Note on the Li^7 Quadrupole Moment*

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Recent measurements unexpectedly determine the sign of the quadrupole moment as positive, although the magnitude is uncertain. On the quasi-atomic model the ground state is $1s^42p^3\,{}^2P_{\frac{1}{2}}$ and of symmetry [3] in Wigner's "first approximation." The corresponding quadrupole moment is -6 in units of $\langle r^2\rangle_{2p}/25$. The explanation of the discrepancy is sought in configuration interaction. A variational calculation of the moment has been made taking into account the P states of symmetry [3] from the low configurations $1s^42p^23p$ and $1s^42p^24f$. The maximum value of the moment is $+2.32$ in the above units; this value is increased by including other configurations. The admixture of [3] P states from excited configurations has no effect on the magnetic moment nor on the matrix element for Be^7 K-capture into the ground state of Li^7; this is satisfactory since the quasi-atomic model agrees reasonably well with experiment in these instances. When the [3] D states from the above configurations are included in the variational calculation, the maximum is raised to 8.03. The configuration interaction need not be large in this case to produce a positive moment; however the inclusion of the [3] D states does impair the agreement with the magnetic moment and the K-capture data. Considering the wide margin of uncertainty in the estimated experimental quadrupole moment, we find that all the data are adequately explained without assuming large spin-orbit interaction or large departures from partition symmetry.

I. INTRODUCTION

RECENT measurements[1] of the nuclear quadrupole interaction with the molecular fields in Li_2 and the lithium halides indicate a positive quadrupole moment for the Li^7 nucleus. The estimated magnitude (2×10^{-26} cm^2) is uncertain, however, because of lack of resolution in the Li_2 experimental line and the uncertain value of the electric field gradient. The quasi-atomic nuclear model predicts a negative quadrupole moment for the ground state ($1s^42p^3\,{}^2P_{\frac{1}{2}}$) of Li^7. A value of -2.7×10^{-26} cm^2 was calculated by Welles[2] using a wave function determined from energy calculations on the quasi-atomic central force model[3] together with a reasonable estimate of $\langle r^2\rangle_{2p}$ (the mean square radius of a $2p$ nucleon in Li^7). Nearly the same wave function and quadrupole moment are obtained in the first approximation of Wigner's theory,[4] which is based on a central-force symmetric Hamiltonian with spin-exchange terms and Coulomb forces neglected. In this approximation the orbital wave function is characterized by a definite symmetry with respect to permutations of the nucleons. The orbital wave function belongs to an irreducible representation of the symmetric group and is labeled by a partition. It is then possible to determine the coefficients in the wave function from the quantum numbers and symmetry considerations, without having recourse to the usual perturbation theory procedure which entails the calculation of energy matrix elements and the solution of secular equations.[5] Assume first that Wigner's approximation is valid and that the partitions make good quantum numbers. The completed shell of $1s$ particles has no effect on the symmetry, makes no contribution to the quadrupole moment, and is therefore omitted. The partitions for Li^7 are then [3], [2+1] and [1+1+1] in the order of decreasing symmetry of the orbital wave function and increasing energy of the nuclear state. The ground state orbital wave function, belonging to the partition [3], is symmetric in the coordinates of the three particles outside the s-shell. Assuming the ground state to be $1s^42p^3\,{}^2P_{\frac{1}{2}}$, the wave function is simply determined (see Table I) and the quadrupole moment found to be -6 in units[6] of $\langle r^2\rangle_{2p}/25$.

The discrepancy with the result inferred from Kusch's experiments may be resolved in several possible ways: (a) by configuration interaction (admixture of wave functions for higher configurations), (b) by spin-orbit interaction (admixture of D states), (c) by the breakdown of partition symmetry (admixture of [2+1] and [1+1+1]). These possibilities, of course, are not independent; furthermore, combinations of (a), (b), and (c) are possible. We denote, e.g., by "Case (bc)," the case in which configuration interaction is absent, but in which $L-S$ coupling and partition symmetry both break down.

Case (a): Partition symmetry and $L-S$ coupling are both preserved. Configuration interaction introduces an admixture of ^2P functions of symmetry [3] from low odd-parity configurations. A positive quadrupole moment may arise from the non-diagonal matrix elements of the quadrupole moment operator. This case is considered below in detail.

