function for the electromagnetic field is given by:

$$L_{em} = -1/16\pi E_{\mu\nu}E_{\mu\nu} \equiv 1/8\pi(\mathbf{E}^2 - \mathbf{B}^2) \tag{8a}$$

and the corresponding symmetric energy-momentum tensor is:

$$\theta_{\mu\nu}{}^{em} = 1/4\pi[E_{\mu\rho}E_{\nu\rho} - \tfrac{1}{4}E_{\rho\sigma}E_{\rho\sigma}\delta_{\mu\nu}]. \tag{8b}$$

The Lagrangian function for the interacting gravitational and electromagnetic fields takes the form:

$$L = L_B + \theta_{\mu\nu}{}^{em} h_{\mu\nu} + L_{em}. \tag{8c}$$

The field equations are derived from it with the help of the variational principle (6), where $h_{\mu\nu}$, A_μ are the field variables, so that:

$$\delta \int L d^4x = \int \left(\frac{\delta L}{\delta h_{\mu\nu}} \delta h_{\mu\nu} + \frac{\delta L}{\delta A_\mu} \delta A_\mu \right) d^4x = 0. \tag{9}$$

This variational relation leads to the equations $\delta L/\delta h_{\mu\nu} = 0$ and $\delta L/\delta A_\mu = 0$. The first is the Eq. (2) of the gravitational field with $T_{\mu\nu} = \theta_{\mu\nu}{}^{em}$. The second gives the electromagnetic field equations in the presence of a gravitational field, which take the form:

$$\partial/\partial x^\nu[E_{\mu\nu} - 2h_{\mu\rho}E_{\rho\nu} - 2E_{\mu\rho}h_{\rho\nu} + E_{\mu\nu}h_{\rho\rho}] = 0. \tag{10a}$$

To these equations we must add the field equations which are obtained from the definition[10] of $E_{\mu\nu}$, i.e.:

$$\partial E_{\mu\nu}/\partial x^\rho + \partial E_{\rho\mu}/\partial x^\nu + \partial E_{\nu\rho}/\partial x^\mu = 0. \tag{10b}$$

Equations (10) reduce to the ordinary Maxwell equations if there is no gravitational field, i.e., if $h_{\mu\nu} = 0$.

We are particularly interested in the electromagnetic field equations in the gravitational field outside of a spherical distribution of matter, such as a star. For that purpose we only need to substitute the corresponding field $h_{\mu\nu} = -MG/c^2 r g_{\mu\nu}$ in (10). We write the resulting equations in terms of the vectors $\mathbf{E}$, $\mathbf{B}$ and we obtain:

$$\nabla\cdot(1+2f)\mathbf{E} = 0, \quad \nabla\times(1-2f)\mathbf{B} = 1/c\, \partial/\partial t(1+2f)\mathbf{E} \tag{11a}$$

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} = -1/c\, \partial/\partial t\mathbf{B} \tag{11b}$$

$$f \equiv MG/c^2 r.$$

We compare these equations with the electromagnetic equations in an inhomogeneous material medium in the

[10] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, London, 1934) p. 97.

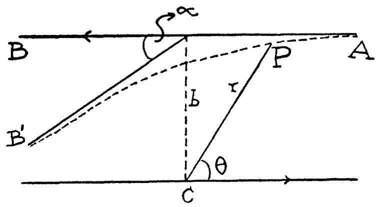

FIG. 1. Bending of a light ray in a gravitational field.

absence of currents and free charges,[11] which are:

$$\nabla\cdot\mathbf{D}=0,\quad \nabla\times\mathbf{H}=1/c\ \partial/\partial t\mathbf{D} \tag{12a}$$

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{E}=-1/c\ \partial/\partial t\mathbf{B} \tag{12b}$$

$$\mathbf{D}=\epsilon\mathbf{E},\quad \mathbf{B}=\mu\mathbf{H}. \tag{12c}$$

From (11) and (12), it is clear that the gravitational field of the star makes the space surrounding it behave, from an electromagnetic standpoint, as an inhomogeneous material medium. In fact, (11) and (12) become identical if we set:

$$\epsilon=1+2M'/r,\quad \mu=1/(1-2M'/r). \tag{13}$$

As the dielectric constant and magnetic permeability are functions of the distance from the star, the index of refraction in the space surrouning the star, which is given by[11]

$$n=(\epsilon\mu)^{\frac{1}{2}}=\left(\frac{1+2M'/r}{1-2M'/r}\right)^{\frac{1}{2}} \tag{14}$$

is also a function of the distance from the star. Light that passes in the neighborhood of a star will therefore be bent by its gravitational field.

