The Forces Exerted on Dislocations and the Stress Fields Produced by Them

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It is shown that the force $d\mathbf{F}$ exerted on a line element $\mathbf{v}d\sigma$ of a dislocation with Burgers vector f by a stress τ is given by $d\mathbf{F}=-\mathbf{v}\times(\mathbf{f}\cdot\tau)d\sigma$. An analogy is drawn between the behavior of a closed line dislocation in a stress field and the behavior of a closed current-carrying loop in a field of magnetic induction. Then formulas for the stress components caused at any point of an infinite elastically isotropic crystal by a line element of a general Burgers dislocation are deduced from Burger's expressions for the displacements (see Sec. III(C)). These formulas bear a close analogy to the Biot-Savart formula of electromagnetic theory. Both of these results taken together constitute a complete system for the investigation of the mutual interaction of dislocations in an infinite elastically isotropic crystal.

I. INTRODUCTION

THIS paper aims to answer the following two questions. (1) How does a given dislocation interact with a given stress field? (2) How do two dislocations interact with each other? The first problem is treated with complete generality. The second problem is discussed for the case of an infinite isotropically elastic crystal.

In Section I, we describe the Burgers dislocation. In Section II, we obtain a result which gives the force on an element of a Burgers dislocation in a given external stress field. This force bears a certain analogy to the formula giving the force on a current element in a magnetic field. In Section III expressions are given for the stresses produced by a line element of a Burgers dislocation. These expressions are obtained from previously published formulas[1] giving the displacements produced by an element of a dislocation.

The results given constitute a complete system for the investigation of the behavior of a dislocation in a stress field and, within the limits mentioned, the interaction of dislocations with one another.

Let us create a Burgers dislocation in a crystal (Fig. 1). The crystal is cut along a surface (i.e., the hatched surface in Fig. 1). Next remove a cylinder of material of radius r_0 having as axis the line ABCDA bounding the cut surface. Give one face of the cut surface an arbitrary displacement $\mathbf{f}$ relative to the other face. Rejoin the two faces of the cut in their displaced position. Finally replace the atoms originally lying inside the cylinder of radius r_0. The block now contains a Burgers dislocation lying along ABCDA. If $\mathbf{f}$ is not parallel to the surface of the cut it will be necessary either to remove a thin slice of material on the cut surface or to add it when the translation is accomplished. r_0 is chosen large enough so that outside this cylinder the classical theory of elasticity can be used. If $\mathbf{f}$ is one atomic distance r_0 turns out to be two or three lattice parameters.[2]

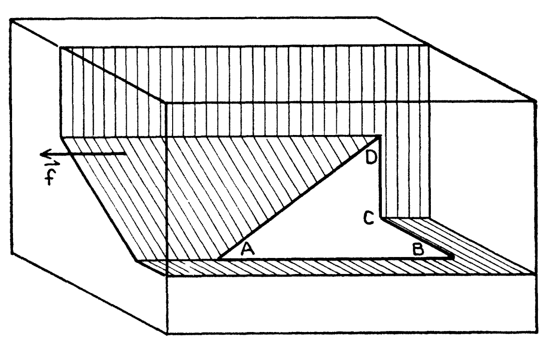

FIG. 1. A Burgers dislocation in a crystal.

The discussion given in the present paper enables one to treat the behavior of that portion of the crystal which behaves elastically. In the case of a Burgers dislocation this excludes only the region inside the cylinder r_0. In the case of two half-dislocations[3] one must exclude a slab of material a few atoms thick lying along that portion of the slip plane which joins the two half-dislocations. In many cases the changes in the arrangement of dislocations and other sources of stress are such that only relatively minor changes in stress occur in the excluded regions. For such situations elastic calculations such as those given in this paper are valuable.