Case (b): It is impossible for D states to be admixed without either configuration interaction or a breakdown

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[1] P. Kusch, Phys. Rev. **76**, 138 (1949).

[2] S. Welles, Phys. Rev. **62**, 197 (1942).

[3] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937); M. Rose and H. Bethe, Phys. Rev. **51**, 205 (1937).

[4] E. Wigner, Phys. Rev. **51**, 106, 947 (1937); F. Hund, Zeits. f. Physik **105**, 202 (1937).

[5] E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937).

[6] This unit, which is used in the following, is roughly estimated to be 0.3×10^{-26} cm^2.

TABLE I. Symmetric wave functions.*

$$\Phi_{2p^3}(P^i)=(2/3)\psi_{P^i}(P_\pi D_\nu)+(\sqrt{5}/3)\psi_{P^i}(P_\pi S_\nu)$$

$$\Phi_{2p^24f}(P^i)=(1/\sqrt{3})\psi_{P^i}(F_\pi D_\nu)+\sqrt{2}/\sqrt{3}\psi_{P^i}(P_\pi \mathfrak{D}_\nu)$$

$$\Phi_{2p^23p}(P^i)=(2\sqrt{5}/3\sqrt{7})\psi_{P^i}(P_\pi{}^*D_\nu)+(2\sqrt{6}/3\sqrt{7})\psi_{P^i}(P_\pi \mathbf{P}_\nu)$$
$$-(1/3\sqrt{7})\psi_{P^i}(P_\pi{}^*S_\nu)+\sqrt{2}/\sqrt{7}\psi_{P^i}(P_\pi \mathbf{S}_\nu)$$

$$\Phi^{\times}{}_{2p^23p}(P^i)=(1/3\sqrt{7})\psi_{P^i}(P_\pi{}^*D_\nu)+(\sqrt{7}/3\sqrt{2})\psi_{P^i}(P_\pi \mathbf{D}_\nu)$$
$$-\sqrt{5}/\sqrt{42}\psi_{P^i}(P_\pi \mathbf{P}_\nu)+(2\sqrt{5}/3\sqrt{7})\psi_{P^i}(P_\pi{}^*S_\nu)$$
$$+(\sqrt{10}/3\sqrt{7})\psi_{P^i}(P_\pi \mathbf{S}_\nu)$$

$$\mathbf{\Phi}(D)=-(1/\sqrt{5})\Phi(D^1)\cdot{}^2\chi_{\frac{1}{2}}+(2/\sqrt{5})\Phi(D^2)\cdot{}^2\chi_{-\frac{1}{2}}$$

$$\Phi_{2p^24f}(D^i)=(1/\sqrt{3})\psi_{D^i}(F_\pi D_\nu)+(2/3)\psi_{D^i}(P_\pi \mathfrak{D}_\nu)$$
$$+(\sqrt{2}/3)\psi_{D^i}(P_\pi \mathfrak{H}_\nu)$$

$$\Phi_{2p^23p}(D^i)=(1/\sqrt{3})\psi_{D^i}(P_\pi{}^*D_\nu)-(1/\sqrt{6})\psi_{D^i}(P_\pi \mathbf{D}_\nu)$$
$$+(1/\sqrt{2})\psi_{D^i}(P_\pi \mathbf{P}_\nu)$$

* See footnote 13.

of partition symmetry, since the $2p^3$ configuration gives rise to no D state of symmetry [3].

Case (c): A large departure from partition symmetry is unlikely without a large spin-orbit interaction. The spin-exchange and Coulomb forces do not cause large departures from symmetry [3] in Li^7. This case has been investigated by Avery and Blanchard.[7] Taking into account the three P states of symmetry [2+1], they are unable to fit the quadrupole moment and other ground state properties of Li^7.

Case (ab): Both ^2P and ^2D functions of symmetry [3] from low odd-parity configurations are mixed in. The admixture of D states implies spin-orbit interaction and this breaks down the partition symmetry. Thus it is somewhat artificial to take into account the above D states and to omit the [2+1]D states from $2p^3$. However, one may reasonably inquire to what extent the properties of light nuclei can be accounted for without assuming a large departure from partition symmetry. This case is also considered below in detail.

