The Scattering of 100-Mev Electrons from a Heavy Nucleus*

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This paper describes and gives the results of a calculation of the scattering of 100-Mev electrons from a heavy nucleus with atomic number $Z=82.2$. For 100-Mev electrons the reduced de Broglie wave-length is about one-fourth of the radius of a lead nucleus and appreciable deviations from pure Coulomb scattering are to be expected. Three different models of the nucleus are considered. In the primary case, the nucleus is assumed to have a uniform charge distribution within its radius. The scattering for this model is calculated by the exact phase-shift method. In the two other models the effects on the scattering of varying the radius and non-uniform charge distributions are investigated. The last two calculations are carried out by perturbation on the first.

I. INTRODUCTION

IT is expected that high energy electron beams, of 100 Mev and higher, will soon be readily available from linear accelerators. It is the purpose of this paper to carry out some exploratory calculations to indicate the effectiveness of this tool in the investigation of the properties of the nucleus; in particular, the effects of the charge distribution within the nucleus and the radius of the nucleus are considered.

At the high energies the non-zero size of the nucleus will cause deviations from Coulomb scattering. Some calculations have already been performed along these lines by Rose.[1] However, these calculations were carried out using the Born approximation and it is known[2,3] that for heavy elements the Born approximation is not valid.

In this paper the exact phase-shift method as applied to the Dirac equation was used. The differential elastic cross section was calculated for the scattering of 100-Mev electrons from lead, $Z=82$ (more accurately, $Z=82.2$ was used in order to be able to use the results of Bartlett and Watson).

Three different cases were considered. In the primary case we assume the nucleus to have a uniform charge distribution within a radius a of the nucleus. The radius a was taken from the paper of Fernbach, Serber, and Taylor[4] and was in this case $a=8.09\times10^{-13}$ cm. For 100-Mev electrons, this radius gives a value of $ka=4.10$, k being the momentum divided by $\hbar$. That is, the de Broglie wave-length of the electron is about one-fourth of the radius of the nucleus. The scattering with this model of the nucleus was calculated exactly.

The scattering effects with two slightly different models of the nucleus were calculated by perturbation on the first case. These two variations were: (1) The radius was left unchanged, the charge density was allowed to increase by 43 percent in going from the center to the outer edge of the nucleus. This charge distribution is similar to that proposed by Feenberg[5] and was taken as $\rho=\rho_0(1+g(r/a)^2)$, where $g=0.43$. (2) The charge distribution was left uniform, but the radius was decreased by five percent. This was done to indicate the sensitivity of the scattering to the radius.

The potential due to a uniform charge distribution can be shown to be, for $r\leq a$,

$$V_0(r)=(Ze/2a)\{3-(r/a)^2\}. \tag{1}$$

For a charge distribution, $\rho=\rho_0[1+g(r/a)^2]$, the potential is

$$V_\rho(r)=\frac{Ze}{a}\left\{1+\frac{1}{2}\frac{10+3g}{10+6g}-\frac{1}{2}\frac{5}{5+3g}\left(\frac{r}{a}\right)^2-\frac{3}{4}\frac{g}{5+3g}\left(\frac{r}{a}\right)^4\right\} \tag{2}$$

and that due to a slightly decreased radius a', $a/a'=1+\epsilon$ where the density is still uniform,

$$\begin{aligned} &V_a(r)=(Ze/2a')\{3-(r/a')^2\}; && r\leq a' \\ &V_a(r)=Ze/r; && r\geq a'. \end{aligned} \tag{3}$$

The potential is very insensitive to the charge distribution, as can be seen by considering the two extreme cases of a uniform charge distribution and one in which the charge is concentrated on the surface.

Since for all charge distribution the value of the potential at $r=a$ is Ze/a, and the potential must increase monotonically as we go in toward the origin, a good way to compare the strengths of different potentials is to compare the value of the potentials at $r=0$, $V(0)$.