II. FORCE ON AN ELEMENT OF A BURGERS DISLOCATION

(A). Sign Convention

The vector $\mathbf{f}$ is the Burgers vector of the dislocation. The sign of this vector can be determined by adopting the following convention. Figure 2 shows the positive sense of description of the dislocation line, the positive outward normal $\mathbf{n}$ of the dislocation surface, and the positive sense of description of a closed curve linking the dislocation line. If one crosses the cut surface going in the direction of the outward normal $\mathbf{n}$, he goes from the cut surface No. 1 to the cut surface No. 2. We shall

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[1] W. G. Burgers, Proc. Akad. v. Wettensch. (Amsterdam), **42**, 293 (1939).

[2] J. S. Koehler, Phys. Rev. **60**, 397 (1941).

[3] R. D. Heidenreich and W. Shockley, Bristol Conference Report on Strength of Solids (1948).

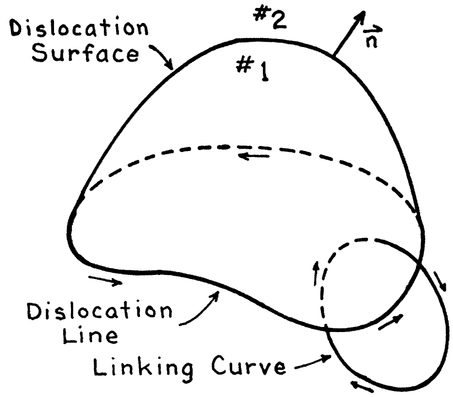

FIG. 2. Sign conventions for general dislocation.

in all cases obtain the dislocation by translating surface No. 1 through the distance **f** while surface No. 2 is held fixed.

(B). Fundamental Principle

If, in constructing the dislocation shown in Fig. 1, we had cut the block along any other surface having ABCDA as its sole internal boundary, the final physical state of the block would have been identical with that produced above. In other words the distribution of displacements and strains in a crystal containing a dislocation depends only on the configuration of the dislocation line and the Burgers vector.

This result is established[4] in Love's treatise. The argument is based on the definition of strain, the compatibility equations, and the assumption of a continuous stress distribution in the dislocated body. It is independent of the elastic properties of the body except that it is assumed that the continuity of stress implies a continuity of the strains. In particular, the result holds for elastically anisotropic crystals as well as isotropic bodies.

(C). Force on a Line Element of Dislocation in a Stress Field

Consider a rigid dislocation Γ having a vector **f**. Let $d\boldsymbol{\sigma}$ be a line element of the dislocation in the direction shown (Fig. 3). The dislocation is immersed in a stress field described by the tensor τ

$$\tau=\begin{pmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \tau_{yy} & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \tau_{zz}\end{pmatrix},$$

where the components τ_{ij} are scalar functions of position.

The dislocation possesses energy W (interaction energy with stress field) by reason of the presence of the stress field τ. The stress field does *not* include the stress

[4] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Cambridge University Press, London, 1927).

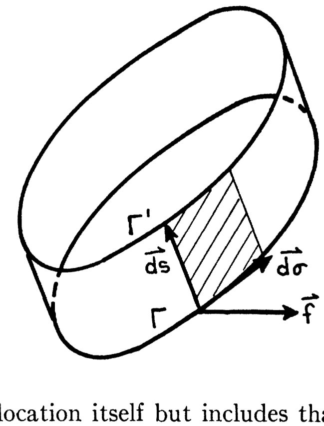

FIG. 3. Infinitesimal displacement of a dislocation.

field caused by the dislocation itself but includes that due to all other sources. We wish to calculate change in energy $\Delta W=W'-W$ caused by giving it an infinitesimal translation $d\mathbf{s}$ in an arbitrary direction, thus bringing it into a new position Γ' where it possesses (interaction) energy W'.

