The integration over S is straightforward and yields

$$\sigma(k_0)=-\frac{16Z^2e^2}{k^2}m^2\cdot S_{\min}^2\int_0^\infty \nu d\nu Q(\nu,m)\{K_1^2(\zeta_m)-K_0(\zeta_m)K_2(\zeta_m)\},$$

where $\zeta_m=S_{\min}\cdot 2m\nu/k_0\cong 2\nu/k_0$.

If $Q(\nu,m)$ is to contain radiative correction $\sim e^2$, one must include the possibility for the emission of a photon, besides the pair. The dependence of the cross section on the energy of this photon is uninteresting, and even a classical effect when the photon has a low energy. $Q(\nu,m)$ should therefore be the cross section for pair creation including possible emission of a photon. As $Q(\nu,m)$ must fall off for high energies ν (as is generally found for such cross sections in quantum electrodynamics), the main contributions to $\sigma(k_0)$ will come from small values of ν/m. We can therefore expand the bracket { } with respect to ζ_m, obtaining

$$\sigma(k_0)=8Z^2e^2\int\frac{d\nu}{\nu}\left(\ln\frac{k}{2m}-\ln\frac{\nu}{2}-\ln\frac{\gamma}{2}-\frac{1}{2}\right)Q(\nu,m)$$
$$=A\ln(k_0/2m)+B,$$

where

$$A=8Z^2e^2\int Q(\nu,m)d\nu/\nu,$$
$$B=8Z^2e^2\int\frac{d\nu}{\nu}\left(\ln\frac{\nu}{m}+\ln\frac{\gamma}{2}+\frac{1}{2}\right)Q(\nu,m),$$
$$\ln\gamma=0.577\cdots.$$

The only change caused by the radiative corrections will therefore consist in a change of A and B of the order of 1/137, which is quite negligible.

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The Inelastic Scattering of High Energy Neutrons by Deuterons According to the Impulse Approximation

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The high energy $n-d$ inelastic scattering problem is formulated in such a way that the amplitudes for $n-n$ and $n-p$ scattering at the same energy appear explicitly. The formulation depends on: (1) the large radius of the deuteron compared with the range of nuclear forces and high energy scattering amplitudes; (2) the high velocity of the incident neutron compared to the zero point motion of the deuteron. The method does not depend on the weakness of nuclear forces compared to the kinetic energy of the neutron and is therefore not equivalent to the Born approximation.

I. INTRODUCTION

IN a previous paper[1] the author discussed the elastic scattering of 90- to 350-Mev neutrons by deuterons and pointed out that because elastic scattering may well represent a considerable fraction of the total *n-d* cross section, it is dangerous to assume that the latter is equal to the sum of the free *n-n* and *n-p* cross sections. This is because binding and interference effects play a large role in elastic scattering. They do not play so important a part in the inelastic scattering, however, and it is possible that experiments which concentrate on the latter rather than on the total *n-d* cross section may yet reveal the magnitude of the *n-n* interaction at high energies. Because of the similar properties of mirror nuclei it appears certain that *n-n* and *p-p* forces are equal for low relative energies. So many surprises have appeared in high energy scattering experiments, however,[2] that it is desirable, if possible, to measure the high energy *n-n* interaction independently.

It is the purpose of this first of two papers to formulate the *n-d* inelastic scattering problem in such a manner that from its measurement one can attempt to deduce the value of the free neutron-neutron cross section. The actual attempted deduction from Berkeley experiments will be carried out in the second paper. A secondary feature of this first part is the demonstration of a phenomenological approach to high energy nuclear reactions in light nuclei, using as a basis the experimentally measured values of nucleon-nucleon cross sections. It is hoped that this general method may eventually be extended to nuclei more complex than the deuteron.

The fundamental assumptions will be twofold: (1) the "collision" time in high energy *n-d* scattering is so short compared to the period of the deuteron that the change in the wave function of the latter during the collision can be described by an "impulse" approximation; (2) the deuteron has such a diffuse structure compared to the range of nuclear forces that the wave function of the incident neutron at one of the two scattering centers within the deuteron is not appreciably perturbed by the presence of the other center. Outgoing waves from both centers are present and must be added together, but individually they will be assumed to be the same as would be produced by a single neutron or proton. In this way the three-body problem is reduced to a superposition of the two-body problems.

