PHYSICAL REVIEW VOLUME 80, NUMBER 3 NOVEMBER 1, 1950

Non-Linear Interactions between Electromagnetic Fields

ROBERT KARPLUS*,† AND MAURICE NEUMAN‡
Institute for Advanced Study, Princeton, New Jersey

(Received June 12, 1950)

The covariant S-matrix formalism of Dyson has been applied to the calculation of the fourth order non-linear polarization of the vacuum, which is related to the lowest order non-linear interaction between electromagnetic fields. The finiteness and the gauge invariance of the interaction are exhibited explicitly by an expression for the fourth-rank vacuum polarization tensor in momentum space.

I. INTRODUCTION

It has long been recognized that higher order corrections in quantum electrodynamics include non-linear interactions between electromagnetic fields.[1] They arise from the polarizability of the vacuum, from the possibility that transitions involve pairs only in intermediate states. Since such a correction depends on the operators of the electromagnetic field alone, it can be thought of as an addition of fourth degree in the field strengths to the electromagnetic Lagrangian density or as a non-diagonal contribution to the scattering matrix between matter-free states. The scattering of light by light, which has received some treatment in the literature,[2–5] is an example of a process which can be described by a specific interaction between photons.

Since the corrections we are discussing are necessarily at least of the order e^4, their calculation has involved considerable complications both because the treatment of effects involving virtual pairs has been traditionally accompanied by divergence and gauge-invariance difficulties and because the expressions encountered were lengthy and tedious to manipulate. With the promise the recent developments in quantum electrodynamics[6–9] give of eliminating the former and reducing the latter of these obstacles, it seemed worth while to re-examine the problem in spite of the smallness of the effects and the consequent difficulties attending their experimental detection. We have therefore rearranged the appropriate portion of the fourth order correction to the S-matrix to display explicitly its finiteness and gauge-invariance. To this end it has been expressed as a sum of terms each of which is a scalar product of derivatives of field strengths multiplied by a finite scalar non-local operator. The Fourier transforms of these are given as integrals over three parameters of rational functions of the momentum variables. The computation of a cross section requires only the well-known manipulations of the appropriate element of the scattering matrix in addition to the evaluation of the above-mentioned integrals for those values of the momenta which are of interest. The length of the expressions involved, however, makes the calculation of cross-sections very tedious except for simple special cases.

A quantity which plays an important role in the calculation is the vacuum polarization tensor $G_{\mu\nu\lambda\sigma}$ of fourth rank,

$$\delta j_\mu(x) = -(\alpha^2/12\hbar)\int G_{\mu\nu\lambda\sigma}(x, x', x'', x''')A_\nu(x') \times A_\lambda(x'')A_\sigma(x''')dx'dx''dx''', \quad (1)$$

where $\delta j_\mu(x)$ is that part of the current induced in the vacuum which is intrinsically cubic in the potential, and which cannot be reduced to lower order effects. This tensor, it will be shown, is finite and divergenceless with respect to all indices,

$$\begin{aligned} &\frac{\partial}{\partial x_\mu} G_{\mu\nu\lambda\sigma}(x, x', x'', x''') = 0, \\ &\frac{\partial}{\partial x_\nu'} G_{\mu\nu\lambda\sigma}(x, x', x'', x''') = 0, \text{ etc.} \end{aligned} \quad (2)$$

It will appear further that the tensor depends only on the mass of the pair field and on the nature of the coupling between the pair and vector fields. Hence Eq. (1) is valid even when $A_\mu(x)$ refers to a neutral vector meson fields, coupled vectorially to the pair field. The effective interaction Lagrangian density, $L(x)$, and the contribution to the scattering matrix, $S^{(4)}$, are simply related to the polarization tensor:

$$\begin{aligned} S^{(4)} &= -\frac{i}{\hbar c}\int d^4x L(x) \\ &= \frac{-i\alpha^2}{(\hbar c)^2}\frac{1}{12}\int G_{\mu\nu\lambda\sigma}(x, x', x'', x''')A_\mu(x)A_\nu(x') \\ &\quad \times A_\lambda(x'')A_\sigma(x''')dxdx'dx''dx'''. \quad (1') \end{aligned}$$

---

* Frank B. Jewett Fellow.
† Now at Harvard University, Cambridge, Massachusetts.
‡ Now at Brookhaven National Laboratory, Upton, Long Island, New York.
[1] For a summary of the literature on this subject the reader is referred to A. Pais, "Developments in the Theory of the Electron" (Institute for Advanced Study and Princeton University, 1948), pp. 21–26.
[2] O. Halpern, Phys. Rev. 44, 855 (1934).
[3] H. Euler, Ann. d. Physik, **26**, 398 (1936).
[4] W. Heisenberg and H. Euler, Zeits. f. Physik **98**, 714 (1936).
[5] A. Achieser, Physik Zeits. Sowjetunion **11**, 263 (1937).
[6] S. Tomonaga, Prog. Theor. Phys. **1**, 27 (1946) and subsequent publications with co-authors.
[7] J. Schwinger, Phys. Rev. **73**, 416 (1948); **74**, 1439 (1948).
[8] R. P. Feynman, Phys. Rev. **76**, 749, 769 (1949).
[9] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

II. THE POLARIZATION TENSOR

The fourth-order term of the electrodynamic S-matrix which describes the non-linear effects in which we are interested is given by the Feynman diagram (Fig. 1) or by the integral over four four-spaces[9]

$$S^{(4)}=-(1/64)(e/\hbar c)^4\int d^4x_1d^4x_2d^4x_3d^4x_4 \times A_\mu(x_1)A_\nu(x_2)A_\lambda(x_3)A_\sigma(x_4) \times Tr\{\gamma_\mu S_F(x_2-x_1)\gamma_\nu S_F(x_3-x_2)\gamma_\lambda \times S_F(x_4-x_3)\gamma_\sigma S_F(x_1-x_4)\}. \quad (3)$$

$A_\mu(x)$ is the vector potential of a photon and/or a fixed electromagnetic field; $S_F(x)$ is the function characteristic of the fluctuations in the pair field,[9]

$$S_F(x)=-\frac{2i}{(2\pi)^4}\lim_{\epsilon\to 0^+}\int d^4p\frac{i(\gamma p)-\kappa}{p^2+\kappa^2-i\epsilon}e^{-ipx} \quad \begin{pmatrix} px=\mathbf{p}\cdot\mathbf{r}-p_0x_0 \\ \kappa=mc/\hbar \end{pmatrix}. \quad (4)$$