III. THE BENDING OF LIGHT RAYS

We can use Fermat's principle to calculate the bending of light rays in the gravitational field of a star, as this principle is a consequence of the electromagnetic field equations[12] in flat space-time.

Let us take a light ray coming from infinity which, in the absence of a gravitational field, would travel along the straight line $[AB]$. Let C be the point at which the gravitating mass is situated, and b the length of the perpendicular from C to $[AB]$ as in Fig. 1. We introduce spherical coordinates with origin at C and polar axis parallel to $[AB]$. The plane CAB is taken as $\varphi=0$.

Fermat's principle states that the path taken by the light ray is given by:

$$\delta\int ndl=0. \tag{15a}$$

As the index of refraction $n=n(r)$ is only a function of the distance from C, the problem has spherical symmetry and a light ray starting in the plane $\varphi=0$ will continue its motion in it. The variational principle (15a) becomes:

$$\delta\int n(r)(dr^2+r^2d\theta^2)^{\frac{1}{2}}$$
$$=\delta\int n(r)(1+r^2\theta'^2)^{\frac{1}{2}}dr\equiv\delta\int\mathcal{L}dr \tag{15b}$$

and the corresponding Euler equation is:

$$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial\theta'}-\frac{\partial\mathcal{L}}{\partial\theta}=\frac{d}{dr}\frac{n(r)r^2\theta'}{(1+r^2\theta'^2)^{\frac{1}{2}}}=0 \tag{15c}$$

or

$$n(r)r^2\theta'(1+r^2\theta'^2)^{-\frac{1}{2}}=\text{const}=a. \tag{15d}$$

Introducing $u=1/r$, the Eq. (15d) becomes:

$$n(u)[(du/d\theta)^2+u^2]^{-\frac{1}{2}}=-a \tag{16a}$$

where from (14) $n^2(u)=1+2M'u/1-2M'u\simeq1+4M'\mu$. This last approximation is valid because r must be larger than the radius R of the star so that $M'u<M'/R$ and for all stars M'/R is small, e.g., for the sun it is $\simeq10^{-6}$. With this value of $n^2(u)$ the equation (16a) can be immediately integrated, giving:

$$u=1/a[2M'/a+(1+4M'^2/a^2)^{\frac{1}{2}}\sin(\theta+\delta)]. \tag{16b}$$

The path described by the light ray is then a hyperbola, and the two integration constants a, δ are determined by the condition that the hyperbola tends asymptotically to the line $[AB]$ whose equation is $u=b^{-1}\sin\theta$. When $\theta\rightarrow0$ we must have then $u\rightarrow0$ and $du/d\theta\rightarrow b^{-1}$, and assuming that $(2M'/b)^2$ can be disregarded as compared with unity, we have the relations:

$$a=b\quad\text{and}\quad\sin\delta=-2M'/b\quad\text{or}\quad\delta\simeq-2M'/b \tag{16c}$$

as again $M'/b<M/R\ll1$. The equation for the hyperbola becomes:

$$u=1/b[2M'/b+\sin(\theta-2M'/b)] \tag{16d}$$

and the second asymptote is given by $\theta=\pi+4M'/b$ as $u\simeq0$ for this value. The angle α between the asymptotes, shown in Fig. 1, is given then by:

$$\alpha=4M'/b \tag{17}$$

which is the same value of the general theory of relativity.

IV. INTERACTION WITH THE PAIR FIELD

The Lagrangian function for the pair field is given by:[7]

$$L_D=-\hbar c/2i(\psi^+\gamma_\mu\partial\psi/\partial x^\mu-\partial\psi^+/\partial x^\mu\gamma_\mu\psi)+imc^2\psi^+\psi \tag{18a}$$

and the corresponding energy momentum tensor is:

$$T_{\mu\nu}{}^D=\hbar c/2i(\psi^+\gamma_\mu\partial\psi/\partial x^\nu-\partial\psi^+/\partial x^\nu\gamma_\mu\psi). \tag{18b}$$

This tensor is not symmetric, but it is known[7] that the symmetric energy momentum tensor has the form: $\theta_{\mu\nu}{}^D=\frac{1}{2}(T_{\mu\nu}{}^D+T_{\nu\mu}{}^D)$. The interaction with the gravita-

[11] J. H. Van Vleck, *The Theory of Electric and Magnetic Susceptibilities* (Oxford University Press, London, 1932), pp. 1, 13.
[12] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), First Edition, p. 343.

tional field is given by $\theta_{\mu\nu}{}^D h_{\mu\nu}$ and as $h_{\mu\nu}$ is symmetric, this is equivalent to $T_{\mu\nu}{}^D h_{\mu\nu}$.