Thus for a uniform charge distribution, $V_0(0)=1.5Ze/a$. For a surface charge, $V(0)=Ze/a$. For a potential due to $\rho=\rho_0[1+g(r/a)^2]$ and taking $g=0.43$, $V_\rho(0)=1.45Ze/a$. This is only a change of five percent from $V_0(0)$ for the uniform charge distribution, although the density varies by almost 50 percent, showing the insensitivity of the potential to the charge distribution.

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[1] M. E. Rose, Phys. Rev. **73**, 279 (1948).
[2] Bartlett and Watson, Proc. Am. Acad. **74**, 53 (1940).
[3] W. McKinley and H. Feshbach, Phys. Rev. **74**, 1759 (1948).
[4] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).
[5] E. Feenberg, Phys. Rev. **59**, 593 (1941).

In general, we might note the useful relation which will give $V(0)$ directly from the charge distribution.

$$V(0)=4\pi\cdot\int_0^a \rho(r)\cdot r\cdot dr. \tag{4}$$

Thus if $\rho=\rho_0 u(r/a)$, then

$$V(0)=\frac{Ze}{a}\cdot\int_0^1 u(x)\cdot x\cdot dx\Big/\int_0^1 u(x)\cdot x^2\cdot dx. \tag{5}$$

II. METHOD OF CALCULATING CROSS SECTION FROM PHASE SHIFTS

If the scattering potential falls off at infinity like $-Ze^2/r$, then the radial part of the wave function will have the asymptotic form given by[6] I, Eq. (3),

$$g_l(r)\underset{r\to\infty}{\longrightarrow}\cos[kr+q\ln 2kr-\tfrac{1}{2}(l+1)\pi+\sigma_l] \tag{6}$$

where $q=Ze^2/\hbar v$, $v=$speed of electron. In terms of the phase shifts σ_l, the cross section $\sigma(\theta)$ for an unpolarized incident beam can be written as,[7]

$$\sigma(\theta)=|f_3(\theta)|^2+|f_4(\theta)|^2, \tag{7}$$

where

$$f_3(\theta)=(1/i2k)\sum_{l=0}\{(l+1)\exp(2i\sigma_l)+l\exp(2i\sigma_{-l-1})\}P_l(\cos\theta) \tag{8a}$$

and

$$f_4(\theta)=(1/i2k)\sum_{l=1}\{\exp(2i\sigma_{-l-1})-\exp(2i\sigma_l)\}P_l^1(\cos\theta) \tag{8b}$$

and

$$P_l^1=\sin\theta dp_l/d(\cos\theta).$$

In summing (8), which is a divergent series, one must first subtract out the scattering amplitude due to a pure Coulomb field. Let σ_l^c be the phase shifts for the Coulomb potential $-Ze^2/r$; then we can rewrite (8) as

$$f_3(\theta)=f_3^c(\theta)+(1/i2k)\sum_{l=0}\times\{(l+1)[\exp(2i\sigma_l)-\exp(2i\sigma_l^c)]+l[\exp(2i\sigma_{-l-1})-\exp(2i\sigma_{-l-1}^c)]\}P_l(\cos\theta) \tag{9a}$$

$$f_4(\theta)=f_4^c(\theta)+(1/i2k)\sum_{l=1}\times\{[\exp(2i\sigma_{-l-1})-\exp(2i\sigma_{-l-1}^c)]-[\exp(2i\sigma_l)-\exp(2i\sigma_l^c)]\}P_l^1(\cos\theta) \tag{9b}$$

where f_3^c, f_4^c are the Coulomb scattering amplitude.

The problem is now to calculate the phase shifts σ_l which appear in (4a) and (4b). This is done by matching at the point $r=a$, beyond which the potential is pure Coulomb. Thus we need to determine the regular solution of the wave equation for $r\leq a$, and both regular and irregular Coulomb wave functions for $r\geq a$.

The Coulomb wave functions were given by Gordon.[8] Instead of using the functions $g_l(r)$, $f_l(r)$, it is more convenient to use the function $S_l(r)$, defined by I, Eq. (35), and which has the asymptotic form, $S_l(r)\to\exp[i(kr+q\ln 2kr-\frac{1}{2}(l+1)\pi+\sigma_l)]$.