It should be noted that these assumptions are not completely equivalent to the Born approximation,

[1] G. F. Chew, Phys. Rev. **74**, 809 (1948).
[2] Hadley, Kelly, Leith, Segrè, Wiegand, and York, Phys. Rev. **75**, 351 (1949); O. Chamberlain and C. Wiegand, Phys. Rev. **79**, 81 (1950).

which has been used previously in this problem.[3,4] It will not be assumed here that the perturbation of the incident wave is small (which is true only at energies still unobtainable with accelerators) but only that the two perturbing centers act independently and "suddenly." The two-body scattering amplitudes can be treated as empirical quantities, and the calculations described in Sections III and IV simply have to do with the "quantum kinematics" of a three-body problem in which each of the three particles has spin one-half and two of the three are identical.

II. THE IMPULSE APPROXIMATION

The formulation of the approximation outlined in Section I is to be applied here to the non-relativistic case although the inclusion of relativistic effects should not be difficult. Before taking on all of the complications of the actual neutron-deuteron problem, it is instructive to consider a simplified situation in which the proton is bound to a fixed short-range force, $U(\mathbf{r}_p)$, which can accommodate one bound state $\varphi_0(\mathbf{r}_p)$ and which interacts with the incident neutron through a potential, $V(\mathbf{r}_n-\mathbf{r}_p)$. The interaction of the neutron with U will be ignored. The Schrödinger equation for this problem is

$$[-\hbar^2/2m\Delta_{r_n}-\hbar^2/2m\Delta_{r_p}+U(\mathbf{r}_p)+V(\mathbf{r}_n-\mathbf{r}_p)]\times\Psi(\mathbf{r}_n,\mathbf{r}_p)=E_0\Psi(\mathbf{r}_n,\mathbf{r}_p),\quad(1)$$

where $E_0=\hbar^2k_0^2/2m-B$, $\mathbf{k}_0$ being the wave number of the incident neutron and B the binding energy of the proton. A solution is desired of the form

$$\Psi(\mathbf{r}_n,\mathbf{r}_p)=(1/2\pi)^{\frac{3}{2}}e^{i\mathbf{k}_0\cdot\mathbf{r}_n}\varphi(\mathbf{r}_p)+\Psi_s(\mathbf{r}_n,\mathbf{r}_p),\quad(2)$$

where $\Psi_s(\mathbf{r}_n,\mathbf{r}_p)$ will contain various kinds of outgoing waves which represent elastic scattering, inelastic scattering, and pick-up processes.[5] Substitution into the wave equation and some rather conventional manipulations[6] allow one to derive an integral equation for $\Psi_s(\mathbf{r}_n,\mathbf{r}_p)$ into which the appropriate boundary conditions are incorporated. The asymptotic form of this equation for large $\mathbf{r}_n$, large $\mathbf{r}_p$, and large $|\mathbf{r}_n-\mathbf{r}_p|$ is

$$\Psi_s\rightarrow\frac{2m}{4\pi\hbar^2}\int d\mathbf{k}_p\varphi_{\mathbf{k}_p}(\mathbf{r}_p)\frac{e^{ik_nr_n}}{r_n}\int\int d\mathbf{r}_n'd\mathbf{r}_p'\varphi_{\mathbf{k}_p}^*(\mathbf{r}_p')\times e^{-i\mathbf{k}_n\cdot\mathbf{r}_n'}V(\mathbf{r}_n'-\mathbf{r}_p')\Psi(\mathbf{r}_n',\mathbf{r}_p'),\quad(3)$$

where $\varphi_{\mathbf{k}_p}(\mathbf{r}_p)$ is the continuum eigenfunction for the proton in the potential field $U(\mathbf{r}_p)$, which is normalized at infinity to $(1/2\pi)^{\frac{3}{2}}e^{i\mathbf{k}_p\cdot\mathbf{r}_p}$. The magnitude of $\mathbf{k}_n$ is determined by the energy equation, $\hbar^2k_n^2/2m=E_0-\hbar^2k_p^2/2m$. This is the part of the outgoing wave which corresponds to inelastic scattering with emission of the proton.