The Lagrangian function for the interacting gravitational and pair fields takes the form:

$$L=L_B+T_{\mu\nu}{}^D h_{\mu\nu}+L_D. \tag{18c}$$

The field equations are derived from it by means of the variational principle (6), where $h_{\mu\nu}$, ψ, ψ^+ are the field variables, so that:

$$\delta\int Ld^4x=\int\left[\frac{\delta L}{\delta h_{\mu\nu}}\delta h_{\mu\nu}+\delta\psi^+\frac{\delta L}{\delta\psi^+}+\frac{\delta L}{\delta\psi}\delta\psi\right]d^4x=0. \tag{19}$$

This variational relation leads to the equations $\delta L/\delta h_{\mu\nu}=0$, $\delta L/\delta\psi^+=0$, $\delta L/\delta\psi=0$, the first of which is Eq. (2) with $T_{\mu\nu}=\theta_{\mu\nu}{}^D$ and the others are the pair field equations in the presence of a gravitational field, that take the form:

$$\gamma_\mu\partial\psi/\partial x^\mu+mc/\hbar\psi-\tfrac{1}{2}h_{\mu\nu}\gamma_\mu\partial\psi/\partial x^\nu -\tfrac{1}{2}\gamma_\mu\partial h_{\mu\nu}\psi/\partial x^\nu=0, \tag{20a}$$

$$-\partial\psi^+/\partial x^\mu\gamma_\mu+mc/\hbar\psi^++\tfrac{1}{2}\partial\psi^+h_{\mu\nu}/\partial x^\nu\gamma_\mu +\tfrac{1}{2}\partial\psi^+/\partial x^\nu\gamma_\mu h_{\mu\nu}=0. \tag{20b}$$

Introducing the operators $P_\mu=\hbar/i\ \partial/\partial x^\mu$, we see that Eq. (20a) could be derived from the free particle Dirac equation by the substitution:

$$P_\mu\to P_\mu-\tfrac{1}{2}h_{\mu\nu}P_\nu-\tfrac{1}{2}P_\nu h_{\mu\nu}. \tag{21}$$

In the classical picture, where the energy-momentum vector P_μ and $h_{\mu\nu}$ commute, (21) implies that P_μ must be substituted by $P_\mu-h_{\mu\nu}P_\nu$, which is somewhat similar to what happens in the electromagnetic case where $P_\mu\to P_\mu+e/cA_\mu$. This result could not be derived directly from the equations of motion (1) as they do not admit in general, a Hamiltonian formulation.

We are interested in the form of the charge-current density 4-vector in the presence of a gravitational field. For that purpose we multiply (20a) by ψ^+ to the left and (20b) by ψ to the right and subtract, and making use of the symmetric form of $h_{\mu\nu}$ we obtain:

$$\partial/\partial x^\mu(\psi^+\gamma_\mu\psi-h_{\mu\nu}\psi^+\gamma_\nu\psi)=0. \tag{22a}$$

It is clear that the charge-current 4-vector for an electron field becomes:

$$j_\mu=ec(\psi^+\gamma_\mu\psi-h_{\mu\nu}\psi^+\gamma_\nu\psi) \tag{22b}$$

as it satisfies the continuity equation (22a), and in the absence of the gravitational field, i.e., $h_{\nu\tau}=0$, it reduces to usual form for the charge-current 4-vector.

The problems which interest us are those in which the electron moves in a combined gravitational and electromagnetic field. For that purpose we must add to the free pair field Lagrangian besides the term $T_{\mu\nu}{}^D h_{\mu\nu}$, a term of the form $j_\mu A^\mu/c=-j_\mu A_\mu/c$ which gives the interaction with the electromagnetic field. In this term, we should take the current j_μ as modified by the presence of the gravitational field as in (22b). The Lagrangian function for the pair field with these interactions, becomes:

$$L=L_D+T_{\mu\nu}{}^D h_{\mu\nu}-e(\psi^+\gamma_\mu\psi-h_{\mu\nu}\psi^+\gamma_\nu\psi)A_\mu. \tag{22c}$$

