Variational Methods in Collision Problems

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RECENTLY variational methods have been applied successfully to scattering problems by several authors. However, it appears to be rather strange that so many different methods have been proposed for the one simple problem of scattering by a center of force. In the present note we wish to point out some simple relations existing among these apparently independent procedures.

For simplicity consider the radial wave equation for S-scattering

$$L[u]\equiv d^2u/dr^2+k^2u+W(r)u=0. \qquad (1)$$

Let us normalize the trial wave function by the conditions

$$u(0)=0,\quad u(r)\rightarrow\cos(kr+\theta)+\lambda\sin(kr+\theta),\quad r\rightarrow\infty, \qquad (2)$$

in which θ is a fixed constant (usually zero or $\pi/2$) and λ is an adjustable parameter. For the correct wave function λ is connected with the phase, η, by the relation $\lambda=\cot(\eta-\theta)$.

It is easily shown that

$$\delta\int_0^\infty uL[u]dr=k\delta\lambda$$

for the correct wave function, $u=u_0$. In other words, the functional

$$F\equiv F[u]=k\lambda-\int_0^\infty uL[u]dr \qquad (3)$$

is stationary for $u=u_0$ and is equal to $k\cot(\eta-\theta)$ since $L[u_0]=0$. If we set $\theta=\pi/2$ (hence $\lambda=-\tan\eta$), we have the variational method originally due to Kohn[1] and equivalent also to Huang's method[2] if we make a slight modification in the latter.

The methods of Hulthén[3] and Schwinger[4] can also be derived directly from (3) with $\theta=0$ (hence $\lambda=\cot\eta$). To show this, it is convenient to set

$$u=\cos(kr)-y+\lambda\sin(kr),\quad y(0)=1,\quad y\rightarrow 0 \text{ as } r\rightarrow\infty. \qquad (4)$$

Substitution of (4) into (3) and integration by parts yields

$$F=F_H\equiv -J+2(k-N)\lambda-k\eta_B\lambda^2, \qquad (5)$$

with

$$J=\int_0^\infty [k^2y^2-(dy/dr)^2+W(\cos(kr)-y)^2]dr,$$
$$N=\int_0^\infty \sin(kr)\cdot[\cos(kr)-y]Wdr,\quad k\eta_B=\int_0^\infty \sin^2kr\cdot Wdr. \qquad (6)$$

Here y contains some adjustable constants $c_1, c_2, \cdots$. To make F_H stationary we set

$$\partial F_H/\partial c_n=0,\quad \partial F_H/\partial\lambda=0,$$

or

$$\partial J/\partial c_n+2\lambda\partial N/\partial c_n=0,\quad (n=1,2,\cdots), \qquad (7)$$
$$k\eta_B\lambda=k-N. \qquad (8)$$

These equations are in complete agreement with those of Hulthén's second method.[5] Furthermore, we have by (5) and (8)

$$F_H=-J+(k-N)\lambda=k\lambda-\Delta,\quad \Delta=J+\lambda N. \qquad (9)$$

Since (9) should give $k\cot\eta$ correctly up to the first order, this also coincides with his result:[6] $\cot\eta=\lambda-k^{-1}\Delta$. Thus Hulthén's second method is equivalent to our (3) with $\theta=0$, and hence it is based on the correct variation principle, although his original derivation is not very simple. Also it will be noted that his method gives $k\cot\eta$ *explicitly* as the stationary value[7] of (3).

