The Electrical Conductivity of an Ionized Gas*

ROBERT S. COHEN**
Sloane Physics Laboratory, Yale University, New Haven, Connecticut

AND

LYMAN SPITZER, JR., AND PAUL MCR. ROUTLY
Princeton University Observatory, Princeton, New Jersey

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The interaction term in the Boltzmann equation for an ionized gas is expressed as the sum of two terms: a term of the usual form for close encounters and a diffusion term for distant encounters. Since distant encounters, producing small deflections, are more important than close encounters, consideration of only the diffusion term gives a reasonably good approximation in most cases and approaches exactness as the temperature increases or the density decreases. It is shown that in evaluating the coefficients in this diffusion term, the integral must be cut off at the Debye shielding distance, not at the mean interionic distance.

The integro-differential equation obtained with the use of this diffusion term permits a more precise solution of the Boltzmann equation than is feasible with the Chapman-Cowling theory. While one pair of coefficients in this equation has been neglected, the remaining coefficients have all been evaluated, and the resultant equation solved numerically for the velocity distribution function in a gas of electrons and singly ionized atoms subject to a weak electrical field. Special techniques were required for this numerical integration, since solutions of the differential equation proved to be unstable in both directions. For high temperatures and low densities the computed electrical conductivity is about 60 percent of the value given by Cowling's second approximation.

INTRODUCTION

QUANTITATIVE analyses of non-uniform gases have naturally been developed along lines relevant to laboratory experiments. The theory of Enskog and of Chapman, systematically expounded by Chapman and Cowling,[1] is primarily concerned with the properties of gases composed predominantly of neutral atoms. While this theory has been applied[1–3] to the conductivity of a completely ionized gas (a gas containing no neutral atoms), the theory is in fact not well suited to handle inverse-square forces between the particles in a gas, and the accuracy of the results obtained is uncertain. In view of the great astrophysical importance of completely ionized gases, as, for example, in stellar interiors, stellar envelopes, and interstellar matter, a reconsideration of this problem has been undertaken.

A new approach to this subject is provided through the work of Chandrasekhar[4] on stellar dynamics. This work is based on the fact that when particles interact according to inverse-square forces, the velocity distribution function is affected primarily by the many small deflections produced by relatively distant encounters. There will be many such encounters during the time a particle travels over its mean free path, and the change in the particle velocity can be computed in the same way as is the change of the position of a particle in Brownian motion. On the assumption that the large deflections produced by the relatively close encounters may be neglected, Chandrasekhar therefore employs a diffusion equation for the velocity distribution function, similar to the equations describing the spatial distribution function in Brownian motion. A similar but incomplete approach was made somewhat earlier by Landau.[5] As we shall see below, the appropriate generalized diffusion equation may be solved numerically when a completely ionized gas is subject to a small electric field or a small temperature gradient.

In Part I, prepared by L. Spitzer, the basic principles of the present paper are developed. Part II, prepared by R. S. Cohen, applies this analysis to a singly ionized gas in a weak electric field, and evaluates certain coefficients in the appropriate integro-differential equation. In Part III, prepared by P. McR. Routly, the numerical solution of the resultant equation is briefly summarized. The final formulas for the electrical conductivity are given in Part IV.

I. GENERAL PRINCIPLES

The velocity distribution function f_r for particles of type r, interacting with particles of different types s, is determined by Boltzmann's equation (reference 1, Eq. (8.1_1))

$$\frac{\partial f_r}{\partial t}+\sum_i v_{ri}\frac{\partial f_r}{\partial x_i}+\sum_i F_{ri}\frac{\partial f_r}{\partial v_{ri}}=\sum_s\left(\frac{\partial_e f_r}{\partial t}\right)_s, \qquad (1)$$

where the notation is similar to that used by Chapman

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[1] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1939).

[2] T. G. Cowling, Proc. Roy. Soc. **A, 183**, 453 (1945).

[3] R. Landshoff, Phys. Rev. **76**, 904 (1949).

[4] S. Chandrasekhar, Astrophys. J. **97**, 255, 263 (1943).

[5] L. Landau, Physik Zeits. Sowjetunion **10**, 154 (1936). In this reference, the important terms representing dynamical friction, which should appear in the diffusion equation, are set equal to zero as a result of certain approximations.

and Cowling, except that v_{ri} is used for the component of velocity of an rth particle in direction i; it should be noted that F_r is the force per unit mass on a particle of type r. The quantity $(\partial_e f_r/\partial t)_s$ gives the change in f_r produced by encounters of r particles with particles of type s.

1. Evaluation of $\partial_e f_r/\partial t$ for Inverse-Square Forces

In the classical theory of non-uniform gases, the assumption is made that only the relatively close encounters are important, and that the forces between particles at greater distances have no effect. On this basis, $(\partial_e f_r/\partial t)dv_x dv_y dv_z$ is the sum of two terms, one representing the number of encounters which place particles of type r in the volume element of velocity space $dv_x dv_y dv_z$, the other, the number taking particles out of this volume element. Thus we have (reference 1, Sections 3.5 and 3.52)

$$(\partial_e f_r/\partial t)_s = \int\int\int (f_r' f_s' - f_r f_s) g b db d\epsilon d\mathbf{v}_s, \qquad (2)$$

where g is the relative velocity $|\mathbf{v}_r - \mathbf{v}_s|$ of the two types of particle before the encounter, b is the so-called impact parameter,—the distance of closest approach if no interaction forces were present,—and ϵ is the angle between the orbital plane and the plane containing the velocities of the two particles before encounter. The corresponding quantities in Chandrasekhar's[6] analysis of stellar dynamics are V, D, and Θ. The quantities f_r' and f_s' are the values of f_r and f_s for velocities such that a particle of type r will be left after the encounter within the volume element $dv_x dv_y dv_z$.