The field equations are derived as usual, from the variational principle (6), where ψ, ψ^+ are taken as the variables, and we obtain for ψ the equation:

$$\gamma_\mu[(\hbar/i\ \partial/\partial x^\mu+e/cA_\mu)\psi-(\hbar/2i\ h_{\mu\nu}\ \partial\psi/\partial x^\nu +\hbar/2i\ \partial h_{\mu\nu}\ \psi/\partial x_\nu+e/cA_\nu h_{\mu\nu}\psi)]-imc\psi=0. \tag{23}$$

Equation (23) is then the Dirac wave equation for an electron moving in a combined gravitational and electromagnetic field. This equation can be obtained from the Dirac free particle equation if we replace P_μ by $(P_\mu+e/cA_\mu)-h_{\mu\nu}(P_\nu+e/cA_\nu)$ and symmetrize the expression to allow for lack of commutability between P_μ and $h_{\mu\nu}$ in the quantum picture. It can also be obtained from (20a) if we replace P_μ by the form $P_\mu+e/cA_\mu$ it takes in the presence of an electromagnetic field.

With the help of (23) we can study the effect of a gravitational field on atoms and electrons. The gravitational interaction between the elementary particles themselves, is of the order of $m^2G/e^2\simeq10^{-40}$ times smaller than the electromagnetic interaction, and therefore, is not observable. On the other hand, the effect of an external gravitational field, such as that of a star, on atoms and electrons does give rise to observable phenomena which merit discussion.

V. THE HYDROGEN ATOM IN A GRAVITATIONAL FIELD

Let us consider a hydrogen atom near the surface of a gravitating mass M (such as a star) of radius R. The hydrogen atom will then be acted on by a constant gravitational field $h_{\mu\nu}=-MG/c^2Rg_{\mu\nu}\equiv-fg_{\mu\nu}$ and f is small for all gravitating bodies; i.e., $f\ll1$. We assume furthermore, that the hydrogen atom is acted by an external electromagnetic field represented by the vector potential $\mathbf{A}$.

From (23) we see that the wave equation for the electron of this hydrogen atom becomes:

$$\gamma_\mu(\delta_{\mu\nu}+fg_{\mu\nu})(\hbar/i\ \partial/\partial x^\nu+e/cA_\nu)\psi-imc\psi=0 \tag{24}$$

where $(A_i)=\mathbf{A}$ and $A_4=i\phi$ where ϕ is the electrostatic field of the proton. In the absence of the gravitational field, the electrostatic field of the proton takes the usual form $\phi=e/r$ where r is the distance from the proton. When a gravitating mass is present, we recall that it changes the dielectric constant of the space surrounding it, so that ϵ takes the value (13), i.e., $\epsilon=1+2f$. The electrostatic field of the proton takes then the form:

$$\phi=e/\epsilon r=e/(1+2f)r. \tag{25}$$

Multiplying Eq. (24) by $i\gamma_4/1+f$ and using the relations[9] $i\gamma_4\gamma_i=\alpha_i$, $\gamma_4=\beta$, we obtain:

$$(i\hbar/c\ \partial/\partial t+e/c\phi)\psi =(1-f)/(1+f)\boldsymbol{\alpha}\cdot(\hbar/i\nabla+e/c\mathbf{A})\psi+mc/1+f\beta\psi. \tag{26}$$

In the case $\mathbf{A}=0$ we can solve this equation and show that it leads to a shift of the spectral lines toward the red. Our physical picture would be simpler though, if we work in the Heisenberg representation. As the Hamiltonian is $\mathcal{H}=i\hbar\partial/\partial t$, Eq. (26) determines it by the relation:

$$\mathcal{H}/c+e/c\phi=(1-f)/(1+f)\boldsymbol{\alpha}\cdot(-i\hbar\nabla+e/c\mathbf{A})+(mc/1+f)\beta. \quad (27)$$

Squaring both sides of (27) and using the anticommuting properties of α_i, β we obtain by a procedure similar to that of Dirac:[13]

$$(\mathcal{H}/c+e/c\phi)^2=\frac{(1-f)^2}{(1+f)^2}\times[(\mathbf{P}+e/c\mathbf{A})^2+e\hbar/c(\boldsymbol{\sigma}\cdot\mathbf{B})]+\frac{m^2c^2}{(1+f)^2}, \quad (28)$$

where $\mathbf{B}=\nabla\times\mathbf{A}$, and σ_i are the spin matrices.