We may write the regular Coulomb wave function as

$$S_{l,R}=\exp(i\eta_l)\cdot\zeta_l(x;\rho_l) \tag{10}$$

where $x=kr$,

$$\zeta_l(x;\rho_l)=\frac{\Gamma(\rho_l+1+iq)}{\Gamma(2\rho_l+1)}\exp(\tfrac{1}{2}q\pi)\cdot\exp(i\tfrac{1}{2}\rho_l\pi)(2x)^{\rho_l}\cdot\exp(-ix)\cdot F(\rho_l+iq+1,\,2\rho_l+1;\,2ix), \tag{11}$$

and

$$\rho_l=[(l+1)^2-\alpha^2]^{\frac{1}{2}},\quad q=Ze^2/\hbar v,\quad \alpha=Ze^2/\hbar c.$$

$F(\alpha,\beta;x)$ is the confluent hypergeometric function[9] and

$$\exp(2i\eta_l)=\frac{l+1+iq'}{\rho_l-iq}\cdot\frac{\Gamma(\rho_l-iq)}{\Gamma(\rho_l+iq)}\cdot\exp(-i\pi\rho_l) \tag{12}$$

where $q'=(Ze^2/\hbar v)[1-(v/c)^2]^{\frac{1}{2}}$.

$\zeta_l(x;\rho_l)$ has the asymptotic form $\zeta_l\to\exp[i(kr+q\ln 2kr)]$. Thus $S_{l,R}\to\exp[i(kr+q\ln 2kr+\eta_l)]$ and so the Coulomb phase shifts are given by

$$\sigma_l^c=\eta_l+\tfrac{1}{2}(l+1)\pi. \tag{13}$$

The irregular solutions is obtained from (10) by simply replacing ρ_l by $-\rho_l$,

$$S_{l,I}=\exp(i\eta_l')\zeta_l(x;-\rho_l), \tag{14}$$

and

$$\exp(2i\eta_l')=\frac{l+1+iq'}{-\rho_l-iq}\frac{\Gamma(-\rho_l-iq)}{\Gamma(-\rho_l+iq)}\exp(i\pi\rho_l). \tag{15}$$

This procedure for obtaining the irregular solution breaks down in the non-relativistic theory since there $\rho_l=l+1$ as $\alpha\to 0$ and the hypergeometric function $F(\alpha,\beta;x)$ becomes infinite when β is a negative integer. Because of this there is no easily seen transition from some of our calculations to the corresponding non-relativistic calculations.

Thus, in contrast with the non-relativistic case η_l and η_l' do not differ by $\pi/2$, but approach each other for high l. In fact, we have after using the relation

$$\Gamma(-Z)=-\pi/\Gamma(Z+1)\sin\pi Z, \tag{16}$$

$$\exp[2i(\eta_l'-\eta_l)]=-\exp(2i\pi\rho_l)\times\sin\pi(\rho_l-iq)/\sin\pi(\rho_l+iq), \tag{17}$$

from which we can calculate $\eta_l'-\eta_l$.

[6] For some of the results used here the reader is referred to a previous paper, G. Parzen, Phys. Rev. **80**, 261 (1950), hereafter denoted by I.

[7] N. F. Mott, Proc. Roy. Soc. **A124**, 426 (1929).

[8] W. Gordon, Zeits. f. Physik **48**, 11 (1928).

[9] G. N. Watson, *Theory of Bessel Functions* (The Macmillan Company, New York, 1945), second edition, p. 100.

Having the above properties of the Coulomb functions, we can now match at $r=a$ to determine the phase shifts. Let $f_l(r)$, $g_l(r)$ be the regular solutions for $r\leq a$; let $f_{l,R}$, $g_{l,R}$ and $f_{l,I}$, $g_{l,I}$ be the regular and irregular Coulomb solutions.