Case (bc): This case has been investigated completely by Avery and Blanchard.[7] Using the eight independent functions of the $2p^3$ configuration, they find it possible to secure reasonable agreement with the quadrupole moment and magnetic moment of Li^7 and with the Be^7 K-capture data. A fairly unique wave function is determined in this way; it appears impossible to fit the data with any very different choice of coefficients. In order to obtain a positive quadrupole moment and retain good agreement with the other data, the wave function is required to be predominantly of symmetry [2+1]. Their final wave function,[8] which corresponds to a quadrupole moment of 3.5 in the above units, is 6 percent [3] and 94 percent [2+1].

Case (ac): In this case there is an admixture of P states of all partition symmetries from all low odd-parity configurations. This is somewhat artificial for the reason given under Case (c), and has not been considered.

Case (abc): The discrepancy is here attributed to a combination of all three explanations. This last alternative, while probably the most nearly correct, is so general and provides so many adjustable constants that it can be reasonably expected to explain all the experimental results without, however, providing much insight into the reliability of the various nuclear approximations.

II. CONFIGURATION INTERACTION

In the following it is assumed that all terms in the wave function are of symmetry [3]. Configurations that interact with $1s^42p^3$ must be of odd-parity; the lowest of these are $1s^42p^23p$, $1s^42p^24f$, $1s^32s2p^3$, and $1s^32p^33d$. If the individual nucleons are assumed to move independently in zero-order in a suitable "auxiliary potential," the zero-order energy is the sum of the eigenvalues for the individual particle states. In the case of the isotropic space oscillator potential, the four excited configurations listed above are lowest in energy and have all the same zero-order energy. Since the approximation of the auxiliary potential (Hartree) is not very suitable for the type of force found in nuclei, the description in terms of single configurations is correspondingly inaccurate and an appreciable interaction between neighboring configurations of the same parity is expected. For simplicity we consider only the $2p^23p$ and $2p^24f$ configurations. Quartet states are excluded by the symmetry [3] requirement; since the total nuclear angular momentum J is $\frac{3}{2}$, this excludes S and F states. The enumeration of the states of symmetry [3] arising from each configuration is elementary. For example, in the case of $2p^23p$ there is only one way to write a symmetric function corresponding to an orbital angular momentum component $M_L=\sum m_l=3$, hence one F state; there are two linearly independent symmetric functions with $M_L=2$, hence one D state; and, since there are four such functions with $M_L=1$, this configuration gives two P states. The number of linearly independent symmetric functions for each value of $\sum m_l$ is just the number of different ways in which values of m_l can be assigned to the individual nucleons. As noted in the previous section, the configuration $2p^3$ gives rise to no D states of symmetry [3] and to but one [3] P state. In the case of $2p^24f$ there is one P state and one D state of this

TABLE II. Matrix elements of Q in units of $\langle r^2\rangle_{2p}/25$.

$2p^3$ P	$2p^24f$ P	$2p^23p$ P	$2p^23p$ $P^\times$	$2p^24f$ D	$2p^23p$ D
-6	$2\sqrt{14}/\sqrt{5}$	$2\sqrt{2}/3\sqrt{7}$	$-34\sqrt{2}/3\sqrt{35}$	$4\sqrt{7}/\sqrt{15}$	$2\sqrt{2}/\sqrt{15}$
	$-46/15$	$8/\sqrt{5}$	$4/5$	$28\sqrt{6}/15$	$4\sqrt{21}/15$
		$-38/21$	$-32\sqrt{5}/35$	$8\sqrt{2}/\sqrt{15}$	$16\sqrt{3}/\sqrt{35}$
			$-748/105$	$4\sqrt{2}/5\sqrt{3}$	$-46/5\sqrt{21}$
				$22/15$	$-4\sqrt{14}/5$
					$28/15$

[7] R. Avery and C. Blanchard, Phys. Rev. **77**, 756A (1950); **78**, 704 (1950). A detailed article describing this work was kindly sent to the writer by R. G. Sachs.

[8] This function does not conform to any nuclear model hitherto proposed. The breakdown of partition symmetry through large axial dipole interaction should lead to comparable proportions of [3] and [2+1]. The above function preserves the partition symmetry, but the symmetry is not that appropriate to the Li^7 ground state.

symmetry. Taking into account all the P and D states arising from $2p^3$, $2p^23p$, and $2p^24f$, there are six symmetric wave functions to be calculated. Details of the calculation are to be found in Appendix I and the results are given in Table I.