Assuming now that the energy of the first term on the right of (28) is small compared with the rest energy, which is actually the case for the hydrogen atom, we can take square root of both sides and obtain:

$$\mathcal{H}/c+e/c\phi=\frac{mc}{1+f}+\frac{(1-f)^2}{2mc(1+f)}\times[(\mathbf{P}+e/c\mathbf{A})^2+e\hbar/c\boldsymbol{\sigma}\cdot\mathbf{B}]. \quad (29)$$

The non-relativistic Hamiltonian is given by (29), when we disregard the modified rest energy term $mc/1+f$. Introducing (25) in (29) and making use of $f\ll 1$, so that $(1-f)^2/(1+f)\simeq 1/1+3f$ we obtain:

$$\mathcal{H}=-e^2/(1+2f)r+1/2m(1+3f)\times[(\mathbf{P}+e/c\mathbf{A})^2+e\hbar/c\boldsymbol{\sigma}\cdot\mathbf{B}]. \quad (30)$$

With the help of this Hamiltonian we can discuss the effect of an external gravitational field on the hydrogen atom. Let us first assume that there is no external magnetic field; i.e., $\mathbf{A}=0$,

$$\mathcal{H}=-e'^2/r+1/2m'\mathbf{P}^2, \quad (31)$$

where $e'^2=e^2/1+2f$, $m'=m(1+3f)$. The Schroedinger equation with this Hamiltonian gives for sth energy level E_s' the value[14]

$$E_s'=-m'e'^4/2\hbar^2s^2=-(1+3f)me^4/2(1+2f)^2\hbar^2s^2\simeq(1-f)E_s \quad (32a)$$

where E_s is the corresponding energy level in the absence of a gravitational field. We see, therefore, that there is a shift in the energy levels which gives a corresponding shift in frequencies:

$$\nu_{sp}'=h^{-1}(E_s'-E_p')=(1-f)h^{-1}(E_s-E_p)=(1-f)\nu_{sp} \quad (32b)$$

which leads to: $(\nu'-\nu)/\nu=\delta\nu/\nu=-f$. The corresponding shift in wave-lengths, observed at large distances from the gravitating mass, so that the relation $\lambda\nu=c$ holds, is given then by:

$$\delta\lambda/\lambda=f=M'/R. \quad (33)$$

We seen then, that the gravitational field leads to shift of the wave-length of the emitted light toward the red, which has the same value as that predicted by the general theory of relativity.

VI. GRAVITATIONAL CORRECTION TO THE MAGNETIC MOMENT OF THE ELECTRON

In the Hamiltonian expression (30), the last term corresponds to the additional potential energy of the spin of the electron. We can replace in this term $\mathbf{B}$ by $\mu\mathbf{H}$, and because of the presence of the gravitational field, μ takes the form (13), i.e., $\mu=1/1+2f$. As $f\ll 1$, we can write the last term of (30) in the form $(1-f)e\hbar/2mc\boldsymbol{\sigma}\cdot\mathbf{H}$. This potential energy may be interpreted as arising from the electron having a magnetic moment:

$$-(1-f)e\hbar/2mc\boldsymbol{\sigma}, \quad (34)$$

which reduces to the usual magnetic moment[13] when the gravitational field disappears, i.e., $f=0$.

The gravitational correction to the magnetic moment of the electron is very small, as on the surface of the earth, where we could attempt to detect it, f is only $\simeq 10^{-9}$. This gravitational effect is therefore, masked by the correction of quantum electrodynamical origin,[15] which is of the order of 10^{-3}. It would be of interest though, to study the effect on the magnetic moment of the electron of the gravitational field of rotating bodies, in connection with the recent suggestion of Blackett[16] concerning the magnetic moment of rotating masses.

The gravitational correction affects the orbital as well as the spin magnetic moment of the electron in the hydrogen atom. In fact, if we assume a constant magnetic field $\mathbf{B}_0=\mu\mathbf{H}_0$ we can write as usual $\mathbf{A}=\frac{1}{2}(\mu\mathbf{H}_0\times\mathbf{r})$, and introducing this in the Hamiltonian (30), we have $\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2$ where $\mathcal{H}_1$ is given by (31) and $\mathcal{H}_2$ becomes:

$$\mathcal{H}_2=(1-f)e/2mc[(\mathbf{r}\times\mathbf{P})+\hbar\boldsymbol{\sigma}]\cdot\mathbf{H}_0+e^2/8mc^2(1+f)(\mathbf{r}\times\mathbf{H}_0)^2. \quad (35)$$

The magnetic moment associated with the orbital motion of the electron is $-(1-f)(e/2mc)(\mathbf{r}\times\mathbf{P})$ and it is affected by the same gravitational correction as is the spin magnetic moment.