Then at the point $r=a$,

$$\begin{aligned} f_l&=Cf_{l,R}+Df_{l,I}\\ g_l&=Cg_{l,R}+Dg_{l,I}. \end{aligned} \tag{18}$$

g_l would then have the asymptotic form,

$$\begin{aligned} g_l\rightarrow& C\cos(kr+q\ln 2kr+\eta_l)\\ &+D\cos(kr+q\ln 2kr+\eta_l')\\ &\rightarrow\text{const.}\cos(kr+q\ln 2kr+\eta_l+\delta_l) \end{aligned} \tag{19}$$

where

$$\tan\delta_l=\sin(\eta_l'-\eta_l)/[-(C/D)+\cos(\eta_l'-\eta_l)]. \tag{20}$$

From (18), we find for C/D,

$$C/D=-(g_{l,I}-\gamma_l f_{l,I})/(g_{l,R}-\gamma_l f_{l,R}),\quad \text{at}\quad r=a \tag{21}$$

where

$$\gamma_l=(g_l/f_l)\big|_{r=a}. \tag{22}$$

The above equations determine δ_l which is the phase shift in addition to the Coulomb phase shift. The phase shift defined in (6) is given then by

$$\sigma_l=\sigma_l^c+\delta_l. \tag{23}$$

III. COMPUTATIONAL PROCEDURE

The quantities to be calculated are the regular and irregular Coulomb functions and the interior regular solution, all at $r=a$. These functions are all calculated by finding a suitable series and summing it. Because the parameter involved in these series, which is ka, is of the order of 4, the series involved converge rather slowly and require from 12 to 20 terms to sum.

In calculating the Coulomb wave functions, the confluent hypergeometric function $F(\alpha,\beta;x)$ which appears converges too slowly requiring about 25 to 30 terms. It is more convenient to treat the function,[10]

$$H(\gamma,b;x)=\exp(-ix)F(\alpha,\gamma;2ix), \tag{24}$$

where $b=i(2\alpha-\gamma)$.

Since $F(\alpha,\gamma;u)$ satisfies the differential equation

$$\left(u\frac{d^2}{du^2}+(\gamma-u)\frac{d}{du}-\alpha\right)\cdot F=0, \tag{25}$$

then $H(\gamma,b;x)=\exp(-ix)\cdot F(\alpha,\gamma;2ix)$ satisfies the equation,

$$\left(x\frac{d^2}{dx^2}+\gamma\frac{d}{dx}+(x-b)\right)\cdot H=0 \tag{26}$$

where $b=i\cdot(2\alpha-\gamma)$.

[10] A. N. Lowen and W. Horenstein, J. Math. Phys. **21**, 264 (1942).

Note that if b is real, then H is a real function. This is the case in the non-relativistic theory. But in our case, $\gamma=2\rho_l+1$, $\alpha=\rho_l+iq+1$, so $2b=-q+i$.

Putting

$$H=\sum_{n=0}^{\infty}C_n\cdot x^n/n! \tag{27}$$

we find,

$$C_{n+1}=[1/(n+\gamma)]\{-nC_{n-1}+2bC_n\} \tag{28}$$

and $C_0=1$.

The C_n are complex. Writing $C_n=\alpha_n+i\beta_n$, we get the recurrence relations for the real quantities α_n, β_n,

$$\alpha_{n+1}=[1/(n+\gamma)]\{-n\alpha_{n-1}-q\alpha_n-\beta_n\}. \tag{29a}$$

$$\beta_{n+1}=[1/(n+\gamma)]\{-n\beta_{n-1}-q\beta_n+\alpha_n\}, \tag{29b}$$

and $\alpha_0=1$, $\beta_0=0$.

From (29) we can also show that we have the simple recurrence relation

$$\beta_{n+1}=(n+1)\alpha_n/(n+\gamma), \tag{30}$$

which can be used as a check.

It is only necessary to sum the series to calculate S_l; the function S_{-l-1} can be derived from S_l. For

$$S_{-l-1}=\exp(i\eta_{-l-1})\zeta_{-l-1}(x;\rho_{-l-1}), \tag{31}$$

and ζ_l depends on l only through $\rho_l=[(l+1)^2-\alpha^2]^{\frac{1}{2}}$.