The transition to momentum space, with

$$A_\mu(x)=(\hbar c)^{\frac{1}{2}}\int A_\mu(k)e^{ikx}d^4k, \quad (5)$$

yields

$$S^{(4)}=-\frac{i}{4}\alpha^2\int d^4x\int d^4k^{(1)}d^4k^{(2)}d^4k^{(3)}d^4k^{(4)} \times \exp[i(k^{(1)}+k^{(2)}+k^{(3)}+k^{(4)})x]A_\mu(k^{(1)})A_\nu(k^{(2)}) \times A_\lambda(k^{(3)})A_\sigma(k^{(4)})\tfrac{1}{3}G_{\mu\nu\lambda\sigma}{}^{(\kappa)}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}), \quad (6)$$

where

$$G_{\mu\nu\lambda\sigma}{}^{(\kappa)}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) = T_{\mu\nu\lambda\sigma}{}^{(\kappa)}(k^{(1)},k^{(2)},k^{(3)},k^{(4)})+T_{\mu\nu\sigma\lambda}{}^{(\kappa)}(k^{(1)},k^{(2)},k^{(4)},k^{(3)}) +T_{\mu\lambda\nu\sigma}{}^{(\kappa)}(k^{(1)},k^{(3)},k^{(2)},k^{(4)}) \quad (7)$$

and

$$T_{\mu\nu\lambda\sigma}{}^{(\kappa)}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) =\frac{1}{i\pi^2}\int d^4p Tr\left\{\gamma_\mu\frac{i\gamma p-\kappa}{p^2+\kappa^2-i\epsilon}\gamma_\nu\frac{i\gamma(p-k^{(2)})-\kappa}{(p-k^{(2)})^2+\kappa^2-i\epsilon} \times\gamma_\lambda\frac{i\gamma(p-k^{(2)}-k^{(3)})-\kappa}{(p-k^{(2)}-k^{(3)})^2+\kappa^2-i\epsilon} \times\gamma_\sigma\frac{i\gamma(p-k^{(2)}-k^{(3)}-k^{(4)})-\kappa}{(p-k^{(2)}-k^{(3)}-k^{(4)})^2+\kappa^2-i\epsilon}\right\}. \quad (8)$$

It will be understood from now on that integrals which depend on ϵ are to be evaluated in the limit as ϵ tends to zero through positive values. $\frac{1}{3}G_{\mu\nu\lambda\sigma}{}^{(\kappa)}$ is equivalent to $T_{\mu\nu\lambda\sigma}{}^{(\kappa)}$ in the integrand of Eq. (6) because of the symmetry of the remaining factors with respect to simultaneous permutations of $k^{(1)}, k^{(2)}\cdots$ and $\mu, \nu, \cdots$. $G_{\mu\nu\lambda\sigma}{}^{(\kappa)}$, of course, is completely symmetric with respect to these operations. Furthermore,

$$G_{\mu\nu\lambda\sigma}{}^{(\kappa)}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}) =G_{\mu\nu\lambda\sigma}{}^{(\kappa)}(-k^{(1)},-k^{(2)},-k^{(3)},-k^{(4)}), \quad (9)$$

because the trace of the spinor product of an odd number of Dirac matrices vanishes.

At this stage the quantities $S^{(4)}$ and $G_{\mu\nu\lambda\sigma}{}^{(\kappa)}$ must be defined more precisely because they depend on the logarithmically divergent tensor $T_{\mu\nu\lambda\sigma}{}^{(\kappa)}$. This is accomplished by regularization:[10,11]

$$\bar{T}_{\mu\nu\lambda\sigma}=T_{\mu\nu\lambda\sigma}{}^{(\kappa)}-T_{\mu\nu\lambda\sigma}{}^{(M)} \quad (10)$$

and, correspondingly,

$$G_{\mu\nu\lambda\sigma}=\lim_{M\to\infty}[G_{\mu\nu\lambda\sigma}{}^{(\kappa)}-G_{\mu\nu\lambda\sigma}{}^{(M)}],$$

$$S^{(4)}=-\frac{i}{12}\alpha^2\int d^4x\int d^4k^{(1)}d^4k^{(2)}d^4k^{(3)}d^4k^{(4)} \times\exp[i(k^{(1)}+k^{(2)}+k^{(3)}+k^{(4)})x]A_\mu(k^{(1)}) \times A_\nu(k^{(2)})A_\lambda(k^{(3)})A_\sigma(k^{(4)}) \times G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)}). \quad (11)$$

To verify the consistency of the theory, the finiteness and gauge-invariance of $S^{(4)}$, one may observe that the polarization tensor $G_{\mu\nu\lambda\sigma}$ is finite and that it satisfies the Fourier transform of Eq. (2),

$$\begin{aligned} k_\mu{}^{(1)}G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)})&=0,\\ k_\nu{}^{(2)}G_{\mu\nu\lambda\sigma}(k^{(1)},k^{(2)},k^{(3)},k^{(4)})&=0, \text{ etc.}\end{aligned} \quad (12)$$

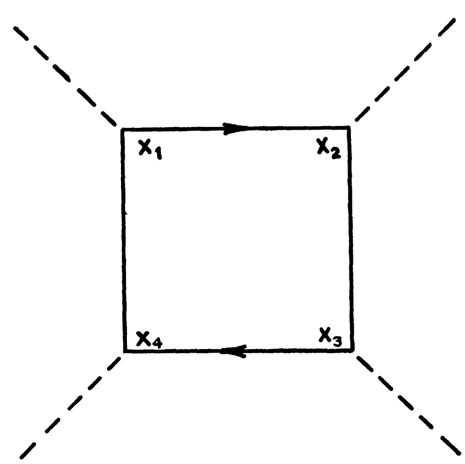

FIG. 1. The Feynman diagram.

[10] W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).

[11] Alternatively, the ambiguous integrals can be defined by demanding that $S^{(4)}$ be gauge-invariant. Equation (2) would then become a defining equation for $G_{\mu\nu\lambda\sigma}$.

Because $G_{\mu\nu\lambda\sigma}$ is symmetric these four equations are actually equivalent to one another.