From the form of $\mathcal{H}_2$ we see that there will be a gravitational correction to the Zeeman effect, though again, it is very small, and it will be masked by the second term in (35) as well as by the quantum electrodynamical corrections.

[13] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, London, 1947), Third Edition, p. 264.
[14] Reference 13, p. 158.
[15] J. Schwinger, Phys. Rev. **73**, 416 (1948).
[16] P. M. S. Blackett, Nature **159**, 658 (1947).

The observed gravitational effects can be explained quite simply in terms of the interaction of Birkhoff's gravitational field with other fields. The mathematical simplicity of flat space-time gravitational theories, suggest that they could be used with profit in the study of the classical and quantum aspects of field theories.

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Three New Delayed Alpha-Emitters of Low Mass*

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Two new positron active isotopes, B^8 and Na^{20}, have been found to decay to excited states of Be^8 and Ne^{20}, which in turn decay "instantaneously" by alpha-emission. Their half-lives are 0.65 ± 0.1 sec. and $\frac{1}{4}$ sec., respectively. N^{12} is also found to have a low energy positron group which leads to an α-unstable excited state in C^{12}. The masses of B^8 and Na^{20} are 8.027 and 20.015, respectively. B^8 decays by a 13.7 ± 0.3-Mev positron, through the same excited state of Be^8 as does Li^8. Estimates of the energies of the excited state in C^{12} and Ne^{20} are made.

I. INTRODUCTION

UNTIL the present time, the only known light, delayed alpha-emitter,[1] was Li^8. In the terminology of classical radioactivity, "delayed alpha-particles," such as those from Li^8, are called "long-range alpha-particles." They arise from excited states of a daughter nucleus, following a beta-decay, and their real lifetimes are too short to be measured directly. Their apparent lifetimes are those of their parents, with which they are in equilibrium. The expression "delayed neutron emitter," is used for the same reason, to indicate that the observed neutron activity of nuclei such as[2] N^{17}, is not a true neutron radioactivity, but rather the "instantaneous" disintegration of an excited beta-decay daughter nucleus. In both neutron and alpha-decays of the delayed variety, it is possible to determine the lifetime of the actual heavy particle reaction, not by time measurements, but indirectly, from the uncertainty principle, using a measurement of the energy spread of the emitted particles.

Li^8 has been investigated by a number of nuclear physicists,[3-6] and its decay scheme is well understood. The beta-transition is first forbidden, and leaves the Be^8 daughter in a broad excited state about 3.1 Mev above the ground state. The width of the state is 0.8 Mev.

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II. DESCRIPTION OF EXPERIMENTS

The present experiments were started in an attempt to observe an example of delayed proton emission. Although this process has not yet been reported, it would be expected from nuclei such as Ne^{17}, O^{13}, and C^9. In the case of Ne^{17}, the reactions would be

$$_{10}Ne^{17}\rightarrow{}_9F^{17*}+e^+$$
$$_9F^{17*}\rightarrow{}_8O^{16}+{}_1H^1.$$

This pair of reactions is similar to the pair describing the delayed neutron activity of N^{17}:

$$_7N^{17}\rightarrow{}_8O^{17*}+e^-$$
$$_8O^{17*}\rightarrow{}_8O^{16}+n.$$

The 32-Mev proton beam from the Berkeley linear accelerator was used to bombard a proportional counter filled with $B^{10}F_3$. (Protons plus B^{10} could give C^9, and protons plus F^{19} could give Ne^{17}.) The linear accelerator is pulsed 15 times per second, for 300 μsec., and the proportional counter "cleans up" in a few milliseconds from the huge burst of ions formed during the 300-μsec. pulse. It is therefore very convenient to count delayed heavy particles through a gate circuit which eliminates all pulses during the time the counter is paralyzed. Activities may be followed in this manner, through buildup to equilibrium, and after the accelerator is turned off, through decay. A delayed heavy particle activity was observed in BF_3, with a half-life of about $\frac{2}{3}$ sec.

Before giving the reasons for the assignment of this activity to B^8, it will be well to describe other experimental techniques which were used in these investigations. Gaseous targets of CH_4 and Ne were also bom-