Thus,

$$\zeta_{-l-2}=\zeta_l$$

and

$$S_{-l-2}=\exp[i(\eta_{-l-2}-\eta_l)]\cdot S_l. \tag{32}$$

Relation (32) holds for both regular and irregular Coulomb functions.

The solutions for $r\leq a$ are obtained by solving the differential equation for $S_l(r)$ by series. Thus, by **I**, Eq. (37),

$$\frac{d}{dx}S=\left(i-\frac{V}{k}\frac{E}{k}\right)S+\left(\frac{l+1}{x}-i\frac{V}{k}\cdot\frac{1}{k}\right)\bar{S}, \tag{33}$$

and for a uniform charge distribution we can write

$$VE/k^2=A-Bx^2, \tag{34}$$

where

$$A=-\tfrac{3}{2}(Ze/ka)\cdot(E/k)$$

and

$$B=\tfrac{1}{2}(Ze^2/ka)\cdot(E/k)\cdot 1/(ka)^2.$$

Putting

$$S=\sum_n a_n x^n,\quad \bar{S}=\sum_n \bar{a}_n x_n,$$

we get

$$\begin{aligned} a_m(m+s)=&i(1-A)a_{m-1}+iBa_{m-3}+(l+1)\bar{a}_m\\ &-iA'\bar{a}_{m-1}+iB'\bar{a}_{m-3} \end{aligned} \tag{35}$$

where $A'=A/E$, and $B'=B/E$;

$$s=l\quad\text{if}\quad l\geq 0,\quad s=-l\quad\text{if}\quad l<0.$$

Again writing $a_n=\alpha_n+i\beta_n$, we get for $l\geq 0$,

$$\alpha_n=-\frac{1-A+A'}{n}\beta_{n-1}-\frac{(B-B')}{n}\beta_{n-3}, \quad (36a)$$

$$\beta_n=\frac{1-A-A'}{n+2(l+1)}\cdot\alpha_{n-1}+\frac{B+B'}{n+2(l+1)}\alpha_{n-3}, \quad (36b)$$

and $\beta_0=0$, $\alpha_0=1$.

The ratio γ_l is given by,

$$\gamma_l=\left(\frac{E+1}{E-1}\right)^{\frac{1}{2}}\frac{\sum_n \alpha_n x^n}{\sum_n \beta_n x^n} \quad \text{at} \quad x=ka. \quad (37)$$

Also here it is not necessary to sum the series to determine γ_{-l-1}. Because the potential is an even function of x, one can show that for high enough energies

$$\gamma_{-l-1}'\cong>1/\gamma_l', \quad (38)$$

where

$$\gamma_l'=[(E-1)/(E+1)]^{\frac{1}{2}}\gamma_l$$

Relation (38) is not exactly true but numerical calculation of γ_{-1} and γ_{-2} showed it to be good to four figures.

IV. PERTURBATION METHOD

If the potential is changed somewhat in the region $r\leq a$, but remains $-Ze^2/r$ for r larger than a, then in the phase-shift method of the previous sections the only quantity which must be recalculated is γ_l.

To obtain an approximate method for calculating γ_l for a potential $V(r)$ which differs slightly from the uniform charge potential $V_0(r)$, consider the potential $U(r)$ constructed as follows:

$$\begin{aligned} U(r)&=V(r); \quad r\leq a \\ U(r)&=0; \quad r>a. \end{aligned} \quad (39)$$

If Eq. (33) is solved for $S_l(r)$, and thus f_l and g_l, then the ratio $\gamma_l=g_l/f_l$ at $r=a$ is the same for the potential $U(r)$ as for the potential $V(r)$, both potentials being identical for $r\leq a$.