For the wave function which appears in the integrand of (3) the following approximation will now be made:

$$\Psi(\mathbf{r}_n,\mathbf{r}_p)\approx\Psi_a(\mathbf{r}_n,\mathbf{r}_p)=\int d\mathbf{k}_pg_0(\mathbf{k}_p)\psi_{\mathbf{k}_0,\mathbf{k}_p}{}^{np}(\mathbf{r}_n,\mathbf{r}_p),\quad(4)$$

where

$$g_0(\mathbf{k}_p)=(1/2\pi)^{\frac{3}{2}}\int d\mathbf{r}_pe^{-i\mathbf{k}_p\cdot\mathbf{r}_p}\varphi_0(\mathbf{r}_p),$$

and $\psi_{\mathbf{k}_0,\mathbf{k}_p}{}^{np}(\mathbf{r}_n,\mathbf{r}_p)$ is the wave function which represents the scattering of a neutron of momentum $\mathbf{k}_0$ by a free proton of momentum $\mathbf{k}_p$. $\Psi_a(\mathbf{r}_n,\mathbf{r}_p)$ clearly represents the scattering of a neutron of momentum $\mathbf{k}_0$ by a free proton wave packet which has the same momentum distribution as the bound proton function φ_0. The author believes that the replacement of Ψ by Ψ_a corresponds to assuming that for the "duration" of the proton's interaction with the neutron the proton does not interact with the force field $U(\mathbf{r}_p)$. The only effect of the latter is assumed to be the generation of the momentum distribution $g_0(\mathbf{k}_p)$, at a time much earlier than the arrival time of the neutron. It seems reasonable that if the range of the n–p force is ρ and the neutron velocity v_0, then the criterion for the validity of this approximation is essentially that the "collision time," ρ/v_0, be short compared to the period of the deuteron. (This ratio is, for instance, 1/10 for 90-Mev neutrons.) An attempt is now being made to verify this assertion by actually calculating the error made, and a discussion of the limitations of the impulse approximation in general will be given in a forthcoming paper by Wick and the author.

Two additional remarks about the approximate function Ψ_a may be added. In the first place it is always at least as good as the Born approximation, since in the limit of a very weak n–p force

$$\Psi_a\rightarrow(1/2\pi)^{\frac{3}{2}}e^{i\mathbf{k}_0\cdot\mathbf{r}_n}\varphi_0(\mathbf{r}_p).$$

Second, it will always represent correctly the limiting case of a very low binding energy of the proton, since in that case $g_0(\mathbf{k}_p)$ becomes a delta-function and one automatically gets free n–p scattering. Even without the supporting argument above, therefore, Ψ_a might be looked on as an interpolation between these two limits.

Substitution of Ψ_a into (3) leads to a simple result, particularly if one employs the R matrix notation.[7] In this notation, the matrix

$$(\mathbf{k}_n',\mathbf{k}_p'|R_{np}|\mathbf{k}_n,\mathbf{k}_p)=-\frac{i}{(2\pi)^{\frac{3}{2}}}\int\int d\mathbf{r}_n'd\mathbf{r}_p'e^{-i(\mathbf{k}_n'\cdot\mathbf{r}_n'+\mathbf{k}_p'\cdot\mathbf{r}_p')}\times V(\mathbf{r}_n'-\mathbf{r}_p')\psi_{\mathbf{k}_n,\mathbf{k}_p}{}^{np}(\mathbf{r}_n',\mathbf{r}_p')\quad(5)$$

describes free neutron-proton scattering from the initial wave numbers $\mathbf{k}_n$ and $\mathbf{k}_p$ to the final wave numbers $\mathbf{k}_n'$ and $\mathbf{k}_p'$. Since the wave function $\psi_{\mathbf{k}_n,\mathbf{k}_p}{}^{np}(\mathbf{r}_n,\mathbf{r}_p)$

[3] Ta You Wu and J. Ashkin, Phys. Rev. **73**, 986 (1948).
[4] F. DeHoffman, Phys. Rev. **76**, 216 (1950).
[5] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950).
[6] See, for example, G. Breit, Phys. Rev. **71**, 215 (1947).
[7] C. Møller, Kgl. Danske, Vid. **23**, No. 1 (1945).

may be factored, i.e.