When the force between two particles varies as the inverse square of their mutual separation, Eq. (2) is no longer appropriate. As has been shown by Jeans,[7] the cumulative effect of the weak deflections resulting from the relatively distant encounters is more important than the effect of occasional large deflections (relatively close encounters). To illustrate this effect, one may compute the cumulative squared value of the deflection angle χ produced during the time Δt by all those encounters for which the impact parameter b is less than some upper limit b_1. For collisions of electrons with heavy ions, whose space density is n_i per cubic centimeter, it may be shown that

$$\langle \sum_{b<b_1} \sin^2\chi \rangle_{Av} = 4\pi g n_i b_0^2 \Delta t \left\{ \ln\left(1+\frac{b_1^2}{b_0^2}\right) - \frac{1}{1+b_0^2/b_1^2} \right\}, \qquad (3)$$

where b_0 is the value of b for which χ equals $\pi/2$, and is given by

$$b_0 = Z_i e^2/g^2 m, \qquad (4)$$

where m is the electron mass. Values computed from Eq. (3) are given in Table I. It is evident from Table I that the relatively distant encounters outweigh the closer ones. For encounters between charged particles of comparable masses, the formula for $\sin^2\chi$, considered by Chandrasekhar,[6] becomes much more complicated, but the general behavior shown in Table I is not altered.

TABLE I. Cumulative mean-square deflection produced by encounters with $b<b_1$.

Impact parameter b_1/b_0	0	1	2	4	10	10^2	10^4	10^8
Mean-square deflection in arbitrary units	0.00	0.19	0.81	1.89	3.63	8.21	17.4	35.8

While Eq. (2) could possibly be salvaged in this case, this equation is not appropriate for inverse-square forces, and obscures the true physical situation. When $\partial_e f_r/\partial t$ is produced by many small deflections, the total deflection produced in an interval of time is similar to the total distance travelled by a particle in Brownian motion, and the change of f_r by such small collisions is described by a diffusion equation of the Fokker-Planck type.[8] In fact, the value of $\partial_e f_r/\partial t$ resulting from the relatively distant encounters depends almost entirely on the first and second derivatives of f_r, not on values of f_r over the entire range of velocities.

If occasional large deflections were entirely negligible, $\partial_e f_r/\partial t$ would be entirely given by a diffusion equation. Actually, Table I shows this is not the case. If we define as a close encounter one for which b is less than some critical value b_c, then the error introduced by the neglect of these encounters will be appreciable for low values of b_m/b_0 but will gradually decrease as b_m/b_0 increases.

The effects produced by the close encounters are best described by an equation of the form (2), with b integrated only up to b_c. The relatively distant encounters are best described by the Fokker-Planck equation.[8] Thus we have finally

$$(\partial_e f_r/\partial t)_s = -J(f_r f_s) - K(f_r f_s), \qquad (5)$$

with

$$J(f_r f_s) \equiv \int_{\mathbf{v}_s=0}^{\infty}\int_{\epsilon=0}^{2\pi}\int_{b=0}^{b_c} (f_r f_s - f_r' f_s') g b db d\epsilon d\mathbf{v}_s, \qquad (6)$$

and

$$K(f_r f_s) \equiv \sum_i \frac{\partial}{\partial v_i}(f_r\langle \Delta v_{i,s}\rangle) - \tfrac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial v_i \partial v_j}(f_r\langle \Delta v_{i,s}\Delta v_{j,s}\rangle), \qquad (7)$$

where, in general, for any quantity x, $\langle x_s\rangle$ is defined by

$$\langle x_s\rangle = \int_0^{\infty} g f_s dv_s \int_0^{2\pi} d\epsilon \int_{b_c}^{b_m} x b db; \qquad (8)$$

evidently $\langle x_s\rangle dt$ represents the mean value of x resulting

[6] S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago, Illinois, 1942), Chapter II.

[7] J. H. Jeans, *Astronomy and Cosmogony* (Cambridge University Press, London, 1929), p. 318.

[8] A thorough survey of such processes has been given by S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).

from all encounters with particles of type s during the time interval dt. The definition of $J(f_r f_s)$ differs from that given by Chapman and Cowling (reference 1, Eq. 7.11_2) in that only the close encounters are considered, the effect of the distant encounters entering into $K(f_r f_s)$.

The terms of third and higher order in Δv have been neglected in Eq. (7). It is readily shown that these terms are relatively small if b_c much exceeds b_0. Thus Eqs. (5), (6), and (7) should give moderately high accuracy in the evaluation of $\partial_e f_r/\partial t$.

To obtain the velocity distribution function in a first approximation $J(f_r f_s)$ may be neglected. This will involve an appreciable error, as shown in Table I, but should at least provide a considerably more accurate determination of f_r than has hitherto been available. It is probably best in this case not to neglect the close encounters altogether, and we shall therefore let b_c equal zero in the computation of $K(f_r f_s)$. To obtain a higher approximation, a finite b_c could be retained, and $J(f_r f_s)$ could be introduced as a small perturbation to the solution found below.

Accordingly, we derive an equation for f_r on the assumption that $(\partial_e f_r/\partial t)_s$ in Eq. (1) may be set equal to $-K(f_r f_s)$. We shall assume that the gas is in a steady state with no systematic motion, but with an electrical field $\mathbf{E}$ and a temperature gradient ∇T. Following Chapman and Cowling (reference 1, Section 7.1), we shall write

$$f_r = f_r^{(0)} + f_r^{(1)}, \tag{9}$$

where $f_r^{(0)}$ is the Maxwellian velocity distribution function, and obtain, finally (by use of expressions $(8.3_{8,10})$ of reference 1),

$$f_r^{(0)}\left\{\frac{m_r v_r^2}{2kT} - \frac{5}{2}\right\}\sum_i v_{ri}\frac{\partial T}{T\partial x_i} - f_r^{(0)}\frac{eZ_r}{kT}\sum_i E_i v_{ri} + \sum_s K(f_r^{(1)} f_s^{(0)}) + \sum_s K(f_r^{(0)} f_s^{(1)}) = 0. \tag{10}$$

The electrical field E_i and the electrostatic charge e are in e.s.u.; thus for electrons Z_r is -1. Quantities involving the square of $f^{(1)}$ have been neglected in Eq. (10). When Eq. (10) is applied to an electron gas, we shall omit all subscripts from quantities referring to electrons, such as $f^{(0)}$, $f^{(1)}$, m, and v.