The notation in Table I is conventional or self-explanatory except in the following respects: P_π is a $2p$ function and P_π^* a $3p$ function for the proton; D_ν represents a D state for equivalent $2p$ neutrons, $\mathbf{D}_\nu$ a non-equivalent $2p3p$ pair and $\mathfrak{D}_\nu$ a $2p4f$ pair. The two linearly independent P state functions for $2p^23p$ are denoted by Φ and $\Phi^\times$. In calculating the matrix elements of the quadrupole moment operator, the functions ψ are expanded in terms of product functions for the protons and neutrons each with its own magnetic quantum number, e.g.,

$$\psi_{P^1}(P_\pi D_\nu)=(1/\sqrt{10})\{P_\pi^1 D_\nu^0-\sqrt{3}P_\pi^0 D_\nu^1+\sqrt{6}P_\pi^{-1}D_\nu^2\}.$$

The functions in Table I are orthogonal and normalized.

III. THE QUADRUPOLE MOMENT

The quadrupole moment, q, is defined as the expectation value of $Q=\sum_\pi(3z^2-r^2)$, summed over the protons, for the ground state eigenfunction with $M=J$. The $1s$ shell protons make no contribution to q and the sum reduces to one term.[9] The matrix elements of Q are to be calculated with respect to the set of normalized, symmetrized, orthogonal wave functions given in Table I. The radial integrals which occur in the matrix elements require for their evaluation some explicit assumption about the $2p$, $3p$, and $4f$ radial eigenfunctions. Assuming isotropic oscillator functions, these integrals have been expressed in terms of $\langle r^2\rangle_{2p}$. Some details of the evaluation of the matrix elements are given in Appendix II; the results are in Table II.

We seek the greatest possible values of q under the assumptions of Case (a) and of Case (ab) of Section I. The maximum of q is given by the largest root of the secular equation: $|Q_{ij}-\lambda\delta_{ij}|=0$. In Case (a) only the P states are taken into account and the maximum value of the moment is 2.32 in units of $\langle r^2\rangle_{2p}/25$. The coefficients of the normalized wave function for this value of q are 0.344, 0.630, 0.670 and -0.190 in the order: $2p^3\,P$, $2p^24f\,P$, $2p^23p\,P$, and $2p^23p\,P^\times$. The maximum quadrupole moment, while positive, is not large and since the corresponding eigenfunction is only 12 percent $2p^3$, very large configuration interaction is indicated. However the P states from other excited configurations have not been taken into account; their inclusion in the variational calculation can only increase the maximum above 2.32. We conclude that a positive quadrupole moment is consistent with a pure P ground state of symmetry [3], provided that there is large configuration interaction.

The inclusion of the D states helps materially to increase the moment. In contrast to the P states, the diagonal matrix elements for the D states are positive, and the two D functions together give a maximum q of 4.67. If all six functions of the set given in Table I are taken in combination, the maximum moment is raised to 8.03. The corresponding coefficients of the normalized wave function in the order of Tables I and II are: 0.276, 0.502, 0.523, -0.111, 0.542, and 0.304. This function is 38.6 percent D state and only 7.6 percent $2p^3$. It is interesting to note that a fair-sized positive moment can be obtained from a combination of the functions of Table I which is predominantly $2p^3$. Thus, a function which is 60 percent $2p^3$ and 10 percent each of $2p^24f\,P$, $2p^23p\,P$, $2p^24f\,D$, and $2p^23p\,D$, if the phases are all the same, yields a quadrupole moment of 2.55. We conclude that Case (ab) of the Introduction provides a possible explanation of Kusch's results.