$$\psi_{\mathbf{k}_n,\mathbf{k}_p}{}^{np}(\mathbf{r}_n, \mathbf{r}_p) = (1/2\pi)^{\frac{3}{2}} e^{i\frac{1}{2}(\mathbf{k}_n+\mathbf{k}_p)\cdot(\mathbf{r}_n+\mathbf{r}_p)} \times \varphi_{\frac{1}{2}(\mathbf{k}_n-\mathbf{k}_p)}{}^{np}(\mathbf{r}_n-\mathbf{r}_p), \quad (6)$$

one can also write (5) in the form

$$(\mathbf{k}_n', \mathbf{k}_p'|R_{np}|\mathbf{k}_n, \mathbf{k}_p) = \delta(\mathbf{k}_n'+\mathbf{k}_p'-\mathbf{k}_n-\mathbf{k}_p)(\mathbf{k}'|r_{np}|\mathbf{k}), \quad (7)$$

where $\mathbf{k}=\frac{1}{2}(\mathbf{k}_n-\mathbf{k}_p)$, $\mathbf{k}'=\frac{1}{2}(\mathbf{k}_n'-\mathbf{k}_p')$, and

$$(\mathbf{k}'|r_{np}|\mathbf{k}) = -i\int d\mathbf{r} e^{-i\mathbf{k}'\cdot\mathbf{r}} V(\mathbf{r})\varphi_{\mathbf{k}}{}^{np}(\mathbf{r}). \quad (8)$$

To find the cross section for free n–p scattering into any wave number interval, $d\mathbf{k}'$, the formula

$$\sigma_{np} d\mathbf{k}' = 2\pi/\hbar v_0 |(\mathbf{k}'|r_{np}|\mathbf{k})|^2 \delta(E_n'+E_p'-E_n-E_p)d\mathbf{k}'$$

is to be used, in which v_0 is the initial relative velocity, $E_n=\hbar^2k_n^2/2m$, etc., and where all energies and momenta are to be expressed in terms of $\mathbf{k}'$, via the momentum conservation condition.

Similarly one can describe the inelastic scattering by an R matrix. From Eq. (3) it is seen that, if

$$(\mathbf{k}_n, \mathbf{k}_p|R_{in}|\mathbf{k}_0, 0) = -i\int\int d\mathbf{r}_n d\mathbf{r}_p \varphi_{\mathbf{k}_p}{}^*(\mathbf{r}_p) e^{-i\mathbf{k}_n\cdot\mathbf{r}_n} V(\mathbf{r}_n-\mathbf{r}_p)\Psi(\mathbf{r}_n, \mathbf{r}_p), \quad (9)$$

then the cross section for emission of the proton into $d\mathbf{k}_p$ while the neutron emerges in $d\mathbf{k}_n$ is

$$\sigma_{in} d\mathbf{k}_n d\mathbf{k}_p = 2\pi/\hbar v_0 |(\mathbf{k}_n, \mathbf{k}_p|R_{in}|\mathbf{k}_0, 0)|^2 \times\delta(E_n+E_p-E_0)d\mathbf{k}_n d\mathbf{k}_p \quad (10)$$

with no condition of momentum conservation. Replacing Ψ by Ψ_a, it follows from the definition (5) that

$$(\mathbf{k}_n, \mathbf{k}_p|R_{in}|\mathbf{k}_0, 0) \approx \int\int d\mathbf{k}_p' d\mathbf{k}_p{}^0 g_{\mathbf{k}_p}{}^*(\mathbf{k}_p')(\mathbf{k}_n, \mathbf{k}_p'|R_{np}|\mathbf{k}_0, \mathbf{k}_p{}^0) g_0(\mathbf{k}_p{}^0) \quad (11a)$$

$$= \int d\mathbf{k}_p' g_{\mathbf{k}_p}{}^*(\mathbf{k}_p') \left(\frac{\mathbf{k}_n-\mathbf{k}_p'}{2}\middle| r_{np} \middle|\frac{\mathbf{k}_0-\mathbf{k}_p{}^0}{2}\right) g_0(\mathbf{k}_p{}^0), \quad (11b)$$

where

$$g_{\mathbf{k}_p}(\mathbf{k}_p') = (1/2\pi)^{\frac{3}{2}} \int d\mathbf{r}_p e^{-i\mathbf{k}_p'\cdot\mathbf{r}_p} \varphi_{\mathbf{k}_p}(\mathbf{r}_p),$$

and in (11b), $\mathbf{k}_p{}^0=\mathbf{k}_n+\mathbf{k}_p'-\mathbf{k}_0$.