2. The Cut-Off Parameter b_m

The quantities $\langle \Delta v_{i,s} \rangle$ and $\langle \Delta v_{i,s} \Delta v_{j,s} \rangle$ in Eq. (7) are expressible in terms of integrals over encounters with different values of b. As is well known, these integrals diverge logarithmically, and the integration must be terminated at some maximum b_m to give a finite result. According to Cowling,[2] Chandrasekhar,[4] Spitzer[9] and others, b_m should be set equal to the interionic distance, but according to Persico,[10] Landau[5] and others, b_m should equal the Debye distance, h, at which the electron-ion plasma shields any particular charge; this cut-off has also been discussed and used by Bohm and Aller.[11] For a gas composed of electrons with a particle density n_e and of ions with an average charge $Z_i e$

$$h^2 = kT/4\pi n_e e^2(1+Z_i). \tag{11}$$

The factor $1+Z_i$ in the denominator takes into account shielding by heavy ions as well as by electrons.

We shall show that h is almost certainly the proper cut-off distance. Let us consider the mean square value of the velocity change Δv for a single electron during the time Δt. We shall consider that the electrons and ions all move in straight lines, a legitimate assumption for the distant encounters. Under these assumptions we shall then show that for a particle initially at rest $\langle (\Delta v)^2 \rangle$ is given by an equation of the form (3), even for values of b arbitrarily large compared to the interionic distance. It follows that some further assumption is needed to give a finite result, and the introduction of shielding gives h as the natural value of b_m to be used.

To obtain this result we consider the general statistics of the electrostatic field, a subject similar to that already treated by Chandrasekhar and von Neumann.[12] If Δv is the change in velocity experienced by an electron of charge $-e$ and mass m during the time interval Δt, then

$$(\Delta \mathbf{v})^2 = (e^2/m^2)\int_0^{\Delta t}\int \mathbf{E}(t)\cdot\mathbf{E}(t')\,dt\,dt' \tag{12}$$

where $\mathbf{E}(t)$ is the electrical field acting on the particle. If now Δt becomes large and both sides are averaged over all complexions of the gas, the integrand of Eq. (12) is seen to involve the autocorrelation coefficient of the electrical field $\mathbf{E}(t)$. If each electron is assumed to move in a straight line, this autocorrelation coefficient can be evaluated exactly; averaging also over a Maxwellian velocity distribution for the electrons, we have, in an obvious notation,

$$(\mathbf{E}(t)\cdot\mathbf{E}(t+\tau))_{\mathrm{Av}} = 8\pi^{\frac{1}{2}} n_e e^2 j/|\tau|, \tag{13}$$

where n_e is the particle density of electrons, and where

$$j^2 \equiv m/2kT. \tag{14}$$

Finally we obtain for Eq. (12)

$$\langle (\Delta v)^2 \rangle = (16\pi^{\frac{1}{2}} n_e e^4 j/m^2)\ln(\tau_2/\tau_1). \tag{15}$$

Since the integral of the autocorrelation coefficient over $d\tau$ diverges at both limits of integration, we have replaced the limits 0 and ∞ by τ_1 and τ_2, respectively. Apart from the argument of the logarithm, Eq. (15) agrees exactly with the $\langle (\Delta v)^2 \rangle$ found by Chandrasekhar[13]

[9] L. Spitzer, Jr., M. N. R. A. S. **100**, 396 (1940).
[10] E. Persico, M. N. R. A. S. **86**, 294 (1926).
[11] D. Bohm and L. H. Aller, Astrophys. J. **105**, 131 (1947).
[12] S. Chandrasekhar and J. von Neumann, Astrophys. J. **95**, 489 (1941); **97**, 1 (1943).
[13] Reference 6. The sum of the two Eqs. (5.724) gives $\langle (\Delta v)^2 \rangle dt$; $(G+H)/v_2$ equals $2j/\pi^{\frac{1}{2}}$ when v_2 is small.

for a particle at rest. It is evident from the derivation that Eq. (15) is valid for particles whose distance of closest approach exceeds the interionic distance, and the correct upper limit for τ must be about the period of a plasma oscillation. The divergence for low τ_1 is the natural result of extending the assumption of straight-line motion to the very close encounters; evidently τ_1 should be about b_0/v.

A simple physical argument shows the reasonableness of this result. The effects produced by electrons beyond the interionic distance may be attributed to statistical fluctuations in the electron density; i.e., the electron density is greater on one side than on the other. For electrons outside a sphere of radius r, the effective number of electrons which contribute to the force at the center of the sphere will be proportional to $n_e r^3$, and the fluctuations in this number will vary as $n_e^{\frac{1}{2}} r^{\frac{3}{2}}$, yielding a net electrical field at the center which is proportional to $e n_e^{\frac{1}{2}} r^{-\frac{1}{2}}$, but which is random in direction. This field will not change appreciably in direction for a time somewhat less than r/v, but after a time somewhat greater than r/v will be in a new random direction. Greater fields last a shorter time, while fields produced by more distant electrons are smaller. Thus this field from electrons outside a sphere of radius r will be primarily responsible for the value of the autocorrelation coefficient of $\mathbf{E}$ when t equals r/v. It follows that this autocorrelation coefficient is proportional to $(e n_e^{\frac{1}{2}} r^{-\frac{1}{2}})^2$ or to $e^2 n_e/vt$, in agreement with Eq. (13).

While the proof has been carried through only for electrons at rest, it seems most improbable that the results will be qualitatively different for electrons in motion. We shall therefore set b_m equal to h in all integrals for the diffusion coefficients, and these coefficients will then contain as a factor $\ln(h/b_0)$. While the quantity h/b_0 will in fact vary with velocity, the effect of such variations is no greater than that of other neglected terms. For the interaction of electrons with particles of average charge $Z_i e$ we may write

$$\ln(h/b_0)=\ln(qC^2) \tag{16}$$

where C^2 is the mean square electron velocity and

$$q\equiv(m/2e^3Z_i)[kT/\pi n_e(1+Z_i)]^{\frac{1}{2}}. \tag{17}$$

At high temperatures one must consider the wave character of the electrons, an effect pointed out in this connection by Marshak.[14] An electron passing through a circular aperture of radius a will be spread out by diffraction through angles of about $\lambda/2\pi a$, where λ is the electron wave-length h/mv. If this angle exceeds the classical deflection angle for an electron passing by at a distance a from an ion of charge $Z_i e$, the deflections produced by the most distant encounters will be materially increased. The ratio of the quantum mechanical to the classical deflection is $2Z_i\alpha c/v$, where α is the fine-structure constant 1/137. If Z_i is unity, this ratio equals one for a velocity of 4.4×10^8 cm/sec., corresponding to an electron temperature of about 4×10^5 degrees. For lower temperatures one may conclude that the classical formulas are valid. The corrections introduced by quantum mechanics will not be large except at temperatures substantially above 10^6 degrees.