Next, consider the functional

$$F_s=F_H+\int_0^\infty (L[u])^2W^{-1}dr. \qquad (10)$$

An application of the variation principle to F_s leads to the same results as those given above, *for the second term on the right-hand side of (10) vanishes for the correct solution, together with its variation.* On substituting (4) into (10) and integrating by parts, we obtain, setting $z=\cos(kr)-y$,

$$F_s=\int_0^\infty (d^2z/dr^2+k^2z)^2W^{-1}dr+\int_0^\infty z(d^2z/dr^2+k^2z)dr. \qquad (11)$$

Setting $d^2z/dr^2+k^2z=-Wv$, and noting that $z\rightarrow\cos(kr)$ as $r\rightarrow\infty$, we obtain from Green's theorem

$$z(r)=\int_0^\infty G(r,r')W(r')v(r')dr', \qquad (12)$$

where $G(r,r')$ is the Green's function used in Schwinger's method.[8] In this way we find

$$F_s=\int_0^\infty Wv^2dr-\int_0^\infty Wvdr\int_0^\infty G(r,r')W(r')v(r')dr'. \qquad (13)$$

It will be noted that

$$k^{-1}\int_0^\infty Wv\sin(kr)dr=1$$

by virtue of the condition $y(0)=1$. If we remove this restriction on v and write (13) in a homogeneous form by the well-known method, F_s becomes just the expression for $k\cot\eta$ in Schwinger's method,[9] which is thus connected with Hulthén's by the simple relation (10). Sometimes it would be convenient to use the former in the form (10) which does not contain a double integral, in contrast to (13). *An interesting consequence of (10) is that the Schwinger method always gives a larger (smaller) value of $k\cot\eta$ than does Hulthén's if $W(r)\geqq 0$ $[W(r)\leqq 0]$ everywhere.* It can even be shown that it gives an upper (lower) bound for $k\cot\eta$ if W is not too strong (more precisely, if $|\eta|<\pi$).[10]

These results can be extended to more general cases (higher angular momentum, inclusion of Coulomb potential).

[1] W. Kohn, Phys. Rev. **74**, 1763 (1948), Eq. (2.14).
[2] S. Huang, Phys. Rev. **76**, 1878 (1949).
[3] L. Hulthén, K. Fysiogr. Sällsk. Lund Föohandl. **14**, No. 21 (1944); Arkiv. Mat. Astr. Fys. **35A**, No. 25 (1948).
[4] J. Schwinger, Phys. Rev. **72**, 742 (1947); **78**, 135 (1950). J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).
[5] See reference 3, second paper, Eqs. (23) and (24).
[6] See reference 5, Eq. (36).
[7] His first method (reference 3, first paper) can also be derived easily from our standpoint. We have only to replace (8) by $\int uL[u]dr=0$; this changes the trial function, u, only by a first-order quantity and hence F is unaffected to the first order. Since F reduces to $k\lambda$ in this case, this is exactly equivalent to the first method of Hulthén.
[8] J. M. Blatt and J. D. Jackson, reference 4, Eq. (2.7).
[9] See reference 8, Eq. (2.11). It will be remarked that our function, v, does not necessarily satisfy the boundary condition $v(0)=0$. But this is not essential, for it follows for the correct solution automatically from the variation principle (natural boundary condition).
[10] The proof will be given elsewhere together with generalizations.

Mass Assignments of Alpha-Active Isotopes in the Rare-Earth Region*

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IN a previous communication from this laboratory, the production of alpha-radioactivity in the rare-earth elements was reported.[1] The suggestion was made that this might be due to the influence of the stable configuration of 82 neutrons on the daughter nuclides and some likely isotopic assignments were proposed on this basis. We have succeeded in testing this suggestion through the use of the mass spectrograph to make an isotopic assignment for one of the major artificial rare-earth alpha-activities with the result that this explanation seems to be confirmed.

The mass assignment of the alpha-emitting terbium isotope of 4.0-hr. half-life and 4.0-Mev alpha-particle energy was made by performing a mass spectrographic separation of terbium activity onto a photographic plate and detecting alpha-activity by a transfer plate technique. The terbium activity (6×10^7 alpha-disintegrations per minute at end of a 5-hr. bombardment) was produced by bombardment of 30 mg of gadolinium oxide with 150-Mev protons in the 184-inch cyclotron, and rapid chemical separation was made by elution from cation exchange columns