Now the potential $U(r)$ will cause a certain phase shift which is connected to γ_l by the relation **I**, Eq. (64),

$$\gamma_l=\left(\frac{E+1}{E-1}\right)^{\frac{1}{2}}\frac{j_l(ka)-\tan\delta_l\cdot n_l(ka)}{j_{l+1}(ka)-\tan\delta_l\cdot n_{l+1}(ka)}. \quad (40)$$

Thus to determine the γ_l for the potential $V(r)$, one can equally well determine the phase shifts δ_l of the potential $U(r)$. The phase shift δ_l of $U(r)$ can be approximated by means of the variational result of paper **I**. There we showed that, **I**, Eq. (60),

$$\delta_l-\delta_l\cong-\frac{E+1}{2k}\cdot\int_0^\infty 2(V-V_t)\cdot\left\{\frac{g_t^2}{E+1}+\left(\frac{k}{E+1}\right)^2\frac{f_t^2}{E-1}\right\}dr. \quad (41)$$

Here we can choose for the trial potential $V_t(r)$, the potential $V_0(r)$ due to a uniform charge distribution. Further, instead of the wave functions of $V_0(r)$, we can put in (41), as a good approximation, the wave functions of a free particle. The integral then only involves Bessel functions and can be evaluated more easily. Thus (41) becomes

$$\delta_l-\delta_l{}^0\cong\tfrac{1}{2}(E+1)\{\Delta_l+[k/(E+1)]^2\cdot\Delta_{l+1}\}, \quad (42)$$

and

$$\Delta_l=-k\int_0^\infty 2\cdot(V-V_0)\cdot j_l^2(kr)\cdot r^2dr.$$

The application of (42) requires the evaluation of integrals of the form

$$k_{n,m}=\int_0^x t^m J_n{}^2(t)dt. \quad (43)$$

In some cases, for m even, the indefinite integral can be done. However, it is generally simpler to use a recursion formula which will be developed now. This will give $k_{n+1,m}$ in terms of $k_{n,m}$.

Using the Bessel function relation,

$$J_n(t)=(t/2n)(J_{n-1}+J_{n+1}),$$

we get from (43)

$$k_{n,m}=(1/2n)\{L_{n,m}+L_{n+1,m}\}, \quad (44)$$

where

$$L_{n,m}=\int_0^x t^{m+1}\cdot J_n(t)\cdot J_{n-1}(t)\cdot dt. \quad (45)$$

Now, using the relation $dJ_n/dt=\frac{1}{2}\{J_{n-1}-J_{n+1}\}$ we get for $L_{n,m}$ from (45)

$$L_{n,m}=L_{n+1,m}+\int_0^x t^{m+1}(dJ_m{}^2/dt)dt. \quad (46)$$

Integration by parts yields

$$L_{n,m}=L_{n+1,m}+x^{m+1}J_n{}^2(x)-(m+1)k_{n,m}. \quad (47)$$

Comparison of the expressions for $k_{n,m}$ as obtained from (47) and (44) gives the recursion formula for $L_{n,m}$,

$$L_{n+1,m}=\frac{2n+m+1}{2n-m-1}L_{n,m}-\frac{2n}{2n-m-1}x^m\cdot J_n{}^2(x). \quad (48)$$

Relation (48) together with (44) gives a recursion relation for $k_{n,m}$.

The recursion chain may be interrupted if we happen to arrive at a value of n for which $2n=m+1$.

The integral (45) for low values of n can usually be calculated easily in our case since we have Bessel functions of half-odd integer orders which involve only sines and cosines. For higher n, the recursion formula can be applied. The one disadvantage in the above procedure is that cancellation occurs in applying the

TABLE I. Phase shifts for the scattering of 100-Mev electrons from a lead nucleus with a uniform charge distribution. δ_l is the phase shift with the Coulomb phase shift subtracted.

l	$\delta_l=\sigma_l-\sigma_l^c$	$\delta_{-l-1}=\sigma_{-l-1}-\sigma_{-l-1}^c$
0	$-.5339$	
1	$.2064-\pi$	$-.5307$
2	$.2507-\pi$	$.2091-\pi$
3	$.0678-\pi$	$.2572-\pi$
4	$.0115-\pi$	$.0711-\pi$
5	$.0013-\pi$	$.0124-\pi$
6	$.0001-\pi$	$.0015-\pi$
7		$.0001-\pi$

recursion formula and one must start out with a great many figures in order to get a significant answer for large values of n.