That $G_{\mu\nu\lambda\sigma}$ is indeed finite may be seen more readily by separating the divergent terms in $T_{\mu\nu\lambda\sigma}{}^{(\kappa)}$ and $T_{\mu\nu\lambda\sigma}{}^{(M)}$:

$$\bar{T}_{\mu\nu\lambda\sigma}=T_{\mu\nu\lambda\sigma}{}^0+[T_{\mu\nu\lambda\sigma}{}^{(\kappa)}-T_{\mu\nu\lambda\sigma}{}^{(M)}-T_{\mu\nu\lambda\sigma}{}^0], \quad (13)$$

where

$$T_{\mu\nu\lambda\sigma}{}^0=\frac{1}{i\pi^2}\int d^4p Tr\left\{\gamma_\mu\frac{i\gamma p-\kappa}{p^2+\kappa^2-i\epsilon}\gamma_\nu\frac{i\gamma p-\kappa}{p^2+\kappa^2-i\epsilon}\times\gamma_\lambda\frac{i\gamma p-\kappa}{p^2+\kappa^2-i\epsilon}\gamma_\sigma\frac{i\gamma p-\kappa}{p^2+\kappa^2-i\epsilon}-\gamma_\mu\frac{i\gamma p-M}{p^2+M^2-i\epsilon}\gamma_\nu\frac{i\gamma p-M}{p^2+M^2-i\epsilon}\times\gamma_\lambda\frac{i\gamma p-M}{p^2+M^2-i\epsilon}\gamma_\sigma\frac{i\gamma p-M}{p^2+M^2-i\epsilon}\right\}. \quad (14)$$

The quantity in brackets in Eq. (13) clearly approaches a finite limit as M becomes infinite. The tensor $T_{\mu\nu\lambda\sigma}{}^0$ is easily evaluated to be[12]

$$T_{\mu\nu\lambda\sigma}{}^0=\frac{1}{i\pi^2}\int d^4p\left\{(\delta_{\mu\nu}\delta_{\lambda\sigma}+\delta_{\mu\sigma}\delta_{\nu\lambda}-2\delta_{\mu\lambda}\delta_{\nu\sigma})\times\left[\frac{(4/3)(p^2)^2+4\kappa^2p^2}{[p^2+\kappa^2-i\epsilon]^4}-\frac{(4/3)(p^2)^2+4M^2p^2}{[p^2+M^2-i\epsilon]^4}\right]+(\delta_{\mu\nu}\delta_{\lambda\sigma}+\delta_{\mu\sigma}\delta_{\nu\lambda}-\delta_{\mu\lambda}\delta_{\nu\sigma})\times\left[\frac{\kappa^4}{[p^2+\kappa^2-i\epsilon]^4}-\frac{M^4}{[p^2+M^2-i\epsilon]^4}\right]\right\} \quad (15)$$

and so does not contribute to $G_{\mu\nu\lambda\sigma}$ because one term vanishes on symmetrization and the other on integration over the momentum p_μ.

One can verify Eq. (12), now, by the use of Eqs. (7), (8), (9), and (11), if he notes that

$$k_\mu{}^{(1)}\gamma_\mu=\frac{1}{i}[i\gamma(p-k^{(2)}-k^{(3)}-k^{(4)})+\kappa]-\frac{1}{i}[i\gamma p+\kappa]$$
$$=\frac{1}{i}[i\gamma p+\kappa]-\frac{1}{i}[i\gamma(p+k^2+k^3+k^4)+\kappa], \text{ etc.}; \quad (16)$$

the resulting terms cancel in sets of four. A characteristic group is

$$(1/\pi^2)\int d^4pTr\left\{\frac{i\gamma p-\kappa}{p^2+\kappa^2-i\epsilon}\gamma_\nu\frac{i\gamma(p-k^{(2)})-\kappa}{(p-k^{(2)})^2+\kappa^2-i\epsilon}\gamma_\lambda\times\frac{i\gamma(p-k^{(2)}-k^{(3)})-\kappa}{(p-k^{(2)}-k^{(3)})^2+\kappa^2-i\epsilon}\gamma_\sigma-[\text{same with }\kappa\to M]-\frac{i\gamma p-\kappa}{p^2+\kappa^2-i\epsilon}\gamma_\lambda\frac{i\gamma(p+k^{(3)})-\kappa}{(p+k^{(3)})^2+\kappa^2-i\epsilon}\gamma_\nu\times\frac{i\gamma(p+k^{(2)}+k^{(3)})-\kappa}{(p+k^{(2)}+k^{(3)})^2+\kappa^2-i\epsilon}\gamma_\sigma+[\text{same with }\kappa\to M]\right\}=0. \quad (17)$$

These four terms come from $T_{\mu\nu\lambda\sigma}{}^{(\kappa)}(1234)$,

$$-T_{\mu\nu\lambda\sigma}{}^{(M)}(1234),\quad T_{\mu\lambda\nu\sigma}{}^{(\kappa)}(-1,-3,-2,-4),$$

and $-T_{\mu\lambda\nu\sigma}{}^{(M)}(-1,-3,-2,-4)$ in that order. (For simplicity, the $k^{(i)}$ have been replaced by i in the arguments of the tensors.) The displacement $p_\mu\to p_\mu-k_\mu{}^{(2)}-k_\mu{}^{(3)}$ in the third and fourth makes them equal to the first and second, respectively, except for sign.

III. METHOD OF CALCULATION

From Eqs. (7), (8) and (11) it is evident that the polarization tensor can be written

$$G_{\mu\nu\lambda\sigma}(1234)=\sum_{\substack{i=2,3,4\\ j=1,3,4}}\sum_{\substack{l=1,2,4\\ m=1,2,3}}A^{ijlm}(1234)k_\mu{}^{(i)}k_\nu{}^{(j)}k_\lambda{}^{(l)}k_\sigma{}^{(m)}+\sum_{\substack{l=1,2,4\\ m=1,2,3}}B_1{}^{lm}(1234)\delta_{\mu\nu}k_\lambda{}^{(l)}k_\sigma{}^{(m)}+\sum_{\substack{j=1,3,4\\ m=1,2,3}}B_2{}^{jm}(1234)\delta_{\mu\lambda}k_\nu{}^{(j)}k_\sigma{}^{(m)}+\cdots+C_1(1234)\delta_{\mu\nu}\delta_{\lambda\sigma}+C_2(1234)\delta_{\mu\lambda}\delta_{\nu\sigma}+C_3(1234)\delta_{\mu\sigma}\delta_{\nu\lambda}, \quad (18)$$

where the A, B, C are invariants which depend on scalar products of the four momenta. An expression of this form is obtained if the spinor summation and integration over p_μ, Eq. (8), are carried out. Because the four momenta are connected by the conservation equation, only three are independent; to maintain a symmetrical appearance of $G_{\mu\nu\lambda\sigma}$, the three were chosen in a way which is dependent on the vector index they carry, as illustrated by the restrictions on the summations in Eq. (18). The problem of obtaining an explicit expression for $G_{\mu\nu\lambda\sigma}$, and so $S^{(4)}$, can now be solved in a straight forward manner by obtaining the coefficients in Eq. (18) from Eqs. (7) to (11). This task is not as extensive as it seems, because the symmetry of the polarization tensor means that many invariants can be