IV. CONCLUSIONS

As a further test of the adequacy of the wave functions discussed in the preceding section, one can use them to calculate the magnetic moment of Li^7 and the matrix element for Be^7 K-electron capture leading to the ground state of Li^7. The symmetry [3] of the space wave functions makes the magnetic moment calculation very simple. The admixture of P states from excited configurations has no effect on the magnetic moment, essentially because the magnetic moment is given by the vector model and depends only on the quantum numbers. More explicitly, the non-diagonal terms in the spin part of the moment vanish through orthogonality of the space functions and the diagonal terms are all the same because all states have the same quantum numbers. The orbital part of the moment is unaffected by P state admixture since the symmetry of the wave function permits l_z for the proton to be replaced by $L_z/3$ and M_L is a good quantum number. The magnetic moment in Case (a) remains at the value predicted by the quasi-atomic model: $\mu_P+(1/3)=3.123$ nuclear magnetons, in reasonable agreement with the experimental value[10] of 3.253. Admixture of symmetry [3] D states can only decrease the magnetic moment. All cross terms vanish and the moment is the weighted sum of contributions from P^1, D^1, and D^2. The result is $3.123-4.197p_D$ where p_D denotes the percentage of D state. Thus the magnetic moment favors Case (a) over Case (ab).

Avery and Blanchard[7] have pointed out that an additional condition on the Li^7 wave function is provided by the experimental results[11] on the lifetime and branching ratio for Be^7 K-capture. The absolute square

[9] This is proved immediately by observing that, (a) Q is a one-particle operator, (b) the one-particle integrals vanish for the $1s$ protons because the angular dependence of the operator is given by $P_2(\cos\theta)$.

[10] Rabi, Millman, Kusch, and Zacharias, Phys. Rev. **55**, 526 (1939).

[11] E. Segrè and C. Wiegand, Phys. Rev. **75**, 39 (1949); R. Williamson and H. Richards, Phys. Rev. **76**, 614 (1949).

of the matrix element of the Gamow-Teller interaction operator $\sum_i \tau_1^{(i)}\sigma^{(i)}$ has been inferred[7] from experiment to have the value 1.76. The theoretical value[12] is 5/3 for a P state of symmetry [3]. This value is independent of the detailed properties of the wave function, just as in the case of the magnetic moment, and is unaffected by P state admixture through configuration interaction. The agreement is very good and provides additional evidence in favor of Case (a). Admixture of [3] D states can only decrease the matrix element and impair the agreement, since the absolute square of the matrix element is 3/5 for a [3] D state.

Considering the wide margin of uncertainty in Kusch's estimated value for the quadrupole moment, we may conclude that Case (a) is reasonably consistent with all the data. While Case (ab) provides the possibility of explaining a larger quadrupole moment than can be obtained from Case (a), this can only be secured by sacrificing the good agreement with the other data. It appears from the calculations that the experimental results to date can be understood without assuming large spin-orbit interaction and, especially, without invoking any large departure from partition symmetry.

This problem was suggested by E. Feenberg of Washington University. I wish to record my appreciation for this suggestion and to acknowledge some useful preliminary discussion.

APPENDIX I. CALCULATION OF WAVE FUNCTIONS

The functions needed for the calculation of the quadrupole moment are those with $M=J=\frac{3}{2}$. Hence the P states have $M_L=1$ and $M_S=\frac{1}{2}$, while the D state functions are given by the linear combination: $-5^{-\frac{1}{2}}\Phi(D^1)\cdot\chi_{\frac{1}{2}}+(4/5)^{\frac{1}{2}}\Phi(D^2)\cdot\chi_{-\frac{1}{2}}$ where the superscript is M_L and the subscript M_S. Since we are neglecting spin-spin forces between unlike particles, the neutrons have their own multiplicity and are in a singlet state. The orbital functions Φ for each state are linear combinations of functions ψ with the same total L and M_L and corresponding to a proton orbital angular momentum L_π and a neutron angular momentum L_ν, the coefficients being determined by the symmetry requirement. The ψ-functions are expansible in terms of product functions which assign separate magnetic quantum numbers to protons and neutrons; the coefficients are given by the well-known transformation formulas for angular momentum eigenfunctions.[13] The final step in the decomposition is to express the separate proton and neutron functions in terms of one-particle functions with definite values of n, l, and m_l. The neutron functions for non-equivalent particles require symmetrization. After the decomposition is effected and each ψ expressed in terms of one-particle functions, the coefficients of the ψ's given in Table I are obtained by requiring that Φ be completely symmetrical in all particles. This can be done in the simpler cases by inspection; in others, e.g., the $2p^23p$ configuration, some algebra is required.