[14] R. E. Marshak, Ann. N. Y. Acad. Sci. **41**, 49 (1941).

3. Diffusion Equation in Spherical Coordinates

Before Eq. (10) can be solved, $K(ff)$ must be expressed in spherical coordinates. The derivation of the Fokker-Planck equation by Chandrasekhar is readily carried over to the spherical case, provided we substitute for the particle density f the quantity h, defined by

$$h(\mathbf{v})\equiv f(\mathbf{v},t)v^2\sin\theta. \tag{18}$$

By standard methods it follows that

$$\begin{aligned}K(ff)&=-[\partial_e f(\mathbf{v},t)]/\partial t\\&=\frac{1}{v^2\sin\theta}\Big\{\sum_i\frac{\partial}{\partial x_i}(h(\mathbf{v})\langle x_i\rangle)\\&\qquad-\tfrac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(h(\mathbf{v})\langle x_ix_j\rangle)\Big\},\end{aligned} \tag{19}$$

where x_i stands for the changes in the three coordinates; i.e., for Δv, $\Delta\theta$, and $\Delta\phi$ as i goes from 1 to 3.

To apply Eq. (19), the coefficients $\langle x_i\rangle$ must be expressed in terms of the velocity shifts in rectangular coordinates, since it is these which can be evaluated by the theory of binary encounters. We consider rectangular axes ξ, η and ζ, where ξ is in the direction of the velocity $\mathbf{v}$ before the encounter, while η and ζ are in the directions of increasing θ and ϕ, respectively. Thus the η, ζ axes are tangent to the circles of constant θ and constant ϕ at the point $\mathbf{v}$. We let Δ_ξ, Δ_η, and Δ_ζ represent the velocity displacements in these local rectangular coordinates.

The complicated general relations between Δv, $\Delta\theta$, and $\Delta\phi$ on the one hand and Δ_ξ, Δ_η, and Δ_ζ on the other simplify when these quantities are averaged over all collisions. When a small electrical field is present, for example, then (reference 1, Section 8.31) $f^{(1)}(v)$ varies as $\cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{E}$. In such a case it is evident from the symmetry of the problem that the coefficients $\langle\Delta_\zeta\rangle$, $\langle\Delta_\zeta\Delta_\xi\rangle$ and $\langle\Delta_\zeta\Delta_\eta\rangle$ all vanish.

We shall also show that under these conditions $\langle\Delta_\zeta^2\rangle$ equals $\langle\Delta_\eta^2\rangle$. Consider encounters between particles of velocity $\mathbf{v}$ and those of velocity $\mathbf{v}_1$. We shall keep the angle between these velocities fixed, but shall vary the angle Θ between the fundamental plane, containing $\mathbf{v}$ and $\mathbf{v}_1$, and the ξ, η plane fixed by the direction of $\mathbf{v}$ and $\mathbf{E}$. Let Δ_F be the change of $\mathbf{v}$ perpendicular to the original value of $\mathbf{v}$ and lying in the fundamental plane, and let Δ_G be the change of $\mathbf{v}$ perpendicular to the

fundamental plane. Evidently,

$$\Delta_\eta=\Delta_F\cos\Theta+\Delta_G\sin\Theta \tag{20}$$

$$\Delta_\xi=\Delta_F\sin\Theta-\Delta_G\cos\Theta. \tag{21}$$

As Θ varies, Δ_F and Δ_G per encounter remain unchanged since the relative velocity g, the impact parameter b, etc., are all unaffected. It follows that $\langle\Delta\eta^2\rangle$ will equal $\langle\Delta_\xi^2\rangle$ provided that $\langle\sin2\Theta\rangle$ and $\langle\cos2\Theta\rangle$ are both zero. Since $f^{(1)}$ will have components varying only as $\cos\Theta$, these averages are in fact zero, and the result follows. Similar results hold when a small temperature gradient is present.

The relations we require then reduce to the simple form

$$\begin{aligned}
\langle\Delta v\rangle&=\langle\Delta_\xi\rangle+(\langle\Delta_\eta^2\rangle/v),\\
\langle\Delta\theta\rangle&=\frac{\langle\Delta_\eta\rangle}{v}-\frac{\langle\Delta_\xi\Delta_\eta\rangle}{v^2}+\frac{\cot\theta}{2v^2}\langle\Delta_\eta^2\rangle,\\
\langle(\Delta v)^2\rangle&=\langle\Delta_\xi^2\rangle,\\
\langle(\Delta v)(\Delta\theta)\rangle&=\langle\Delta_\xi\Delta_\eta\rangle/v,\\
\langle(\Delta\theta)^2\rangle&=\langle\Delta_\eta^2\rangle/v^2.
\end{aligned} \tag{22}$$

If we substitute relations (22) into Eq. (19), we have, after some rearrangement,

$$\begin{aligned}
K(ff)=&\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2f\left\{\langle\Delta_\xi\rangle+\frac{1}{v}\langle\Delta_\eta^2\rangle-\frac{1}{2v^2}\frac{\partial}{\partial v}(v^2\langle\Delta_\xi^2\rangle)\right\}\right]\\
&+\frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\left[f\sin\theta\left\{\langle\Delta_\eta\rangle-\frac{1}{2v}\frac{\partial}{\partial\theta}\langle\Delta_\eta^2\rangle\right\}\right]\\
&-\frac{1}{2v^2}\frac{\partial}{\partial v}\left[v^2\langle\Delta_\xi^2\rangle\frac{\partial f}{\partial v}\right]-\frac{1}{2v^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\langle\Delta_\eta^2\rangle\frac{\partial f}{\partial\theta}\right]\\
&-\frac{1}{v^2\sin\theta}\frac{\partial}{\partial\theta}\left[f\sin\theta\langle\Delta_\xi\Delta_\eta\rangle+\frac{\partial}{\partial v}(vf\sin\theta\langle\Delta_\xi\Delta_\eta\rangle)\right].
\end{aligned} \tag{23}$$

Equation (23) will give $K(f_r^{(n)}f_s^{(m)})$, provided that on the right-hand side $f_r^{(n)}$ replaces f wherever this occurs explicitly, and the averages of Δ_ξ, Δ_ζ, etc., are evaluated over $f_s^{(m)}$ rather than over f.