Erect a cylinder on Γ with generators parallel to $d\mathbf{s}$ and extending to the surface of the crystal. We know we can construct the dislocation Γ by cutting the crystal along the surface of this cylinder and displacing the material inside the cylinder by a translation **f**. We know further that the interaction energy W of Γ with the stress field τ can be computed[5] by evaluating the work done on the surface of this cylinder during this displacement by the stress field τ. Similarly, the energy W' is the work done by τ on the surface of the cylinder erected on Γ' during the displacement **f**. It follows that $\Delta W=W'-W$ is the negative of the work done by τ on that portion of the cylindrical surface having Γ and Γ' as bases. Consider an element of this surface, $d\boldsymbol{\sigma}\times d\mathbf{s}$, a vector element (Fig. 3) of area with normal pointing outward from the cylinder. The force on this area[6] is $\tau\cdot(d\boldsymbol{\sigma}\times d\mathbf{s})$. Then the work done is $\mathbf{f}\cdot\{\tau\cdot(d\boldsymbol{\sigma}\times d\mathbf{s})\}$ on this element. The total work $-\Delta W=\int_\sigma \mathbf{f}\cdot\{\tau\cdot(d\boldsymbol{\sigma}\times d\mathbf{s})\}$. The force acting on the dislocation in the direction of $d\mathbf{s}$ is then $-\partial W/\partial s=+\int_\sigma \mathbf{f}\cdot\{\tau\cdot(d\boldsymbol{\sigma}\times\boldsymbol{\lambda})\}$, where $d\boldsymbol{\sigma}=\boldsymbol{\nu}d\sigma$ and $d\mathbf{s}=\boldsymbol{\lambda}ds$, $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ being unit vectors. Then, since τ is symmetric,

$$\mathbf{F}_\lambda=-\int_\sigma \{(\mathbf{f}\cdot\tau)\cdot(\boldsymbol{\lambda}\times\boldsymbol{\nu})\}d\sigma.$$

Since the integration is on $d\sigma$ we may interpret the integrand as being the λ-component of force acting on an element of length $d\sigma$ of the dislocation. This can be written as

$$d\mathbf{F}_\lambda=-\boldsymbol{\lambda}\cdot\{\boldsymbol{\nu}\times(\mathbf{f}\cdot\tau)\}d\sigma.$$

It follows that the resultant force $d\mathbf{F}$, in a stress field τ, on an element $\boldsymbol{\nu}d\sigma$ of a dislocation having a

[5] A. H. Cottrell, *Progress in Metal Physics* (Interscience Publishers, Inc., New York, 1949), p. 85.

[6] For notation see L. Page, *Introduction of Theoretical Physics* (D. Van Nostrand Company, Inc., New York, 1928), p. 34.

vector $\mathbf{f}$ is given by the expression[7]

$$d\mathbf{F}=-\boldsymbol{\nu}\times(\mathbf{f}\cdot\tau)d\sigma.$$

It should be noted that this formula is very general.[8] The stress field may arise from any source whatever. The dislocation may be of the most general Burgers type. The crystal may be finite or infinite. The crystal may be elastically isotropic or elastically anisotropic.

(D). Immediate Deductions

The quantity $\mathbf{f}\cdot\tau$ is simply the vector traction $\boldsymbol{\tau}_f$ across an element of area perpendicular to $\mathbf{f}$ multiplied by $|\mathbf{f}|$. Since $\mathbf{f}$ is a constant vector there is no loss of generality in choosing the x-axis parallel to $\mathbf{f}$. Then $\boldsymbol{\tau}_f=\mathbf{i}\tau_{xx}+\mathbf{j}\tau_{yx}+\mathbf{k}\tau_{zx}$. Moreover $\boldsymbol{\tau}_f$ is a solenoidal vector, for

$$div\boldsymbol{\tau}_f=\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}$$