The procedure is illustrated in the case of $\Phi_{2p^24f}(P^1)$. The component functions can only be $\psi_{P^1}(F_\pi D_\nu)$ and $\psi_{P^1}(P_\pi \mathfrak{D}_\nu)$, the former corresponding to $2p$ neutrons and the latter to a $2p4f$ pair. The transformation formulas give

$$\psi_{P^1}(F_\pi D_\nu)=35^{-\frac{1}{2}}\{15^{\frac{1}{2}}F_\pi{}^3D_\nu{}^{-2}-10^{\frac{1}{2}}F_\pi{}^2D_\nu{}^{-1}+6^{\frac{1}{2}}F_\pi{}^1D_\nu{}^0-3^{\frac{1}{2}}F_\pi{}^0D_\nu{}^1+F_\pi{}^{-1}D_\nu{}^2\}$$

$$\psi_{P^1}(P_\pi \mathfrak{D}_\nu)=10^{-\frac{1}{2}}\{P_\pi{}^1\mathfrak{D}_\nu{}^0-3^{\frac{1}{2}}P_\pi{}^0\mathfrak{D}_\nu{}^1+6^{\frac{1}{2}}P_\pi{}^{-1}\mathfrak{D}_\nu{}^2\}.$$

The normalized one-particle functions are denoted by

$$(2p^1)\equiv -a,\quad (2p^0)\equiv b,\quad (2p^{-1})\equiv c$$
$$(4f^3)\equiv -d,\quad (4f^2)\equiv e,\quad (4f^1)\equiv -f,\quad (4f^0)\equiv g,\quad (4f^{-1})\equiv h,$$

where the phases have been chosen to agree with the Condon-Shortley convention. The normalized neutron functions are given by

$$D^2=aa,\quad D^1=-2^{-\frac{1}{2}}(ab+ba),\quad D^0=6^{-\frac{1}{2}}(2bb-ac-ca),$$
$$D^{-1}=2^{-\frac{1}{2}}(cb+bc),\quad D^{-2}=cc$$
$$\mathfrak{D}^2=42^{-\frac{1}{2}}\{-15^{\frac{1}{2}}(dc+cd)-5^{\frac{1}{2}}(eb+be)+(fa+af)\}$$
$$\mathfrak{D}^1=42^{-\frac{1}{2}}\{10^{\frac{1}{2}}(ec+ce)+8^{\frac{1}{2}}(fb+bf)-3^{\frac{1}{2}}(ga+ag)\}$$
$$\mathfrak{D}^0=42^{-\frac{1}{2}}\{-6^{\frac{1}{2}}(fc+cf)-3(gb+bg)-6^{\frac{1}{2}}(ha+ah)\}.$$

Hence

$$\psi_{P^1}(F_\pi D_\nu)=70^{-\frac{1}{2}}\{-30^{\frac{1}{2}}dcc-10^{\frac{1}{2}}(ebc+ecb)-8^{\frac{1}{2}}fbb+2^{\frac{1}{2}}(fac+fca)+3^{\frac{1}{2}}(gab+gba)+2^{\frac{1}{2}}haa\}$$

$$\psi_{P^1}(P_\pi \mathfrak{D}_\nu)=140^{-\frac{1}{2}}\{-30^{\frac{1}{2}}(cdc+ccd)-10^{\frac{1}{2}}(bec+bce+ceb+cbe)+2^{\frac{1}{2}}(afc+acf+cfa+caf)-8^{\frac{1}{2}}(bfb+bbf)+3^{\frac{1}{2}}(agb+abg+bga+bag)+2^{\frac{1}{2}}(aha+aah)\}$$

and inspection suffices to give the coefficients of Table I. In the case of $2p^23pP^1$ there are two independent solutions to the equations for symmetrization; these give the orthogonal functions $\Phi_{2p^23p}(P^1)$ and $\Phi^{\times}{}_{2p^23p}(P^1)$ of Table I.