When this analysis was first carried out, it was thought that the cross-product term $\langle\Delta_\xi\Delta_\eta\rangle$ would have no effect on the velocity distribution function, and this term has been ignored throughout the remainder of this paper. It now appears that this term may be appreciable; to evaluate this term, however, an extension of Chandrasekhar's analysis of two-body encounters does not suffice, and a new approach to the statistics of such encounters is required.

II. DERIVATION OF EQUATION

When the average energy imparted to the electrons between encounters is small compared with their kinetic energy, we may write (reference 1, Sections 7.31 and 8.31)

$$f^{(1)}(\mathbf{v})=f^{(0)}(v)D(jv)\cos\theta. \tag{24}$$

The function $f^{(0)}(v)$ is the Maxwellian distribution function, given by the equation

$$f^{(0)}(v)=(n_ej^3/\pi^{\frac{3}{2}})\exp(-j^2v^2), \tag{25}$$

where n_e is the number of electrons per cm^3 and j is defined by Eq. (14). For subsequent convenience, we define D to be a function of the dimensionless variable jv. The quantity θ is again the polar angle measured from an axis parallel to the electric field **E**.

For an electron-proton gas, Z is -1 for electrons, and if no temperature gradient is assumed, Eq. (10) becomes, in the present notation,

$$(2j^2ef^{(0)}/m)Ev\cos\theta+K(ff_p)+K(ff)=0. \tag{26}$$

1. Electron-Proton Interaction

The proton interaction term $K(ff_p)$ is found from Eq. (23). We assume the protons are at rest; all the terms in Eq. (23) for $K(ff_p)$ then cancel out or vanish except one, and we have

$$K(ff_p)=-\frac{1}{2v^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\langle\Delta_{\eta,p}^2\rangle\frac{\partial f}{\partial\theta}\right). \tag{27}$$

The diffusion coefficient $\langle\Delta_{\eta,p}^2\rangle$ may be taken from Chandrasekhar.[15] If now we substitute Eq. (9) and (24) into (27), and carry out the differentiation with respect to θ, we have

$$K(ff_p)=[3Lf^{(0)}D(jv)\cos\theta]/2v^3, \tag{28}$$

where we have written

$$L\equiv(8\pi e^4n_e/3m^2)\ln(qC^2). \tag{29}$$

2. Electron-Electron Interaction

Derivation of $K(ff)$ from Eq. (23) is more lengthy. Adopting the terminology of Chandrasekhar,[16] we denote as "test" particles, with a velocity v, those electrons whose change of velocity is being considered, and as "field" particles, with velocity v_1, those of all velocities whose perturbing effect on the test particles is being investigated. The velocities of the field particles fall in the two ranges $0<v_1<v$ and $v<v_1<\infty$.

There are nine terms in Eq. (23), each involving one diffusion coefficient. The first, second, third, and sixth terms, upon use of (24), will each yield one part due to interaction of $f^{(0)}(v_1)$, the spherically symmetric component of the field particles' velocity distribution, with the asymmetric component, $D(jv)$, of the test distribution, and a second part due to the corresponding inter-

[15] See reference 6, Eq. (5.724); the quantity $\Sigma\Delta v_\perp^2/dt$ equals $\langle\Delta_\xi^2\rangle+\langle\Delta_\eta^2\rangle$ and is thus twice $\langle\Delta_\eta^2\rangle$. For electron-proton interaction x_0 is large and $H(x_0)$ equals unity.

[16] See reference 6, paragraph 2.3.

action of $D(jv_1)$ with the symmetric component $f^{(0)}(v)$ of the test distribution. The second of these parts each, in turn, consists of integrals over dv_1 of $D(jv_1)$, multiplied by functions of v_1. The fifth and seventh terms each yield one part only, and the eighth and ninth terms are neglected.

The fourth term can apparently not be evaluated directly by any simple extension of Chandrasekhar's analysis. To evaluate this term by an indirect method, we multiply Eq. (23) by $\cos\theta \sin\theta d\theta$ and integrate over all θ. If we first subtract a term $f\langle\Delta_\xi\rangle/v$, adding this same quantity to the first term, then the fourth term becomes $-P/v$, where

$$P\equiv\int_0^\pi f \sin\theta d\theta\{\cos\theta\langle\Delta_\xi\rangle-\sin\theta\langle\Delta_\eta\rangle\}, \tag{30}$$

which will be evaluated separately.

3. Evaluation of the Diffusion Coefficients

We follow Chandrasekhar's general method,[17] but we replace his spherically symmetric distribution $f^{(0)}$ by the modified function $f^{(0)}(1+D\cos\theta)$, in accordance with Eqs. (9) and (24). The integrals obtained are all straightforward. If we let

$$\begin{aligned}\langle\Delta_\xi^2\rangle&=p_0+p_1\cos\theta,\\ \langle\Delta_\eta^2\rangle&=q_0+q_1\cos\theta,\\ \langle\Delta_\xi\rangle&=r_0+r_1\cos\theta,\end{aligned} \tag{31}$$

we have, first, Chandrasekhar's results

$$\begin{aligned}p_0&=(3LjG(x)/x);\\ q_0&=(3LjH(x)/2x);\\ r_0&=-6Lj^2G(x);\end{aligned} \tag{32}$$

where $G(x)$ and $H(x)$ are functions defined by Chandrasekhar.[18] The terms p_1, q_1, and r_1 may be expressed in the form

$$p_1=\frac{12}{5\pi^{\frac{1}{2}}}Lj\left[\frac{I_5(x)}{x^4}+x\{I_0(\infty)-I_0(x)\}\right]; \tag{33}$$

$$q_1=\frac{2}{\pi^{\frac{1}{2}}}Lj\left[\frac{I_3(x)}{x^2}-\frac{3I_5(x)}{5x^4}+\frac{2x}{5}\{I_0(\infty)-I_0(x)\}\right]; \tag{34}$$

$$r_1=-\frac{4}{\pi^{\frac{1}{2}}}Lj^2\left[\frac{2I_3(x)}{x^3}-I_0(\infty)+I_0(x)\right]; \tag{35}$$

the quantities x and $I_n(x)$ are defined by

$$x\equiv jv, \tag{36}$$

$$I_n(x)\equiv\int_0^x y^nD(y)\exp(-y^2)dy. \tag{37}$$

[17] See reference 6, Chapter II, Section 2.3. We use, in place of his gravitational factor, $G^2m_1^2$, the electrical analog, e^4/m^2. In place of his q, Eq. (17) is used.