[12] With the aid of the identities

(1) $\int p_\alpha f(p^2)d^4p=\int p_\alpha p_\beta p_\gamma f(p^2)d^4p=0,$

(2) $\int p_\alpha p_\beta f(p^2)d^4p=\frac{1}{4}\delta_{\alpha\beta}\int p^2f(p^2)d^4p,$

(3) $\int p_\alpha p_\beta p_\gamma p_\eta f(p^2)d^4p=\frac{1}{24}[\delta_{\alpha\beta}\delta_{\gamma\eta}+\delta_{\alpha\gamma}\delta_{\beta\eta}+\delta_{\alpha\eta}\delta_{\beta\gamma}]\times\int(p^2)^2f(p^2)d^4p.$

obtained from others by merely interchanging some of the momenta. Since $G_{\mu\nu\lambda\sigma}(k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)})$ is invariant under simultaneous permutations of the arguments and indices, the following relations hold among the A's:

$$\begin{aligned} A^{2111}(1234) &= A^{4441}(2341) = A^{3343}(3412) = A^{2322}(4123) \\ &= A^{4111}(1432) = A^{2122}(2143) = A^{3323}(3214) = A^{4443}(4321) \\ &= A^{3111}(1324) = A^{3313}(2413) = A^{2422}(3142) = A^{4442}(4231). \end{aligned} \quad (19a)$$

$$\begin{aligned} A^{2121}(1234) &= A^{4141}(2341) = A^{4343}(3412) = A^{2323}(4123) \\ &= A^{2112}(1243) = A^{3113}(2314) = A^{3443}(3421) = A^{2442}(4132) \\ &= A^{4422}(3142) = A^{3322}(4213) = A^{3311}(2413) = A^{4411}(3124). \end{aligned} \quad (19b)$$

$$\begin{aligned} A^{2123}(1234) &= A^{4121}(2341) = A^{4143}(3412) = A^{4323}(4123) \\ &= A^{4341}(1432) = A^{2141}(2143) = A^{2321}(3214) = A^{2343}(4321) \\ &= A^{3312}(1324) = A^{4311}(2431) = A^{4412}(3142) = A^{4322}(4213) \\ &= A^{4421}(1342) = A^{3411}(2413) = A^{3321}(3124) = A^{3422}(4231) \\ &= A^{2142}(1243) = A^{3112}(2314) = A^{3143}(3421) = A^{3422}(4132) \\ &= A^{3413}(1423) = A^{2113}(2134) = A^{2412}(3241) = A^{2443}(4312). \end{aligned} \quad (19c)$$

$$\begin{aligned} A^{2311}(1234) &= A^{2441}(2341) = A^{3341}(3412) = A^{2342}(4123) \\ &= A^{4113}(1432) = A^{4122}(2143) = A^{3123}(3214) = A^{4423}(4321) \\ &= A^{3121}(1324) = A^{3441}(2431) = A^{2421}(3142) = A^{3423}(4213) \\ &= A^{4112}(1342) = A^{4313}(2413) = A^{2312}(3124) = A^{4342}(4231) \\ &= A^{2411}(1243) = A^{2313}(2314) = A^{4413}(3421) = A^{2423}(4132) \\ &= A^{3141}(1423) = A^{3122}(2134) = A^{4142}(3241) = A^{3342}(4321). \end{aligned} \quad (19d)$$

$$A^{2143}(1234) = A^{4321}(2341) = A^{3412}(1324). \quad (19e)$$

$$\begin{aligned} A^{2341}(1234) &= A^{4123}(4321) = A^{2413}(1243) = A^{4312}(3124) \\ &= A^{3142}(4312) = A^{3421}(2431). \end{aligned} \quad (19f)$$

Some of these invariants, of course, are unchanged by certain interchanges of the momenta. Similar equations hold for the coefficients B and C, Eq. (18). These relations, therefore, considerably reduce the labor of finding all the coefficients.

One can, moreover, take advantage of the fact that the polarization tensor satisfies Eq. (12), in other words, that it is gauge-invariant. It is easy to show that a gauge-invariant tensor of the form Eq. (18) vanishes identically if all the coefficients A vanish. Hence, knowledge of the A's is sufficient to determine $G_{\mu\nu\lambda\sigma}$ completely and it becomes unnecessary to calculate the coefficients B and C at all.[13] They will therefore be ignored in the subsequent work. The terms which involve an A will be called "heads" or "leading terms."

One can take further advantage of the gauge-invariance of $G_{\mu\nu\lambda\sigma}$ by expressing this quantity as far as possible as the sum of simpler gauge-invariant tensors. These can be constructed from a consideration of scalar products of four field strengths, which will certainly give rise to gauge-invariant expressions. Thus one can define the tensors $g_{\mu\nu\lambda\sigma}{}^{(i)}$ as follows:

$$\begin{aligned} F_{\alpha\beta}(1)F_{\beta\alpha}(2)F_{\gamma\delta}(3)F_{\delta\gamma}(4) &= 4A_\mu(1)A_\nu(2)A_\lambda(3)A_\sigma(4)g_{\mu\nu\lambda\sigma}{}^{(1)}(1234) \\ g_{\mu\nu\lambda\sigma}{}^{(1)}(1234) &\sim k_\mu{}^{(2)}k_\nu{}^{(1)}k_\lambda{}^{(4)}k_\sigma{}^{(3)}, \end{aligned} \quad (20)$$

$$\begin{aligned} F_{\alpha\beta}(1)F_{\beta\gamma}(2)F_{\gamma\delta}(3)F_{\delta\alpha}(4) &= A_\mu(1)A_\nu(2)A_\lambda(3)A_\sigma(4)g_{\mu\nu\lambda\sigma}{}^{(2)}(1234) \\ g_{\mu\nu\lambda\sigma}{}^{(2)}(1234) &\sim k_\mu{}^{(2)}k_\nu{}^{(3)}k_\lambda{}^{(4)}k_\sigma{}^{(1)} + k_\mu{}^{(4)}k_\nu{}^{(1)}k_\lambda{}^{(2)}k_\sigma{}^{(3)}, \end{aligned} \quad (21)$$

$$\begin{aligned} F_{\alpha\beta}(1)F_{\beta\alpha}(2)k_\gamma{}^{(1)}F_{\gamma\delta}(3)F_{\delta\epsilon}(4)k_\epsilon{}^{(1)} &= -2A_\mu(1)A_\nu(2)A_\lambda(3)A_\sigma(4)g_{\mu\nu\lambda\sigma}{}^{(3)}(1234) \\ g_{\mu\nu\lambda\sigma}{}^{(3)}(1234) &\sim [(k^{(3)}k^{(4)})k_\mu{}^{(2)}k_\nu{}^{(1)}k_\lambda{}^{(1)}k_\sigma{}^{(1)} \\ &\quad -(k^{(1)}k^{(3)})k_\mu{}^{(2)}k_\nu{}^{(1)}k_\lambda{}^{(4)}k_\sigma{}^{(1)} \\ &\quad -(k^{(1)}k^{(4)})k_\mu{}^{(2)}k_\nu{}^{(1)}k_\lambda{}^{(1)}k_\sigma{}^{(3)}], \end{aligned} \quad (22a)$$