Equation (11a) is the formal expression of the impulse approximation as applied to inelastic scattering. Replacement of $g_{\mathbf{k}_p}{}^*(\mathbf{k}_p')$ by $g_0{}^*(\mathbf{k}_p')$ would give the expression for elastic scattering.[1] One further step is desirable to complete the phenomenological theory; the n–p scattering amplitude should be taken outside the integral. This is possible if one of two conditions is satisfied. (1) If v_{np} varies only a little over the important range of $\mathbf{k}_p'$ it is legitimate to replace it by the value it takes where $g_{\mathbf{k}_p}{}^*(\mathbf{k}_p')$ is singular, which will always be at $\mathbf{k}_p'=\mathbf{k}_p$. (2) If r_{np} depends only on the *difference* of initial and final relative momenta, rather than on the individual values, then it is independent of $\mathbf{k}_p'$. For $\frac{1}{2}(\mathbf{k}_n-\mathbf{k}_p')-\frac{1}{2}(\mathbf{k}_0-\mathbf{k}_p{}^0)=\mathbf{k}_n-\mathbf{k}_0$ in virtue of momentum conservation. There are experimental and theoretical reasons for expecting one or both of these conditions to be well satisfied in the n-d problem. They will be discussed later in the actual calculation.

It is asserted therefore that the scattering amplitude for the three-body problem may be approximated by the product of two factors, each of which has to do with a two-body problem:

$$(\mathbf{k}_n, \mathbf{k}_p|R_{in}|\mathbf{k}_0, 0) \approx \left(\frac{\mathbf{k}_n-\mathbf{k}_p}{2}\middle| r_{np} \middle|\frac{\mathbf{k}_0-(\mathbf{k}_n+\mathbf{k}_p-\mathbf{k}_0)}{2}\right) \times\int d\mathbf{k}_p' g_{\mathbf{k}_p}{}^*(\mathbf{k}_p') g_0(\mathbf{k}_n+\mathbf{k}_p'-\mathbf{k}_0). \quad (12)$$

This factorization allows one to hope that the results of the three-body scattering experiment may be analyzable in terms of two-body scattering at the same energy.

It is interesting that the form (12) is exactly that deduced by Fermi[8] from somewhat different arguments to describe the scattering of slow neutrons by molecularly bound protons. The equivalence is more obvious if the integrand of the second factor of (12) is written in configuration space:

$$\int d\mathbf{k}_p' g_{\mathbf{k}_p}{}^*(\mathbf{k}_p') g_0(\mathbf{k}_n+\mathbf{k}_p'-\mathbf{k}_0) = \int d\mathbf{r}_p \varphi_{\mathbf{k}_p}{}^*(\mathbf{r}_p) e^{-i(\mathbf{k}_n-\mathbf{k}_0)\cdot\mathbf{r}_p} \varphi_0(\mathbf{r}_p).$$

In Fermi's case, r_{np} is a constant, proportional to the so-called scattering length. In the forthcoming general discussion of the impulse approximation, the relation of these two problems will be made clear.

III. APPLICATION OF THE METHOD TO THE NEUTRON-DEUTERON PROBLEM

The arguments of the preceding section will be generalized now to apply to the case of inelastic neutron-deuteron scattering. The additional complications are fourfold. In the first place, the initial binding center is not infinitely heavy. It is well known that this can be accounted for by a center-of-mass transformation in conventional treatments. The approach used here likewise encounters no difficulty. In the second place, the incident neutron interacts both with the proton and with the proton's binding center, i.e., the other neutron. This will introduce into the expressions analogous to (3) and (9) two sources of scattering which are propor-

[8] E. Fermi, Ricerca Scient. VII–II, 13 (1936).