V. DISCUSSION OF RESULTS

The behavior of the phase shifts at high energies is quite different from that met in non-relativistic scattering. As was pointed out in paper **I**, the phase shift δ_l, for a given l, does not approach zero when $k\rightarrow\infty$. By a slight modification of the argument presented in *I*, we can show that for the potentials considered in this paper, which go like $1/r$ at infinity but have no pole at the origin, δ_l approaches the value,

$$\delta_l \underset{k\rightarrow\infty}{\longrightarrow} -\frac{1}{\hbar c}\int_0^a V(r)\cdot dr-\frac{Ze^2}{\hbar c}\ln ka-\sigma_l. \qquad (49)$$

In the case of the uniform charge distribution, this gives an asymptotic value of $\delta_0=-0.91$. Our calculated value of $\delta_0=-0.534$ agrees favorably with this as the deviation of δ_0 from its asymptotic value should be of the order of $1/ka=0.244$.

Our curve, Fig. 1, for the scattering due to a uniformly charged nucleus agrees qualitatively with what one would expect from the Born approximation, though it differs greatly quantitatively. The two minima in the curve should be at 66° and 142° according to the Born approximation but are actually at about 70° and 120°. The only feature not indicated by the Born approximation is the increase in the forward scattering as compared with pure Coulomb scattering at the angles of 15° and 30°. This increased forward scattering is due to the constructive interference of the higher order partial waves and would be a small effect at lower energies where the higher order phase shifts are small.

The effects of varying the density or the radius of the nucleus are almost identical although, for the cases chosen here, in opposite directions. Thus these two effects will tend to mask each other. However, the probable deviation of the charge distribution of the nucleus from a uniform one is equivalent to only a five percent change in the radius in affecting the scattering. The scattering at the larger angles, where the diffraction effects are most marked, is particularly sensitive to the above two factors. The large percentage deviation

TABLE II. Differential cross sections as a function of angle in units of $1/k^2=3.889\times10^{-26}$ cm². σ_c is the pure Coulomb cross section. σ_0 is the cross section in the case of a uniformly charged nucleus; σ_ρ refers to the case in which the density is not uniform; σ_a refers to the case in which the radius is decreased slightly but the charge density is uniform. The results at the angles 75°, 105°, and 135° are not as accurate as the results at the other angles since the Coulomb scattering amplitude at these angles were gotten by interpolation.

θ	σ_c	σ_0	$(\sigma_a-\sigma_0)/\sigma_0$ in %	$(\sigma_\rho-\sigma_0)/\sigma_0$ in %
0			0	0
15	347.8	420.8	0.68	−1.1
30	27.39	39.78	7.1	−7.8
45	7.001	5.421	14	−12
60	2.768	0.576	17	−14
75	1.3	0.40		
90	0.710	0.541	18	−21
105	0.37	0.28		
120	0.207	0.046	68	−45
135	0.10	0.069		
150	0.042	0.062	19	−25

shown in Table II at 120° occurs there since 120° happens to be the position of the second minimum of our scattering curve.

Some discussion of other quantities which will affect the scattering is given in the paper of Rose.[1] The screening of the orbital electrons would not be expected to have any effect except at small angles. Actual numerical calculations were carried out by Bartlett and Welton.[11] For an element like mercury; they found the screening to be important at less than 60° for 100-kev electrons,