[18] See reference 6, pp. 63 and 73.

The integral P which arises from the fourth term of Eq. (23) can be interpreted as $2/m$ times the rate of transfer of momentum per second from the test particles of velocity v in a unit volume of velocity space to field particles of all velocities; the momentum in the direction of the electrical field is considered, and the rate of momentum transfer is averaged over the polar angle θ between **v** and **E**. This rate of momentum transfer can be expressed in terms of $D(jv)$, $D(jv_1)$ and the value of $\langle\Delta_\xi\rangle$ found for a spherical distribution of field particles. To obtain this result we note that no momentum transfer arises from the interactions of $f^{(0)}(v)$ and $f^{(0)}(v_1)$, and also that the interactions of $f^{(1)}(v)$ and $f^{(1)}(v_1)$ are assumed small and are neglected. The interaction of $f^{(1)}(v)$ with $f^{(0)}(v_1)$ can be computed by finding the rate of momentum change for a single test particle on interaction with a spherically symmetrical distribution of field particles, and then integrating this rate over θ. Similarly, the interaction of $f^{(0)}(v)$ with $f^{(1)}(v_1)$ can be obtained by considering the rate of momentum change for a single field particle interacting with a spherically symmetrical distribution of test particles of velocity v, and then integrating over both θ_1 and v_1. Since these rates involve $\langle\Delta_\xi\rangle$ only, we can thus evaluate P without evaluating $\langle\Delta_\xi\rangle$ explicitly. These calculations are much simplified by the fact, first noted by Chandrasekhar,[19] that a test particle loses no momentum on interaction with a spherically symmetrical distribution of field particles if the velocities of the field particles exceed that of the test particle. We obtain

$$P=4Lj^2f^{(0)}\left\{-D(x)G(x)+\frac{2}{\pi^{\frac{1}{2}}}[I_0(\infty)-I_0(x)]\right\}. \tag{38}$$

4. Final Equation for $D(x)$

If now Eqs. (28), (30)–(35), and (38) are substituted into Eq. (26), suitably integrated over $\cos\theta \sin\theta d\theta$, we obtain a final equation for $D(x)$. Before writing this equation we first express the infinite integral $I_0(\infty)$ in closed form. We multiply Eq. (26) by $2\pi v^3\cos\theta \sin\theta dv d\theta$ and integrate over all θ and v. The three terms in (26) then give the total change of momentum arising from the electric field, electron-proton interactions, and electron-electron interactions, respectively. The last term, involving $K(ff)$, must give zero on integration, since the mutual electronic interactions cannot change the total momentum of the electrons; the actual cancellation of all the component parts of this term, on integration, provided a check on the detailed form for $K(ff)$. The second term yields $I_0(\infty)$, and the integration of the first term, representing the effect of the electrical field, is simple; we find, after some straightforward substitution,

$$I_0(\infty)=\int_0^\infty D(x)\exp(-x^2)dx=(3\pi^{\frac{1}{2}}A)/8 \tag{39}$$

[19] See reference 4, p. 260.

TABLE II. Values of velocity distribution function $D(x)$.

x	$D(x)/A$	x	$D(x)/A$	x	$D(x)/A$
0.10	0.01487	0.34	0.2079	1.12	1.790
0.11	0.01840	0.36	0.2324	1.20	2.031
0.12	0.02237	0.38	0.2579	1.28	2.291
0.13	0.02676	0.40	0.2844	1.36	2.573
0.14	0.03159	0.44	0.3401	1.44	2.878
0.15	0.03685	0.48	0.3994	1.52	3.208
0.16	0.04254	0.52	0.4620	1.60	3.567
0.17	0.04865	0.56	0.5278	1.76	4.380
0.18	0.05517	0.60	0.5967	1.92	5.343
0.19	0.06208	0.64	0.6687	2.08	6.484
0.20	0.06940	0.68	0.7438	2.24	7.842
0.22	0.08517	0.72	0.8219	2.40	9.437
0.24	0.1024	0.76	0.9033	2.56	11.34
0.26	0.1210	0.80	0.9876	2.72	13.54
0.28	0.1409	0.88	1.166	2.88	16.13
0.30	0.1621	0.96	1.359	3.04	19.05
0.32	0.1844	1.04	1.566	3.20	22.40

where

$$A\equiv -mE/[2\pi j^2 e^3 n_e \ln(qC^2)]. \tag{40}$$

If Eqs. (39) and (40) are used, the final equation for $D(x)$ becomes

$$D''(x)+P(x)D'(x)+Q(x)D(x)=R(x)+S(x), \tag{41}$$

where

$$P(x)=-2x-\frac{1}{x}+\frac{2x^2\Phi'(x)}{\Lambda}, \tag{42}$$

$$Q(x)=\frac{1}{x^2}-2\frac{1+\Phi(x)-2x^3\Phi'(x)}{\Lambda}, \tag{43}$$

$$R(x)=\frac{-2.4x^2+1.6x^4-2.4x^6}{\Lambda}A, \tag{44}$$

$$S(x)=\frac{16}{\pi^{\frac{1}{2}}\Lambda}\left\{xI_3(x)-2\left(\frac{1+x^2+x^4}{5x^3}\right)I_5(x)+x^2\left(\frac{2-3x^2+2x^4}{5}\right)I_0(x)\right\}. \tag{45}$$

The quantity $\Phi(x)$ is the usual error function, while $\Lambda(x)$ is defined by

$$\Lambda\equiv\Phi(x)-x\Phi'(x). \tag{46}$$

5. Behavior of $D(x)$ for Small and Large x

When x is small, Φ and Φ' may be expressed as power series, and the system of Eqs. (41) to (45) admits a special series solution in ascending powers of x, which we shall denote by $D_{a1}(x)$. For large x, on the other hand, Φ' may be set equal to zero, while Φ equals unity, and we have a special series solution $D_{a2}(x)$ in descending powers of x. Both these series are asymptotic, and will diverge after a certain number of terms.