tional to $[V_{np}(\mathbf{r}_1-\mathbf{r}_p)+V_{nn}(\mathbf{r}_1-\mathbf{r}_2)]\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_p)$ if $\mathbf{r}_1$ is the coordinate of the incident neutron and $\mathbf{r}_2$ that of the initially bound neutron. The approximation now is to assume that when $\mathbf{r}_1$ is close to $\mathbf{r}_p$ the wave function has the form

$$\Psi_a{}^{np}=\int\int d\mathbf{k}_p d\mathbf{k}_2 G_0(\mathbf{k}_p, \mathbf{k}_2)e^{i\mathbf{k}_2\cdot\mathbf{r}_2}\psi_{\mathbf{k}_0, \mathbf{k}_p}{}^{np}(\mathbf{r}_1, \mathbf{r}_p),$$

where

$$G_0(\mathbf{k}_p, \mathbf{k}_2)=(1/2\pi)^{7/2}\int\int d\mathbf{r}_p d\mathbf{r}_2 \times e^{-i(\mathbf{k}_p\cdot\mathbf{r}_p+\mathbf{k}_2\cdot\mathbf{r}_2)}\varphi_0(\mathbf{r}_p-\mathbf{r}_2)$$
$$=\delta(\mathbf{k}_p+\mathbf{k}_2)g_0\left(\frac{\mathbf{k}_p-\mathbf{k}_2}{2}\right)$$

$\varphi_0(\mathbf{r}_p-\mathbf{r}_2)$ being the deuteron wave function and g_0 its Fourier transform. On the other hand, when $\mathbf{r}_1$ is close to $\mathbf{r}_2$, a superposition of neutron-neutron scattering functions will be used. The essence of this approximation is thus to ignore the perturbation of the wave function at one of the two scattering centers which is induced by the other center. Its validity rests on the smallness of the high energy two-body scattering amplitudes in comparison to the dimensions of the deuteron.

A little forethought will reveal that a straightforward application of the procedure outlined in Section II with this approximate function will give a recipe for calculating inelastic *n–d* scattering which is not quite self-consistent. Forgetting the Pauli principle and spin, for a moment, and labeling the final momentum of the incident neutron by $\mathbf{k}_1$ and that of the initially bound neutron by $\mathbf{k}_2$, the formula analogous to (11a) will be

$$(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_p|R_{in}|\mathbf{k}_0, 0)$$
$$\approx\int\int\int\int d\mathbf{k}_p'd\mathbf{k}_2'd\mathbf{k}_p{}^0d\mathbf{k}_2{}^0 G_{\mathbf{k}_p,\mathbf{k}_2}{}^*(\mathbf{k}_p', \mathbf{k}_2')$$
$$\times\{(\mathbf{k}_1, \mathbf{k}_p'|R_{np}|\mathbf{k}_0, \mathbf{k}_p{}^0)\delta(\mathbf{k}_2'-\mathbf{k}_2{}^0)$$
$$+(\mathbf{k}_1, \mathbf{k}_2'|R_{nn}|\mathbf{k}_0, \mathbf{k}_2{}^0)\delta(\mathbf{k}_p'-\mathbf{k}_p{}^0)\}G_0(\mathbf{k}_p{}^0, \mathbf{k}_2{}^0), \quad (13)$$