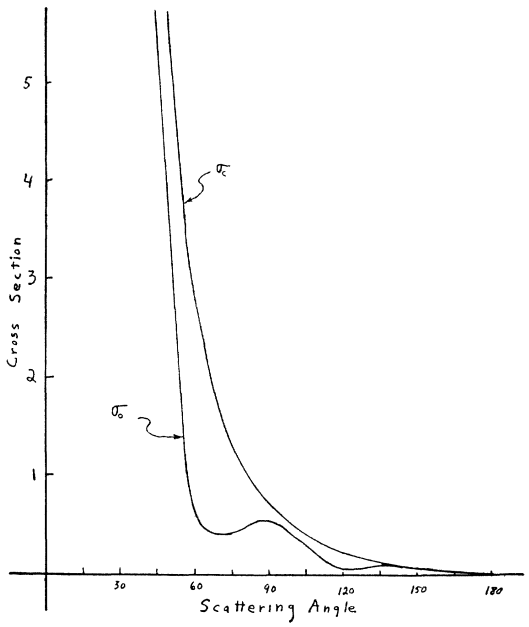

FIG. 1. The scattering of 100-Mev electrons from a lead nucleus ($Z=82.2$). σ_0 is the scattering with a uniform charge distribution. σ_c is the scattering with a pure Coulomb field. Units are $1/k^2=3.889\times10^{-26}$ cm².

[11] J. H. Bartlett, Jr. and T. A. Welton, Phys. Rev. **59**, 281 (1941).

and at less than 15° for 230-kev electrons; at 100 Mev we can expect the angle within which screening is important to be very small.

Radiative effects in the scattering as predicted by quantum electrodynamics may be large enough to mask the effect due to non-uniform charge density or to a slight change in the radius of the nucleus. Calculations for a pure Coulomb field based on Schwinger's[12] results would indicate this to be the case. However, since these results are based on the Born approximation, and give the deviation due to radiative effects from the Born approximation expression for the scattering, not from the actual scattering, their reliability in the case of heavy elements is doubtful. The order of magnitude given by them is probably still correct.

In our calculation, where we have assumed a continuous charge distribution within the nucleus, we have obtained a result which should approximate rather well the coherent elastic scattering of the electrons by the Z protons in the nucleus. The incoherent scattering[13] by the individual protons we expect to be much smaller than the coherent scattering by the nucleus as a whole. If we think of each proton as contributing to the entire scattering amplitude, which is proportional to Ze^2, an amount proportional to e^2, then the coherent scattering will be of the order of Z times larger than the incoherent scattering.

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The Disintegration of Ti^{45}

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The positron and photo-electron spectra of 3.05-hr. Ti^{45} have been studied. At least 96 percent of the positron transitions are to the ground state of Sc^{45} with a maximum positron kinetic energy of 1.022 ± 0.01 Mev. Less than four percent of the positron transitions are to an excited level of Sc^{45} at 450 kev. In addition, certain longer-lived activities are found in the chemically separated titanium fraction produced by both deuteron and proton bombardment of scandium oxide. Two of the activities are shown to be P^{32} and Sc^{46} impurities.

I. INTRODUCTION

BOTH deuteron and proton bombardment of scandium have been reported[1] to result in a radioactive isotope of titanium having a half-life of 3.05 hr. This activity has been assigned to the isotope Ti^{45}. Its positron spectrum, as obtained from a histogram resulting from cloud-chamber studies, has indicated an end-point energy of about 1.2 Mev.

Subsequent work[2] has confirmed the half-life originally reported for this isotope.

In the most recent table of radioactive isotopes by Seaborg and Perlman[3] a private communication is quoted from Dessauer indicating the presence of a second longer-lived isotope of titanium which can be produced by proton bombardment of scandium, and which is also assigned to the isotope Ti^{45}. In further correspondence with the present authors, Dessauer[4] has indicated that the 21-day half-life reported in the compilation of Seaborg and Perlman[3] is a typographical error. What he actually found in the titanium fraction after irradiating scandium with 7-Mev protons were two half-lives, 3.1 hr. and 3.1 days. At some time the 3.1 days was erroneously copied as 21 days.

It would thus appear that a beta-spectrometer study of the radioactive isotopes of titanium produced by either deuteron or proton bombardment of scandium could perhaps lead to interesting results.

In addition to the study of the disintegration schemes, the shape of the positron spectrum can also be of interest. Using the half-life and energy values reported in the literature Konopinski[5] has calculated a ft-value for the 3.05-hr. Ti^{45} corresponding to an allowed transition. Most recent evidence [6–9] indicates that simple allowed spectra should produce linear Fermi-Kurie plots. Since no gamma-rays had been reported,[1] it was

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