and this last expression equated to zero is one of the three equilibrium equations of elasticity theory for the case where body forces can be neglected. The formula for the force on a line element of dislocation becomes $d\mathbf{F}=-|\mathbf{f}|\boldsymbol{\nu}\times\boldsymbol{\tau}_f d\sigma$ and this is exactly analogous to the formula $d\mathbf{F}=(i/c)\boldsymbol{\nu}\times\mathbf{B}d\sigma$ from electromagnetic theory. The behavior of a dislocation in the vector field $\boldsymbol{\tau}_f$ is exactly the same as that of a current-carrying wire in the vector field $\mathbf{B}$. In particular, one can introduce the concept of lines of force and arrive at a result that a dislocation loop will tend to move in such a way as to link with the greatest possible number of lines. Whether it actually will move depends upon the atomic constraints, just as whether a current loop will move depends upon the mechanical constraints imposed upon it. The only motion which is easily possible for a dislocation is motion in its slip plane. Any other motion requires the simultaneous breaking of so many atomic bonds as to be virtually impossible at ordinary temperatures, except in so far as diffusion of holes and extra atoms allows such motions to proceed. Greater freedom may be expected at high temperatures. One motion which is possible for dislocation loops which is not possible for current loops is a motion in which the length of the loop increases. The loop length will increase if sufficient energy is supplied to furnish the increased self energy corresponding to the increase in length of the loop and the energy lost (converted into heat) in overcoming the atomic constraints. If the stress system is reversed the field $\boldsymbol{\tau}_f$ will be reversed, and since the dislocation loop is not free to rotate through 180° to align itself with the field but is bound to the slip plane, it will try to reduce its area as much as possible.

[7] The force on a dislocation given by J. S. Koehler, reference 2, differs slightly from that obtained above because he did not use an energetically closed system. This discrepancy will be dealt with in detail in a later paper, together with other discrepancies which exist in the literature.

[8] Shockley and Read have obtained a formula which gives the component in the slip direction of the force on an edge dislocation produced by any externally applied shear stress. T. W. Read and W. Shockley, Phys. Rev. **78**, 275 (1950).

A difference appears between electromagnetic theory and dislocation theory when one considers a set of dislocations in a crystal. There is only one field $\mathbf{B}$ but there are, for example, six different fields $\boldsymbol{\tau}_f$ in a face centered cubic crystal, since there are six different slip directions. The dislocations may be divided into six groups according to their vectors $\mathbf{f}$.

If electrostatic or other non-mechanical forces act on a crystal the fields $\boldsymbol{\tau}_f$ will not be solenoidal; there will be sources of $\boldsymbol{\tau}_f$ within the crystal.

III. STRESS FIELD CAUSED BY A LINE ELEMENT OF DISLOCATION

(A). Displacements Expressed as Line Integrals

In the paper[1] in which Burgers introduced the concept of a Burgers dislocation he also gave explicit expressions for the components of displacement caused by a general dislocation in an infinite, elastically isotropic crystal. Each expression contained three terms, of which two were given as line integrals over the dislocation curve, and the remaining one as a surface integral over an arbitrary surface bounded by the dislocation curve. He pointed out that the surface integral could be replaced by a line integral but did not express it as such. The problem of replacing the surface integral by a line integral does not have a unique solution, but one can choose a symmetric solution which enables the x-component of displacement to be written as follows;[9]

$$\begin{aligned}u_1=\frac{1}{4\pi}\int_\sigma\Bigg\{&\frac{d\xi_1}{d\sigma}\left[f_1\frac{YZ}{3R}\left(\frac{1}{Z^2+X^2}-\frac{1}{X^2+Y^2}\right)\right.\\&\left.+X\frac{K}{R^3}(-f_2Z+f_3Y)\right]\\&+\frac{d\xi_2}{d\sigma}\left[f_1\frac{ZX}{3R}\left(\frac{1}{X^2+Y^2}-\frac{1}{Y^2+Z^2}\right)\right.\\&\left.+X\frac{K}{R^3}(-f_3X+f_1Z)-\frac{(1-K)}{R}f_3\right]\\&+\frac{d\xi_3}{d\sigma}\left[f_1\frac{XY}{3R}\left(\frac{1}{Y^2+Z^2}-\frac{1}{Z^2+X^2}\right)\right.\\&\left.+X\frac{K}{R^3}(-f_1Y+f_2X)+\frac{(1-K)}{R}f_2\right]\Bigg\}d\sigma.\end{aligned}$$