where $G_{\mathbf{k}_p,\mathbf{k}_2}(\mathbf{k}_p', \mathbf{k}_2')$ is the Fourier transform of the final continuum deuteron wave function. The inconsistency becomes apparent if we formally insert the additional factors $\delta(\mathbf{k}_1-\mathbf{k}_1')$ and $\delta(\mathbf{k}_0-\mathbf{k}_1{}^0)$ and integrate over $d\mathbf{k}_1'$ and $d\mathbf{k}_1{}^0$. This does not change the value of the integral but it allows the following rather obvious interpretation: One starts with a wave function, expressed in momentum space, which is $\delta(\mathbf{k}_1{}^0-\mathbf{k}_0)\times G_0(\mathbf{k}_p{}^0, \mathbf{k}_2{}^0)$ and which represents a free neutron with momentum $\mathbf{k}_0$ and a deuteron at rest in its ground state. The matrix $R_{np}+R_{nn}$ operates on this state (the integration over $\mathbf{k}_1{}^0$, $\mathbf{k}_2{}^0$, $\mathbf{k}_p{}^0$), producing a scattered wave which must then be resolved into its component parts. The resolution is carried out by computing the overlap of the outgoing wave with $G_{\mathbf{k}_2,\mathbf{k}_p}(\mathbf{k}_2', \mathbf{k}_p')\times\delta(\mathbf{k}_1'-\mathbf{k}_1)$, a function which takes into account the interaction between the two particles of which originally the deuteron was composed but not the interaction between the incident particle and either of the other two. The interaction with the proton is presumably already included in the matrix R_{np} and that with the neutron is in R_{nn}, but in each of the two terms the interaction of one of the three pairs of particles in the final state is omitted. Now it is obvious that one pair must be omitted if one is to avoid solution of a three-body problem, and in the spirit of the approximation which is being made such an omission is consistent. However for reasons of common sense, it seems that the following recipe is more appropriate than that expressed by (13): In resolving the scattered wave generated by R_{np}, compute the overlap integral with *either* $G_{\mathbf{k}_2,\mathbf{k}_p}(\mathbf{k}_2'-\mathbf{k}_p')\delta(\mathbf{k}_1'-\mathbf{k}_1)$ *or* $G_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{k}_1', \mathbf{k}_2')\delta(\mathbf{k}_p'-\mathbf{k}_p)$ according to whether $|\mathbf{k}_p-\mathbf{k}_2|$ or $|\mathbf{k}_1-\mathbf{k}_2|$ is the smaller. In other words, take into account that interaction which is the stronger. Because of conservation of energy and momentum $|\mathbf{k}_p-\mathbf{k}_2|$ and $|\mathbf{k}_1-\mathbf{k}_2|$ cannot *both* be small. If they are nearly equal then they must be sufficiently large that both pairs of interactions are negligible. In resolving the wave generated by R_{nn} a similar choice should be made, based on the relative magnitudes of $|\mathbf{k}_p-\mathbf{k}_2|$ and $|\mathbf{k}_p-\mathbf{k}_1|$.

The final two additional complications possessed by the actual *n–d* problem are the spin degrees of freedom and the identity of the two neutrons. Spin variables must be included in the arguments of initial and final wave functions and also in the matrices R_{nn} and R_{np}, and summations over spin must be added to the integrations over momenta. The Pauli principle is satisfied by antisymmetrizing the scattering matrix in the neutron variables.

If the combined momentum and spin variables $\mathbf{k}_i$, σ_i of a particle i be designated by ξ_i and a summation over ξ_i understood to include integration over continuous momentum variables, then according to the above arguments the complete unsymmetrized matrix for *n–d* inelastic scattering is to be written as follows

$$(\xi_1, \xi_2, \xi_p|R_{in}|\xi_0, \xi_0{}^D)=\sum_{\xi_1', \xi_2', \xi_p'}\sum_{\xi_1{}^0, \xi_2{}^0, \xi_p{}^0}\{H_{\xi_p\xi_1,\xi_2}{}^*(\xi_p', \xi_1', \xi_2')(\xi_1', \xi_p'|R_{np}|\xi_1{}^0, \xi_p{}^0)\delta_{\xi_2', \xi_2{}^0}$$
$$+H_{\xi_1\xi_2,\xi_p}(\xi_p', \xi_1', \xi_2')(\xi_1', \xi_2'|R_{nn}|\xi_1{}^0, \xi_2{}^0)\delta_{\xi_p', \xi_p{}^0}\}H_0(\xi_1{}^0, \xi_2{}^0, \xi_p{}^0). \quad (14)$$

In this formula,

$$H_0(\xi_1{}^0, \xi_2{}^0, \xi_p{}^0)=\delta_{\xi_1{}^0, \xi_0}G_{\xi_0}{}^D(\xi_2{}^0, \xi_p{}^0),$$

where ξ_0 represents the momentum and spin of the incident neutron, and $G_{\xi_0}{}^D$ is the wave function (in spin-momentum space) of a deuteron at rest with a well-defined spin. The other symbol requiring definition is

$$H_{\xi_1\xi_p,\,\xi_2}(\xi_1',\xi_2',\xi_p')=\delta_{\xi_1',\,\xi_1}G_{\xi_p,\,\xi_2}(\xi_2',\xi_p') \quad \text{if } |\mathbf{k}_2-\mathbf{k}_p|<|\mathbf{k}_2-\mathbf{k}_1|$$
$$=\delta_{\xi_p',\,\xi_p}G_{\xi_1,\,\xi_2}(\xi_1',\xi_2') \quad \text{if } |\mathbf{k}_2-\mathbf{k}_p|>|\mathbf{k}_2-\mathbf{k}_1|$$

etc., where, for example $G_{\xi_1,\,\xi_2}(\xi_1',\xi_2')$ is the continuum wave function of two neutrons, normalized on the energy shell to $\xi_{\xi_1',\,\xi_1}\delta_{\xi_2',\,\xi_2}$ plus outgoing waves.