$$\begin{aligned} F_{\alpha\beta}(1)F_{\beta\alpha}(2)k_\gamma{}^{(2)}F_{\gamma\delta}(3)F_{\delta\epsilon}(4)k_\epsilon{}^{(1)} &= -2A_\mu(1)A_\nu(2)A_\lambda(3)A_\sigma(4)g_{\mu\nu\lambda\sigma}{}^{(4)}(1234) \\ g_{\mu\nu\lambda\sigma}{}^{(4)}(1234) &\sim [(k^{(3)}k^{(4)})k_\mu{}^{(2)}k_\nu{}^{(1)}k_\lambda{}^{(2)}k_\sigma{}^{(1)} \\ &\quad -(k^{(2)}k^{(3)})k_\mu{}^{(2)}k_\nu{}^{(1)}k_\lambda{}^{(4)}k_\sigma{}^{(1)} \\ &\quad -(k^{(1)}k^{(4)})k_\mu{}^{(2)}k_\nu{}^{(1)}k_\lambda{}^{(2)}k_\sigma{}^{(3)}], \end{aligned} \quad (22b)$$

$$\begin{aligned} k_\alpha{}^{(2)}F_{\alpha\beta}(4)F_{\beta\gamma}(1)[F_{\gamma\delta}(2)F_{\delta\epsilon}(3) - F_{\gamma\delta}(3)F_{\delta\epsilon}(2)]k_\epsilon{}^{(1)} &= A_\mu(1)A_\nu(2)A_\lambda(3)A_\sigma(4)g_{\mu\nu\lambda\sigma}{}^{(5)}(1234) \\ g_{\mu\nu\lambda\sigma}{}^{(5)}(1234) &\sim [(k^{(2)}k^{(4)})(k_\mu{}^{(2)}k_\nu{}^{(3)}k_\lambda{}^{(1)}k_\sigma{}^{(1)} \\ &\quad -k_\mu{}^{(3)}k_\nu{}^{(1)}k_\lambda{}^{(2)}k_\sigma{}^{(1)}) + (k^{(1)}k^{(4)}) \\ &\quad \times(k_\mu{}^{(3)}k_\nu{}^{(1)}k_\lambda{}^{(2)}k_\sigma{}^{(2)} - k_\mu{}^{(2)}k_\nu{}^{(3)}k_\lambda{}^{(1)}k_\sigma{}^{(2)})]. \end{aligned} \quad (23)$$

It should be noticed that Eq. (20) involves only heads in Eq. (19e), Eq. (21) those in Eq. (19f), Eq. (22) those in Eq. (19a–c), and Eq. (23) those in Eq. (19d). These few types of tensors, of course, need not be sufficient to express $G_{\mu\nu\lambda\sigma}$ completely. To find out to what extent they and their permutations are represented in $G_{\mu\nu\lambda\sigma}$ and what remainder is left after this is done, the coefficients A must be known more precisely. For this purpose we return to a consideration of $\bar{T}_{\mu\nu\lambda\sigma}$.

IV. EVALUATION OF THE POLARIZATION TENSOR

By the usual methods of carrying out the integration over the momenta of the virtual particles,[8] Eq. (10) can be transformed to

$$\bar{T}_{\mu\nu\lambda\sigma}(1234) = \frac{6}{i\pi^2}\int d\tau \int d^4p \cdot Tr\left\{ \frac{\gamma_\mu[i\gamma(p-\lambda^{(1)})-\kappa]\gamma_\nu[i\gamma(p-\lambda^{(2)})-\kappa] \times\gamma_\lambda[i\gamma(p-\lambda^{(3)})-\kappa]\gamma_\sigma[i\gamma(p-\lambda^{(4)})-\kappa]}{[p^2+\kappa^2-i\epsilon+(k^{(1)})^2y_4y_1+(k^{(2)})^2y_1y_2+(k^{(3)})^2y_2y_3 +(k^{(4)})^2y_3y_4-(k^{(1)}+k^{(2)})(k^{(3)}+k^{(4)}y_2y_4 -(k^{(1)}+k^{(4)})(k^{(2)}+k^{(3)})y_1y_3]^2} - [\text{same with } \kappa\rightarrow M]\right\}, \quad (24)$$

---

[13] The method of defining the ambiguous integrals by requiring gauge-invariance consists of calculating the coefficients A and constructing B's and C's that make the tensor gauge-invariant. This is the case because the A's are unambiguous while the C's are ambiguous. The method is therefore completely equivalent to regularization.

where

$$\begin{aligned}\lambda_\mu^{(1)}&=-k_\mu^{(1)}y_4+k_\mu^{(2)}(y_2+y_3)+k_\mu^{(3)}y_3,\\ \lambda_\mu^{(2)}&=-k_\mu^{(2)}y_1+k_\mu^{(3)}(y_3+y_4)+k_\mu^{(4)}y_4,\\ \lambda_\mu^{(3)}&=+k_\mu^{(1)}y_1-k_\mu^{(3)}y_2+k_\mu^{(4)}(y_4+y_1),\\ \lambda_\mu^{(4)}&=k_\mu^{(1)}(y_1+y_2)+k_\mu^{(2)}y_2-k_\mu^{(4)}y_3,\end{aligned}\tag{25}$$

and

$$\int d\tau=\underset{y_i>0,\ \Sigma y_i=1}{\int\int\int\int} dy_1dy_2dy_3dy_4.$$

Since only the leading terms are of interest, the quantities $i\gamma p$ and κ or M in the numerators of the two terms may be dropped; as was already pointed out (see Eq. (9) and note to Eq. (15)), they contribute only quadratically and are then accompanied by a factor $\delta_{\mu\nu}$, so that they constitute part of a term B or C, Eq. (18). Then the regularizing term becomes of order M^{-4} and vanishes as the auxiliary mass tends to infinity. Hence, as far as head terms are concerned, the integration over the momentum p_μ in Eq. (24) becomes trivial, and

$$\bar{T}_{\mu\nu\lambda\sigma}(1234)\sim\frac{1}{\kappa^4}\int d\tau D(1234)\, Tr\{\gamma_\mu\gamma\lambda^{(1)}\gamma_\nu\gamma\lambda^{(2)}\gamma_\lambda\gamma\lambda^{(3)}\gamma_\sigma\gamma\lambda^{(4)}\}.\tag{26}$$

Here

$$\begin{aligned}D(1234)=\{1-i\epsilon'+(1/\kappa^2)[(k^{(1)})^2y_4y_1+(k^{(2)})^2y_1y_2\\ +(k^{(3)})^2y_2y_3+(k^{(4)})^2y_3y_4-(k^{(1)}+k^{(2)})(k^{(3)}+k^{(4)})y_2y_4\\ -(k^{(1)}+k^{(4)})(k^{(2)}+k^{(3)})y_1y_3]\}^{-2}\end{aligned}\tag{27}$$

with $\epsilon'=\epsilon/\kappa^2$.