From this the expressions for u_2 and u_3 can be obtained by simultaneous cyclic permutation of the three sets of quantities (ξ_1,ξ_2,ξ_3), (X,Y,Z), and (f_1,f_2,f_3). The quantities appearing in this equation are defined as follows:

$$X=x-\xi_1,\quad Y=y-\xi_2,\quad Z=z-\xi_3,$$
$$R=(X^2+Y^2+Z^2)^{\frac{1}{2}},$$

[9] This formula can be obtained from Eq. (21) and (5) of reference 1 by writing (5) as a line integral. The connection between the line and surface integral is given by Stoke's theorem. Thereafter it is necessary to find a vector potential for the field $\mathbf{r}/r^3$.

x, y, z are coordinates of the field point ξ_1, ξ_2, ξ_3 are coordinates of the source point, i.e., a point on the dislocation line.

f_1, f_2, f_3 are the components of the Burgers vector

$$K=\frac{\lambda+\mu}{\lambda+2\mu},\quad 1-K=\frac{\mu}{\lambda+2\mu}.$$

λ and μ are the Lamé constants of isotropic elasticity theory. (Figure 4.)

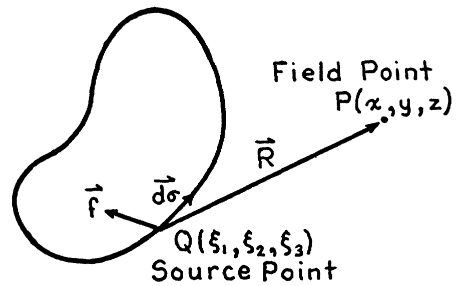

FIG. 4. Sketch showing the quantities which appear in the formulas for displacement, strain, and stress caused by a line element of dislocation.

(B). Strain Components Expressed as Line Integrals

The strain components are defined (Love's definitions)

$$e_{xx}=\partial u_1/\partial x,\ \text{etc.}\quad e_{yz}=\partial u_2/\partial z+\partial u_3/\partial y,\ \text{etc.}$$

An elementary though somewhat tedious calculation then gives the following line integral expressions:

$$\begin{aligned}
e_{xx}=\frac{1}{4\pi}\int_\sigma\Big\{&\frac{d\xi_1}{d\sigma}\Big[Kf_2\Big(\frac{3ZX^2}{R^5}-\frac{Z}{R^3}\Big)-Kf_3\Big(\frac{3YX^2}{R^5}-\frac{Y}{R^3}\Big)\Big]\\
&+\frac{d\xi_2}{d\sigma}\Big[f_3\Big(\{1-3K\}\frac{X}{R^3}+3K\frac{X^3}{R^5}\Big)\\
&\qquad-f_1\Big(\{1-K\}\frac{Z}{R^3}+3K\frac{ZX^2}{R^5}\Big)\Big]\\
&+\frac{d\xi_3}{d\sigma}\Big[f_1\Big(\{1-K\}\frac{Y}{R^3}+3K\frac{YX^2}{R^5}\Big)\\
&\qquad-f_2\Big(\{1-3K\}\frac{X}{R^3}+3K\frac{X^3}{R^5}\Big)\Big]\Big\}d\sigma,
\end{aligned}$$

$$\begin{aligned}
e_{yz}=\frac{1}{4\pi}\int_\sigma\Big\{&\frac{d\xi_1}{d\sigma}\Big[Kf_2\Big(\frac{6YZ^2}{R^5}-\frac{2Y}{R^3}\Big)-Kf_3\Big(\frac{6ZY^2}{R^5}-\frac{2Z}{R^3}\Big)\Big]\\
&+\frac{d\xi_2}{d\sigma}\Big[f_2\Big(\frac{X}{R^3}\Big)+f_3K\Big(\frac{6XYZ}{R^5}\Big)\\
&\qquad-f_1\Big(\{1-2K\}\frac{Y}{R^3}+6K\frac{YZ^2}{R^5}\Big)\Big]\\
&+\frac{d\xi_3}{d\sigma}\Big[f_3\Big(-\frac{X}{R^3}\Big)+f_1\Big(\{1-2K\}\frac{Z}{R^3}+6K\frac{ZY^2}{R^5}\Big)\\
&\qquad-f_3K\Big(\frac{6XYZ}{R^5}\Big)\Big]\Big\}d\sigma.
\end{aligned}$$