To obtain general solutions for $D(x)$ in each of these regions, solutions of the homogeneous equation, with $T(x)$ set equal to zero, must be added. As will be evident from the analysis in the next section, such solutions are of two sorts. When x is small, for example, one of these solutions goes to infinity as $\exp(\alpha/x^{\frac{1}{2}})$, yielding an infinite conductivity, and obviously cannot represent a physical solution. The other goes to zero more rapidly than $D_{a1}(x)$, and therefore becomes negligible as x decreases. Similarly, for large x one of the solutions cannot represent reality, while the other goes to zero more rapidly than the leading term of $D_{a2}(x)$. Hence the boundary conditions on Eq. (41) to (45) are that $D(x)$ approach $D_{a1}(x)$ and $D_{a2}(x)$, respectively, as x approaches zero or infinity.

For a Lorentz gas,[20] in which electron-electron interactions are entirely neglected and the protons are again assumed at rest, $K(ff)$ may be ignored in Eq. (26). In this case we obtain the usual result

$$D(x)=Ax^4. \tag{47}$$

III. SOLUTION OF EQUATION

We wish to find the solution to Eq. (41) which is of physical interest and which therefore agrees with the asymptotic series $D_{a1}(x)$ and $D_{a2}(x)$ at zero and infinity, respectively. This solution will be denoted by the superscript c. The complexity of Eq. (41) is such that a closed analytical solution cannot be expected. It might appear that Eq. (41) could be solved by direct numerical integration, since starting values for small x are known from the asymptotic series and the integrals in $S(x)$ could be evaluated as the integration proceeded. Actually such a direct integration is not possible. The integration of Eq. (41) is in fact unstable for both increasing and decreasing x; i.e., a small deviation from the correct solution increases so very rapidly in the course of integration that any trace of the correct solution soon disappears. This behavior is associated with the singularity of $Q(x)$, which varies as $1/x^3$ for small x. A similar instability is introduced by the dominant term in $Q(x)$ for large x. To obtain $D^c(x)$ the approach described below was developed.

1. Decomposition of Basic Equation

To overcome the difficulties associated with the instability of the basic equation, we let

$$P(x)=P_0(x)+\Delta P(x);\quad Q(x)=Q_0(x)+\Delta Q(x), \tag{48}$$

where $P_0(x)$ and $Q_0(x)$ are chosen to represent the leading parts of $P(x)$ and $Q(x)$ and also to permit analytical solution of the following simplified reduced form of Eq. (41),

$$D''(x)+P_0(x)D'(x)+Q_0(x)D(x)=0. \tag{49}$$

In general, Eq. (49) has two solutions, which we shall denote by $U(x)$ and $V(x)$. We now write

$$D(x)=g(x)U(x)+h(x)V(x) \tag{50}$$

[20] H. A. Lorentz, Proc. Amst. Acad. 7, 438 (1905).

where $g(x)$ and $h(x)$ are functions to be determined. The simultaneous differential equations for $g(x)$ and $h(x)$, which are easily obtained by the standard methods of variation of parameters, may then be solved numerically without any basic difficulty.

Because the leading terms of $P(x)$ and $Q(x)$ are different in the cases of small and large x, it was necessary to consider separately the two ranges of x, $0.10 \leq x \leq 0.80$ and $0.80 \leq x \leq 3.20$. The formulas used in each specific range are given below.

(i). Range $0.10 \leq x \leq 0.80$

All quantities peculiar to this particular range will carry the subscript 1. The functions $P_{01}(x)$ and $Q_{01}(x)$ are defined as follows:

$$P_{01}(x) = 2/x, \quad Q_{01}(x) = (-3\pi^{\frac{1}{2}}/2x^{\frac{3}{2}}) - 2/x^2; \quad (51)$$

Equation (49) then has the solutions

$$U_1(x) = x^{-\frac{1}{2}} I_3(\alpha/x^{\frac{1}{2}}), \quad (52)$$

$$V_1(x) = x^{-\frac{1}{2}} K_3(\alpha/x^{\frac{1}{2}}), \quad (53)$$

where

$$\alpha^2 \equiv 6\pi^{\frac{1}{2}}. \quad (54)$$

The numerical values of these functions and their derivatives were computed by interpolation with the aid of B.A.A.S. Tables.[21]

(ii). Range $0.80 \leq x \leq 3.20$

The corresponding quantities in this range carry the subscript 2 and have the following expressions.

$$P_{02}(x) = -2x, \quad Q_{02}(x) = -4, \quad (55)$$

$$U_2(x) = xe^{x^2}, \quad (56)$$

$$V_2(x) = 1 - 2xe^{x^2} \int_x^\infty e^{-v^2} dy. \quad (57)$$

2. Description of Integration

As a first step, a starting value of $g_1(x)$ at $x=0.10$ was obtained by integration of the appropriate equation for $g_1(x)$ from $x=0$, where $g_1(x)$ vanishes, to $x=0.10$. In this integration the asymptotic series D_{a1} was used to compute g_1'. The "correct" value of $g_1(x)$ at 0.10 is denoted by $g_1^c(0.10)$. In the same way, the accurate determination of $g_2^c(3.20)$ was carried out, with $D_{a2}(x)$ used to compute g_2'.

The starting value $h_1^c(0.10)$ cannot be determined so simply. In general there will be one value of $h_1(0.10)$, which, together with $g_1^c(0.10)$, will yield on numerical integration the calculated value of $g_2(x)$ at $x=3.20$; i.e., $g_2^c(3.20)$. This will be the correct solution. In order to obtain this correct solution, advantage was taken of the linear properties of the above equations. Two arbitrary starting values of $h_1(x)$ at $x=0.10$ yield two linearly independent solutions, each with the correct starting value of $g_1(x)$ at $x=0.10$. The linear combination of these two solutions, which at $x=3.20$ gives $g_2^c(3.20)$, is then the correct solution.