A further simplification results from the consideration of heads only when the spin sum in Eq. (26) is carried out, because all but twenty-four terms may be ignored. Thus

$$\begin{aligned}&Tr\{\gamma_\mu\gamma\lambda^{(1)}\gamma_\nu\gamma\lambda^{(2)}\gamma_\lambda\gamma\lambda^{(3)}\gamma_\sigma\gamma\lambda^{(4)}\}\\ &\sim4[(\lambda_\mu^{(1)}\lambda_\nu^{(2)}+\lambda_\mu^{(2)}\lambda_\nu^{(1)})(\lambda_\lambda^{(3)}\lambda_\sigma^{(4)}+\lambda_\lambda^{(4)}\lambda_\sigma^{(3)})\\ &\quad+(\lambda_\mu^{(1)}\lambda_\nu^{(3)}+\lambda_\mu^{(3)}\lambda_\nu^{(1)})(\lambda_\lambda^{(2)}\lambda_\sigma^{(4)}-\lambda_\lambda^{(4)}\lambda_\sigma^{(2)})\\ &\quad+(\lambda_\mu^{(1)}\lambda_\nu^{(4)}+\lambda_\mu^{(4)}\lambda_\nu^{(1)})(\lambda_\lambda^{(2)}\lambda_\sigma^{(3)}+\lambda_\lambda^{(3)}\lambda_\sigma^{(2)})\\ &\quad+(\lambda_\mu^{(2)}\lambda_\nu^{(3)}-\lambda_\mu^{(3)}\lambda_\nu^{(2)})(\lambda_\lambda^{(4)}\lambda_\sigma^{(1)}-\lambda_\lambda^{(1)}\lambda_\sigma^{(4)})\\ &\quad+(\lambda_\mu^{(4)}\lambda_\nu^{(2)}-\lambda_\mu^{(2)}\lambda_\nu^{(4)})(\lambda_\lambda^{(1)}\lambda_\sigma^{(3)}+\lambda_\lambda^{(3)}\lambda_\sigma^{(1)})\\ &\quad+(\lambda_\mu^{(3)}\lambda_\nu^{(4)}-\lambda_\mu^{(4)}\lambda_\nu^{(3)})(\lambda_\lambda^{(1)}\lambda_\sigma^{(2)}-\lambda_\lambda^{(2)}\lambda_\sigma^{(1)})].\end{aligned}\tag{28}$$

It must be remembered, of course, that the $\lambda^{(i)}$ have to be expressed in terms of the three momenta appropriate to the index the $\lambda^{(i)}$ carries in accordance with the convention adopted with Eq. (18). By substitution of Eq. (25) into Eq. (28) one can then express $\bar{T}_{\mu\nu\lambda\sigma}$ in the form

$$\bar{T}_{\mu\nu\lambda\sigma}(1234)\sim\sum_{\substack{i=2,3,4\\ j=1,3,4}}\ \sum_{\substack{l=1,2,4\\ m=1,2,3}} A_1^{ijlm}(1234)k_\mu^{(i)}k_\nu^{(j)}k_\lambda^{(l)}k_\sigma^{(m)},\tag{29}$$

where the $A_1^{ijlm}(1234)$ are the contributions of $\bar{T}_{\mu\nu\lambda\sigma}(1234)$ to the coefficients A in Eq. (18). Because $\bar{T}_{\mu\nu\lambda\sigma}$ is less symmetric than $G_{\mu\nu\lambda\sigma}$, as many as fifteen of the A_1 must be obtained before the remainder can be generated by symmetry operations. Such a set may consist of the following:

$$A_1^{2111}(1234)=\frac{8}{\kappa^4}\int d\tau y_1(y_2+y_3+y_4)(y_1+y_2-y_3-y_4)\times(y_1+y_2+y_3-y_4)D(1234),\tag{30a}$$

$$A_1^{3111}(1234)=\frac{-8}{\kappa^4}\int d\tau(y_1+y_2)(y_3+y_4)(y_2+y_3+y_4-y_1)\times(y_1+y_2+y_3-y_4)D(1234),\tag{30b}$$

$$A_1^{2121}(1234)=\frac{-8}{\kappa^4}\int d\tau y_1(y_2+y_3+y_4)(y_1+y_3+y_4-y_2)\times(y_1+y_2+y_3-y_4)D(1234),\tag{31a}$$

$$A_1^{2112}(1234)=\frac{8}{\kappa^4}\int d\tau y_1(y_2+y_3+y_4)(y_1+y_2-y_3-y_4)\times(y_2+y_3-y_1-y_4)D(1234),\tag{31b}$$

$$A_1^{3311}(1234)=\frac{8}{\kappa^4}\int d\tau(y_1+y_2)(y_3+y_4)(y_1+y_3+y_4-y_2)\times(y_1+y_2+y_3-y_4)D(1234),\tag{31c}$$

$$A_1^{2113}(1234)=-\frac{8}{\kappa^4}\int d\tau y_1(y_2+y_3+y_4)(y_1+y_2-y_3-y_4)\times(y_1+y_2+y_4-y_3)D(1234),\tag{32a}$$

$$A_1^{3112}(1234)=-\frac{8}{\kappa^4}\int d\tau(y_1+y_2)(y_3+y_4)(y_2+y_3-y_1-y_4)\times(y_2+y_3+y_4-y_1)D(1234),\tag{32b}$$

$$A_1^{2141}(1234)=\frac{8}{\kappa^4}\int d\tau y_1(y_2+y_3+y_4)(y_1+y_2+y_3-y_4)\times(y_1+y_2+y_4-y_3)D(1234),\tag{32c}$$

$$\begin{aligned}A_1^{2311}(1234)=-A_1^{3121}(1234)=\frac{4}{\kappa^4}\int d\tau(y_1+y_2+y_3-y_4)\\ \times\{(y_1+y_2)[y_1y_2+(1-y_1)(1-y_2)]\\ +(y_3+y_4)[y_1(1-y_2)+y_2(1-y_1)]\}D(1234),\end{aligned}\tag{33a}$$