The other normal strain components can be obtained from e_{xx} by simultaneous cyclic permutation of variables and the other shear strain components can similarly be obtained from e_{yz}.

(C). Stress Components

The stress components are related to the strain components by

$$\tau_{xx}=\lambda\Delta+2\mu e_{xx},\ \text{etc.}\quad \tau_{yz}=\mu e_{yz},\ \text{etc.},$$

where $\Delta=e_{xx}+e_{yy}+e_{zz}$.

Line integral expressions for the stresses are

$$\begin{aligned}
\tau_{xx}=\frac{1}{4\pi}\int_\sigma\Big\{&\frac{d\xi_1}{d\sigma}\Big[f_2\Big(3L\frac{ZX^2}{R^5}-M\frac{Z}{R^3}\Big)\\
&\qquad-f_3\Big(3L\frac{YX^2}{R^5}-M\frac{Y}{R^3}\Big)\Big]\\
&+\frac{d\xi_2}{d\sigma}\Big[f_3\Big(3L\frac{X^3}{R^5}-L\frac{X}{R^3}\Big)-f_1\Big(3L\frac{ZX^2}{R^5}+L\frac{Z}{R^3}\Big)\Big]\\
&+\frac{d\xi_3}{d\sigma}\Big[f_1\Big(3L\frac{YX^2}{R^5}+L\frac{Y}{R^3}\Big)\\
&\qquad-f_2\Big(3L\frac{X^3}{R^5}-L\frac{X}{R^3}\Big)\Big]\Big\}d\sigma,
\end{aligned}$$

$$\begin{aligned}
\tau_{yz}=\frac{1}{4\pi}\int_\sigma\Big\{&\frac{d\xi_1}{d\sigma}\Big[f_2\Big(3L\frac{YZ^2}{R^5}-L\frac{Y}{R^3}\Big)\\
&\qquad-f_3\Big(3L\frac{Y^2Z}{R^5}-L\frac{Z}{R^3}\Big)\Big]+\frac{d\xi_2}{d\sigma}\Big[f_2\Big(\mu\frac{X}{R^3}\Big)\\
&\qquad+f_3\Big(3L\frac{XYZ}{R^5}\Big)-f_1\Big(3L\frac{YZ^2}{R^3}-N\frac{Y}{R^3}\Big)\Big]\\
&+\frac{d\xi_3}{d\sigma}\Big[f_3\Big(-\mu\frac{X}{R^3}\Big)+f_1\Big(3L\frac{ZY^2}{R^5}-N\frac{Z}{R^3}\Big)\\
&\qquad-f_2\Big(3L\frac{XYZ}{R^5}\Big)\Big]\Big\}d\sigma,
\end{aligned}$$

where

$$L=\frac{2\lambda\mu+2\mu^2}{\lambda+2\mu},\quad M=\frac{2\mu^2}{\lambda+2\mu},\quad N=\frac{\lambda\mu}{\lambda+2\mu}.$$

The above integrations are on $d\sigma$. Hence the integrands can be interpreted as the stress produced at a point (x, y, z) by a line element $d\sigma$ of dislocation located at (ξ_1, ξ_2, ξ_3). It is clear that we have reached a result which, for an infinite elastically isotropic crystal, performs the same function as the formula

$$d\mathbf{B}=(\mu i/c)d\boldsymbol{\sigma}\times\mathbf{r}/r^3$$

from electromagnetic theory. This result, together with the result of Section III, provide a complete system for investigating the mutual interaction of dislocations.