APPENDIX II. CALCULATION OF MATRIX ELEMENTS OF Q

The matrix elements of the one-particle (proton) operator $Q=3z_1{}^2-r_1{}^2$ with respect to the set of functions given in Table I are reduced immediately to integrals of the form:

$$\int\int\int Q\bar{\psi}_a\psi_b d\tau_1 d\tau_2 d\tau_3$$

where ψ_a and ψ_b are the functions of Table I. All integrals of this type, in which the neutron state specified in ψ_a does not exactly match the neutron state specified in ψ_b, vanish by orthogonality. Thus all cross terms in the diagonal matrix elements must vanish. In the next step the ψ-functions are expanded in terms of the product functions with separate quantum numbers for neutrons and protons. All integrals vanish in which the magnetic quantum numbers of the two neutron functions do not match; in the remaining integrals the integration over the neutron coordinates gives unity since the functions are normalized. Thus all integrals have been reduced to the form $\int Q\bar{\phi}_a\phi_b d\tau_1$ where ϕ_a and ϕ_b are one particle functions for the proton with the same magnetic quantum numbers. The one-particle functions of Appendix I contain the normalized associated Legendre functions $\Theta_{l,|m|}$ with the Condon-Shortley phases. Since $Q=2r_1{}^2P_2(\mu_1)$ the preceding integral reduces to the product of an angular integral: $I_{l,|m|;l',|m|}=\int_{-1}^{1}d\mu P_2(\mu)\,\Theta_{l,|m|}(\mu)\,\Theta_{l',|m|}(\mu)$ and a radial integral: $(r^2)_{nl,n'l'}$. The eigenfunctions for the isotropic oscillator have been used to evaluate the radial integrals and express them in terms of $(r^2)_{2p,2p}\equiv\langle r^2\rangle_{2p}$. The number of integrals to be evaluated is greatly reduced by observing that, e.g.,

$$\int Q|\psi_{P^1}(P_\pi D_\nu)|^2=\int Q|\psi_{P^1}(P_\pi \mathbf{D}_\nu)|^2=\int Q|\psi_{P^1}(P_\pi \mathfrak{D}_\nu)|^2$$

and that

$$\int Q\bar{\psi}_{P^1}(P_\pi D_\nu)\psi_{P^1}(P_\pi{}^* D_\nu)$$

is obtained from

$$\int Q|\psi_{P^1}(P_\pi D_\nu)|^2$$

by replacing $(r^2)_{2p,2p}$ by $(r^2)_{2p,3p}$.

[12] E. Wigner, Phys. Rev. **56**, 519 (1939).

[13] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, London, 1935), p. 76. The phase conventions 14^{37} and 4^{58} on pp. 78 and 123 have not been used; instead j_2 is taken always to be the lesser angular momentum (when $j_1=j_2$ we use $j_1=L_\pi$, $j_2=L_\nu$ and j_1 for $2p$, j_2 for $3p$ neutrons). The phases of the functions in Table I depend on this convention; the results, of course, do not.

We illustrate by calculating the non-diagonal matrix element Q_{ij} between $2p^24fP$ and $2p^23pD$. After summing over spins this is equal to: $-(1/\sqrt{5})\int Q\bar{\Phi}_{2p^24f}(P^1)\Phi_{2p^23p}(D^1)$. On inserting the expansions of Table I, Q_{ij} further reduces to:

$$-(1/3\sqrt{5})\int Q\bar{\psi}_{P^1}(F_\pi D_\nu)\psi_{D^1}(P_\pi{}^* D_\nu).$$

The decomposition of $\psi_{P^1}(F_\pi D_\nu)$ in terms of product functions was given in Appendix I; the corresponding development for $\psi_{D^1}(P_\pi{}^* D_\nu)$ is: $6^{-\frac{1}{2}}\{-3^{\frac{1}{2}}P_\pi{}^{1*}D_\nu{}^0+P_\pi{}^{0*}D_\nu{}^1+2^{\frac{1}{2}}P_\pi{}^{-1*}D_\nu{}^2\}$. Integration over the neutron coordinates gives

$$\int Q\bar{\psi}_{P^1}(F_\pi D_\nu)\psi_{D^1}(P_\pi{}^* D_\nu)$$
$$=210^{-\frac{1}{2}}\{-3\sqrt{2}\int Q\bar{F}_\pi{}^1P_\pi{}^{1*}-\sqrt{3}\int Q\bar{F}_\pi{}^0P_\pi{}^{0*}$$
$$+\sqrt{2}\int Q\bar{F}_\pi{}^{-1}P_\pi{}^{-1*}\}$$
$$=2\cdot 210^{-\frac{1}{2}}\langle r^2\rangle_{3p,4f}\{-3\sqrt{2}I_{3,1;1,1}-\sqrt{3}I_{3,0;1,0}+\sqrt{2}I_{3,1;1,1}\}$$

where $I_{3,0;1,0}=3\sqrt{21}/35$ and $I_{3,1;1,1}=3\sqrt{14}/35$. Since $\langle r^2\rangle_{3p,4f}=(2\sqrt{14}/5)\langle r^2\rangle_{2p}$, the resulting value of Q_{ij} is $(4\sqrt{21}/15)(\langle r^2\rangle_{2p}/25)$.