$$\begin{aligned}A_1^{2411}(1234)=\frac{4}{\kappa^4}\int d\tau(y_1+y_2-y_3-y_4)\\ \times\{(y_1+y_4)[y_1y_4+(1-y_1)(1-y_4)]\\ +(y_2+y_3)[y_4(1-y_1)+y_1(1-y_4)]\}D(1234),\end{aligned}\tag{33b}$$

$$\begin{aligned}A_1^{2413}(1234)=-\frac{4}{\kappa^4}\int d\tau\{[(y_2+y_3)(y_3+y_4)\\ +(y_1+y_2)(y_1+y_4)][y_1y_3+(1-y_1)(1-y_3)]\\ +[(y_1+y_4)(y_3+y_4)+(y_1+y_2)(y_2+y_3)]\\ \times[y_1(1-y_3)+y_3(1-y_1)]\}D(1234),\end{aligned}\tag{34a}$$

$$\begin{aligned}A_1^{2341}(1234)=\frac{4}{\kappa^4}\int d\tau\{[(1-y_4)(1-y_3)+y_3y_4]\\ \times[(1-y_1)(1-y_2)+y_1y_2]\\ +[(1-y_4)y_3+(1-y_3)y_4]\\ \times[(1-y_2)y_1+(1-y_1)y_2]\}D(1234),\end{aligned}\tag{34b}$$

$$A_1^{2143}(1234)=\frac{32}{\kappa^4}\int d\tau y_1y_3(1-y_1)(1-y_3)D(1234),\tag{35a}$$

$$A_1^{3412}(1234)=\frac{32}{\kappa^4}\int d\tau(y_1+y_2)(y_2+y_3)(y_3+y_4)\times(y_4+y_1)D(1234).\tag{35b}$$

Now, with the help of the symmetrization procedure indicated in Eq. (7), one can write down the coefficients A in terms of the A_1. From the nature of the A_1 one can also deduce certain identities among the A's, and these are indicated in Eqs. (36) to (41).

$$A^{2111}(1234)=A_1^{2111}(1234)+A_1^{2111}(1243)+A_1^{3111}(1324)=A^{2111}(1243),\tag{36}$$

$$A^{2121}(1234)=A_1^{2121}(1234)+A_1^{2112}(1243)+A_1^{3311}(1324)=A^{2121}(2143),\tag{37}$$

$$A^{2123}(1234)=A_1^{2113}(2134)+A_1^{3112}(2314)+A_1^{2141}(2143),\tag{38}$$

$$A^{2311}(1234)=A_1^{2311}(1234)+A_1^{3121}(1324)+A_1^{2411}(1243)=-A^{3121}(1234),\tag{39}$$

$$A^{2143}(1234)=A_1^{2143}(1234)+A_1^{2143}(1243)+A_1^{3412}(1324),\tag{40}$$

$$A^{2341}(1234)=A_1^{2341}(1234)+A_1^{2413}(1243)+A_1^{2413}(4132)=A^{4123}(1234).\tag{41}$$

To apply these relations to $G_{\mu\nu\lambda\sigma}$, it is helpful to consider a piece $G_{\mu\nu\lambda\sigma}'(1234)$ of this tensor:

$$\begin{aligned}G_{\mu\nu\lambda\sigma}'(1234)=&A^{2111}(1234)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(1)}k_\sigma^{(1)}\\ &+A^{2121}(1234)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(1)}+A^{2121}(1243)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(1)}k_\sigma^{(2)}\\ &+A^{2111}(2143)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(2)}+A^{2123}(1234)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(3)}\\ &+A^{2123}(1243)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(4)}k_\sigma^{(2)}+A^{2123}(2143)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(4)}k_\sigma^{(1)}\\ &+A^{2123}(2134)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(1)}k_\sigma^{(3)}+A^{2311}(1234)[k_\mu^{(2)}k_\nu^{(3)}k_\lambda^{(1)}k_\sigma^{(1)}\\ &-k_\mu^{(3)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(1)}]+A^{2311}(2314)[k_\mu^{(2)}k_\nu^{(3)}k_\lambda^{(1)}k_\sigma^{(2)}\\ &-k_\mu^{(3)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(2)}]+A^{2311}(3124)[k_\mu^{(2)}k_\nu^{(3)}k_\lambda^{(1)}k_\sigma^{(3)}\\ &-k_\mu^{(3)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(3)}]+A^{2143}(1234)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(4)}k_\sigma^{(3)}\\ &+A^{2341}(1234)[k_\mu^{(2)}k_\nu^{(3)}k_\lambda^{(4)}k_\sigma^{(1)}+k_\mu^{(4)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(3)}].\end{aligned}\tag{42}$$

A look at Eqs. (20)–(28) suggests that $G_{\mu\nu\lambda\sigma}'(1234)$ be rearranged as follows:

$$
\begin{aligned}
G_{\mu\nu\lambda\sigma}'(1234) = {} & A^{2143}(1234)g_{\mu\nu\lambda\sigma}^{(1)}(1234) \\
& + A^{2341}(1234)g_{\mu\nu\lambda\sigma}^{(2)}(1234) \\
& + \frac{1}{(k^{(3)}k^{(4)})}[A^{2111}(1234)g_{\mu\nu\lambda\sigma}^{(3)}(1234) \\
& + A^{2111}(2134)g_{\nu\mu\lambda\sigma}^{(3)}(2134) + A^{2121}(1234)g_{\mu\nu\lambda\sigma}^{(4)}(1234) \\
& + A^{2121}(1243)g_{\mu\nu\sigma\lambda}^{(4)}(1243)] \\
& + \frac{1}{(k^{(2)}k^{(4)})}[A^{2311}(1234)g_{\mu\nu\lambda\sigma}^{(5)}(1234) \\
& - A^{2311}(3124)g_{\lambda\nu\mu\sigma}^{(5)}(3214)] + a(1234)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(3)} \\
& + a(1243)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(4)}k_\sigma^{(2)} + a(2143)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(4)}k_\sigma^{(1)} \\
& + a(2134)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(1)}k_\sigma^{(3)} + b(2314) \\
& \times (k_\mu^{(2)}k_\nu^{(3)}k_\lambda^{(1)}k_\sigma^{(2)} - k_\mu^{(3)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(2)}), \qquad (43)
\end{aligned}
$$

where

$$
\begin{aligned}
a(1234) = {} & \frac{1}{(k^{(3)}k^{(4)})}[(k^{(3)}k^{(4)})A^{2123}(1234) \\
& + (k^{(2)}k^{(4)})A^{2111}(2143) + (k^{(1)}k^{(4)})A^{2121}(1234)], \qquad (44a)
\end{aligned}
$$

$$
\begin{aligned}
b(2314) = {} & \frac{1}{(k^{(2)}k^{(4)})}[(k^{(2)}k^{(4)})A^{2311}(2314) \\
& + (k^{(1)}k^{(4)})A^{2311}(1234) + (k^{(3)}k^{(4)})A^{2311}(3124)]. \qquad (44b)
\end{aligned}
$$