The starting interval used in the integration was 0.01 and was doubled after every ten steps. Central difference formulas were employed throughout. From the difference tables it was possible to guess ahead the values of $S(x)$ and of the appropriate combination of $D(x)$ and $D'(x)$; then first approximations to $g(x)$ and $h(x)$ at the next integration point could be obtained. These values were then used to obtain more accurate values of $S(x)$, etc., and ultimately second approximations to $g(x)$ and $h(x)$. This cyclic process was carried out at each integration point, in some cases as many as four times, until the values of $g(x)$ and $h(x)$ arising from the last two cycles agreed to five significant figures.

The resulting values of $D^c(x)$ over the entire range from 0.10 to 3.20 are given in Table II. The maximum error in any of these values should not exceed unity in the last digit, except perhaps for the last ten values, where errors as great as two in the last digit are possible.

The integrals $I_n(x)$ were found in the course of numerical integration up to x equal to 3.20. As x increases further these integrals also increase slightly by amounts which obviously depend on the values of $D(x)$ for x greater than 3.20. The special solution $D_{a2}(x)$ was not sufficiently accurate, and a term in $V_2(x)$ was added to yield the approximate general solution in this range. The following values were obtained.

$$I_0(\infty) = 0.66464\text{A}. \quad (58)$$

$$I_3(\infty) = 1.470\text{A}. \quad (59)$$

$$I_5(\infty) = 4.562\text{A}. \quad (60)$$

This computed value of $I_0(\infty)$ may be compared with the exact value, which according to Eq. (39) equals $3\pi^{\frac{1}{2}}\text{A}/8$, or 0.66467A.

TABLE III. Constant in electrical conductivity formula.

Lorentz gas	1.000
Reference 1, first approximation	0.295
Cowling, second approximation	0.578
Present work	0.490

IV. VALUES OF THE ELECTRICAL CONDUCTIVITY

The electrical conductivity σ is simply the total current flowing per cm^3 divided by the electrical field strength E. Electrons with velocities between v and $v+dv$ and with directions between θ and $\theta+d\theta$ and between ϕ and $\phi+d\phi$ will contribute $d\sigma$ to the conductivity per unit volume of physical space, where

$$d\sigma = [-f(v)ev\cos\theta/E]v^2 \sin\theta dv d\theta d\phi. \quad (61)$$

[21] Our sincere thanks are due to Dr. W. G. Bickley of the Imperial College of Science and Technology, London, England, who very kindly sent us the proof sheets of these very complete Bessel function tables, prepared by the Committee for the Calculation of Tables of the British Association for the Advancement of Science.

TABLE IV. Values[a] of $\ln(qC^2)$ and $\ln(q'C^2)$.

Kinetic temperature (°K)	Electron density n_e (cm^{-3}) 1	10^6	10^{12}	10^{18}	10^{24}
10^2	16.3	9.36	—	—	—
	12.4	7.78	3.2		
10^4	23.2	16.3	9.36	—	—
	17.0	12.4	7.78	3.2	
10^6	30.1	23.2	16.3	9.36	—
	21.6	17.0	12.4	7.78	3.2
10^8	37.0	30.1	23.2	16.3	9.36
	26.2	21.6	17.0	12.4	7.78

[a] For each pair of values of n_e and T, the upper figure gives $\ln(qC^2)$, the lower, $\ln(q'C^2)$; the logarithms are to the base e.

The minus sign results from the negative electronic charge. According to Eqs. (9) and (24), $f(v)$ is the sum of a spherically symmetric term $f^{(0)}(v)$, which clearly makes no net contribution to the conductivity, and the term $f^{(0)}(v)D(jv)\cos\theta$. If we substitute this term for $f(v)$, integrate over v, θ, and ϕ, and substitute from Eq. (40) for A we obtain

$$\sigma=[2m/3\pi^{\frac{1}{2}}e^2j^3\ln(qC^2)][I_3(\infty)]/\mathrm{A}. \tag{62}$$

It is convenient to express σ in terms of the conductivity in a Lorentz gas, multiplied by some constant γ. Since $I_3(\infty)/A$ for a Lorentz gas equals three, as may be seen by combining Eqs. (37) and (48), we have

$$\sigma=2\gamma(2/3\pi)^{\frac{3}{2}}mC^3/[e^2\ln(qC^2)], \tag{63}$$

where

$$\gamma=[I_3(\infty)]/3\mathrm{A}. \tag{64}$$

It may be remarked that the mutual electronic interactions do not change the conductivity directly, since the total change of momentum in such interactions is zero. Nevertheless, they alter $D(x)$ and in this way modify the effect which electron-proton collisions have in impeding the current.

The values of γ obtained from various theories are given in Table III below. The value given here, readily obtained on combining Eqs. (59) and (64), is some 15 percent smaller than that found by Cowling[2] in his second approximation.

In addition, the constant q used here is greater than that used by Cowling, since, as was shown in Part I, the cut-off distance should be equated to the Debye shielding radius h, rather than the interionic distance. We shall denote Cowling's value, based on the interionic distance, by q'. If the usual formula for C^2 is taken, and the ionic charge Z_i is set equal to unity, Eq. (17) yields

$$qC^2=(3/\pi^{\frac{1}{2}}n_e^{\frac{1}{2}}e^3)(kT/2)^{\frac{3}{2}} \tag{65}$$

while for q' we have[22]

$$q'C^2=4kT/n_e^{\frac{1}{2}}e^2. \tag{66}$$

Since $\ln(qC^2)$ appears in Eq. (63) for the conductivity, values of $\ln(qC^2)$ are given in Table IV, together with values of $\ln(q'C^2)$ for comparison. For high densities and low kinetic temperatures, q falls below q'; the analysis leading to Eq. (65) breaks down, and q' may be used. For still higher values of $n_e^{\frac{1}{2}}/T$, no values are given in the table; the present theory breaks down completely under such conditions and, moreover, the electron gas tends to become degenerate. It is evident from Table IV that for low densities and high temperatures the change in the cut-off distance has a greater effect on the conductivity than does the change in the value of γ. For these conditions the resultant electrical conductivity is about 60 percent of the value obtained by Cowling.

It is hoped to extend these results in the near future to ionized gases with different average ionic charges, and also to compute thermal conductivities. Further analysis is needed, however, to evaluate the cross-product terms in Eq. (23) which have been neglected in the present work.

[22] See reference 1, Section 10.33. $\mathrm{Ln}(q'C^2)$ equals one-half the function $A_1(2)$ introduced by Chapman and Cowling.