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The Primary Specific Ionization and Intensity of the Cosmic Radiation above the Atmosphere at the Geomagnetic Equator*

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Directional intensities and the primary specific ionization of the charged cosmic-ray flux above the atmosphere were measured by means of a G-M counter telescope in an Aerobee sounding rocket launched at the geomagnetic equator. The intensity at a zenith angle of 45° averaged over all azimuths, was found to be 0.04 particle sec.$^{-1}$ cm^{-2} steradian^{-1}, of which not more than 65 percent can be attributed to primaries, the remainder being due to albedo. The low value (~40 percent) of the observed east-west asymmetry is most directly explainable in terms of positive proton primaries and a large albedo flux at large zenith angles, although a small contribution of negative primaries cannot be excluded.

The primary specific ionization of the radiation above the atmosphere is found to be essentially the same as that of the sea-level radiation, indicating a predominance of singly charged particles of near minimum ionization. This result strongly suggests that the albedo radiation at the equator does not consist of low energy (<100 Mev) protons.

Most of the properties of bursts produced in a small lead block can be accounted for reasonably in terms of known initiating particles and interactions.

I. EXPERIMENTAL ARRANGEMENT

An Aerobee sounding rocket (Round A-11) was fired at the geomagnetic equator about 600 miles off the coast of Peru from the USS Norton Sound on March 22, 1949. The rocket reached an altitude of over 100 km and spent 217 sec. above the appreciable atmosphere. Its trajectory and general flight history were very similar to those[1,2] of Aerobee A-10 which was fired a few days earlier.

This note is mainly concerned with results obtained with an unshielded telescope (Fig. 1) similar to that used in earlier work.[3] In addition to measuring the directional flux of charged radiation, it determined the average primary specific ionization of the radiation, essentially by measuring the efficiency of a low pressure hydrogen-filled counter. This method is not refined enough to determine the ionization of individual particles which traverse the telescope, but does have the advantage of experimental simplicity. All counters[4] were made of relatively thin-walled (0.020 inch) brass tubing with an inside diameter of 2.44 cm. Their effective length was 14.5 cm. As before,[3] the low efficiency counter B' was filled with pure hydrogen, this time to a pressure of only 2.5 cm Hg. Coincidences ABC, $AB'C$, and ACG were telemetered[5] to ground from the

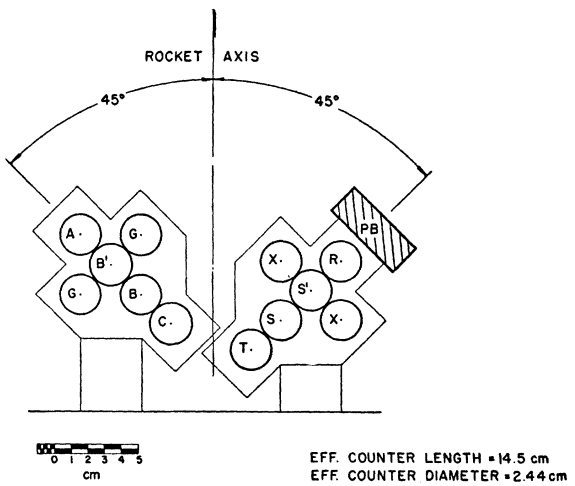

FIG. 1. Disposition in the rocket of telescopes ABC and RST, and lead block.

* Supported by the U. S. Navy, Bureau of Ordnance.
[1] J. A. Van Allen and A. V. Gangnes, Phys. Rev. **78**, 50 (1950).
[2] J. A. Van Allen and A. V. Gangnes, Phys. Rev. **79**, 51 (1950).
[3] S. F. Singer, Phys. Rev. **76**, 701 (1949).
[4] The counters were made up to our specifications by the Nuclear Development Laboratory, Kansas City, Missouri.
[5] G. H. Melton, Electronics **21**, 106 (1948).