The major part of the polarization tensor has now been expressed in terms of the tensors $g_{\mu\nu\lambda\sigma}^{(i)}$ derived from field strengths. The remainder

$$
\begin{aligned}
\sum_{24\ \text{perm}} & a(1234)k_\mu^{(2)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(3)} + b(2314)(k_\mu^{(2)}k_\nu^{(3)}k_\lambda^{(1)}k_\sigma^{(2)} \\
& - k_\mu^{(3)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(2)}) + b(2413)(k_\mu^{(2)}k_\nu^{(4)}k_\lambda^{(2)}k_\sigma^{(1)} - k_\mu^{(4)}k_\nu^{(1)}k_\lambda^{(2)}k_\sigma^{(2)}) \\
& + b(4312)(k_\mu^{(4)}k_\nu^{(4)}k_\lambda^{(1)}k_\sigma^{(3)} - k_\mu^{(3)}k_\nu^{(4)}k_\lambda^{(4)}k_\sigma^{(1)}) \\
& + b(2341)(k_\mu^{(2)}k_\nu^{(3)}k_\lambda^{(4)}k_\sigma^{(2)} - k_\mu^{(2)}k_\nu^{(4)}k_\lambda^{(2)}k_\sigma^{(3)}) \qquad (45)
\end{aligned}
$$

is still gauge-invariant and must satisfy Eq. (12) with appropriate B and C terms. It is therefore identically zero. The reason for this result is the fact that each head in Eq. (45) contains only three different momenta. The third-rank tensor that results from contraction, as in Eq. (12), with the fourth momentum then does not contain this fourth momentum. The third-rank tensors that result from the application of Eq. (12) to terms other than heads, terms that contain a Kronecker delta, will always contain the momentum vector with which they were multiplied. Hence the heads in Eq. (45) must satisfy Eq. (12) alone, and this means that every term in Eq. (45) vanishes. As a check on the calculation, $a(1234)$ and $b(2314)$ were computed explicitly as functions of null-vector momenta; they were indeed found to vanish identically.

The vacuum polarization tensor can therefore be written

$$
\begin{aligned}
G_{\mu\nu\lambda\sigma}(1234) = \sum_{24\ \text{perm}} \Big\{ & \frac{1}{8}A^{2143}(1234)g_{\mu\nu\lambda\sigma}^{(1)}(1234) \\
& + \frac{1}{8}A^{2341}(1234)g_{\mu\nu\lambda\sigma}^{(2)}(1234) \\
& + \frac{1}{2}\frac{1}{(k^{(3)}k^{(4)})}A^{2111}(1234)g_{\mu\nu\lambda\sigma}^{(3)}(1234) \\
& + \frac{1}{2}\frac{1}{(k^{(3)}k^{(4)})}A^{2121}(1234)g_{\mu\nu\lambda\sigma}^{(4)}(1234) \\
& + \frac{1}{3}\frac{1}{(k^{(2)}k^{(4)})}A^{2311}(1234)g_{\mu\nu\lambda\sigma}^{(5)}(1234) \Big\}. \qquad (46)
\end{aligned}
$$

The sum over permutations here refers to simultaneous permutations of the labels of the momentum variables and of the tensor indices of the $g^{(i)}$. The superscripts on the A's define their functional form [Eqs. (30) to (41)].

A calculation based on Eq. (46) of the cross section for the scattering of light by light is being prepared for publication.

V. LOW ENERGY APPROXIMATION

If the interacting fields vary so slowly in space and time that the Fourier transforms have appreciable values only for momenta whose absolute value is much smaller that the mass of the pair field, the function $D(1234)$ in Eq. (27) may be approximated by unity. The evaluation of the integrals Eqs. (30) to (35) then becomes trivial. In particular, the integrals are independent of the momenta. The values of the six basic ones, Eqs. (36) to (41) are

$$
\begin{aligned}
& A^{2111} = A^{2121} = A^{2311} = A^{2123} = 0, \\
& A^{2143} = A^{3412} = A^{4321} = 4/9\kappa^4, \qquad (47) \\
& A^{2341} = A^{2413} = A^{3412} = -14/45\kappa^4.
\end{aligned}
$$

Hence

$$
\begin{aligned}
G_{\mu\nu\lambda\sigma}(1234) = {} & \frac{4}{9\kappa^4}(g_{\mu\nu\lambda\sigma}^{(1)}(1234) + g_{\mu\lambda\nu\sigma}^{(1)}(1324) \\
& + g_{\mu\sigma\nu\lambda}^{(1)}(1423)) - \frac{14}{45\kappa^4}(g_{\mu\nu\lambda\sigma}^{(2)}(1234) \\
& + g_{\mu\lambda\nu\sigma}^{(2)}(1324) + g_{\mu\sigma\nu\lambda}^{(2)}(1423)), \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
S^{(4)} = -i\frac{\alpha^2}{180}\frac{1}{(\hbar c\kappa^2)^2}\int & d^4x[5(F_{\mu\nu}(x)F_{\mu\nu}(x))^2 \\
& - 14(F_{\mu\nu}(x)F_{\nu\lambda}(x)F_{\lambda\sigma}(x)F_{\sigma\mu}(x))], \qquad (49)
\end{aligned}
$$

and the effective Lagrangian density

$$
\begin{aligned}
L(x) = \frac{\alpha^2}{180}\frac{1}{\hbar c\kappa^4} & [5(F_{\mu\nu}(x)F_{\mu\nu}(x))^2 \\
& - 14F_{\mu\nu}(x)F_{\nu\lambda}(x)F_{\lambda\sigma}(x)F_{\sigma\mu}(x)]. \qquad (50)
\end{aligned}
$$

This last quantity, when expressed in terms of electric and magnetic field intensities ($\mathbf{D}$ and $\mathbf{B}$) becomes identical with the result of Euler's calculation:[3]

$$
L(x) = -(2\alpha^2/45\hbar c\kappa^4)[(\mathbf{D}^2 - \mathbf{B}^2)^2 + 7(\mathbf{D}\cdot\mathbf{B})^2]. \qquad (50')
$$

It is a pleasure to thank Professor Oppenheimer for his encouragement and to acknowledge many enlightening discussions with Dr. Res Jost. One of us (M. N.) is indebted to the Institute for Advanced Study for having supported his research by a grant of AEC funds.