This recipe is quite complicated, but from the conventional point of view, in which one assumes that he knows the nature of the two-body interactions, the recipe can in principle be carried out. It contains nothing but the solutions of two-body problems. However, the integrations indicated would be hopelessly tedious if one could not neglect the variation of the R matrices and take them outside the integrals. The latter simplification is certainly essential if we wish to reverse the usual procedure and deduce two-body R matrices from the experimentally observed three-body scattering. It is difficult to make a general argument justifying this approximation which will apply to all terms. It turns out, however, that for all important terms, one or both of the criteria stated at the end of Section II are reasonably well satisfied.

The formulation of (14) is general enough to include any kind of two-body forces, but the very generality conceals most of the physically interesting features of the problem. Therefore it is now proposed to restrict attention to central forces so that spin is independently conserved in the scattering. Tensor forces very likely are important in high energy phenomena; but past experience indicates that most conclusions about scattering, which can be reached from consideration of central forces only, except those which specifically have to do with interchange of spin and orbital angular momentum, also turn out to be true with tensor forces. In the hope that future work will confirm this feature for the present case, and because something is to be learned even if the assumption is incorrect, the remainder of this first paper, and all of the second one, will refer to central forces only. It will be evident, however, that the forces need not be derivable from static potentials.

IV. THE CENTRAL FORCE PROBLEM

With no coupling between spin and orbital angular momentum, the total spin and also the z-component of spin will be constants of the motion. Therefore, instead of considering all six possible initial spin functions and eight possible final spin functions, it suffices to consider two initial states, one quartet and one doublet, and three final, one quartet, and two doublet. These spin states have been discussed by Ashkin and Wu[3] and we depart from their scheme in only one respect. Whereas the initial doublet state for a given z-component of total spin is unique because of the requirement of symmetry in the spins of the two particles making up the deuteron, the final doublet state for the same z-component is twofold degenerate. Ashkin and Wu resolved the degeneracy according to the spin symmetry of the two particles which originally formed the deuteron. Here the degeneracy will be resolved according to the spin symmetry of the two neutrons.

There are three essential scattering amplitudes to compute and these will be designated as follows, departing from the general notation of formula (14). $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_p|{}^4R_{in}{}^S|\mathbf{k}_0,0)$ will be the non-antisymmetrized matrix for inelastic scattering in a quartet state. The latter is symmetric in the spins of the two neutrons, as indicated by the superscript S. Now ${}^4R_{in}{}^S$ must contain the factor $\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_p-\mathbf{k}_0)$, so let the co-factor of this delta-function be designated by $(\mathbf{k}_{12},\mathbf{k}_p|{}^4r_{in}{}^S|\mathbf{k}_0,0)$, where $\mathbf{k}_{12}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$. Then the cross section for quartet inelastic scattering is given by

$$ {}^4\sigma_{in}d\mathbf{k}_pd\mathbf{k}_{12}=2\pi/\hbar v_0|(\mathbf{k}_{12},\mathbf{k}_p|{}^4r_{in}{}^S|\mathbf{k}_0,0)-(\mathbf{k}_{21},\mathbf{k}_p|{}^4r_{in}{}^S|\mathbf{k}_0,0)|^2 \times\delta(E_1+E_2+E_p-E_0)d\mathbf{k}_pd\mathbf{k}_{12}, \quad (15)$$

where

$$E_1+E_2=\hbar^2/4m(\mathbf{k}_0-\mathbf{k}_p)^2+\hbar^2/mk_{12}^2,$$

since $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_0-\mathbf{k}_p$, and

$$E_p=\hbar^2/2mk_p^2,\quad E_0=\hbar^2/2mk_0^2-B.$$

The choice of final variables to be $\mathbf{k}_p$ and $\mathbf{k}_{12}$ is made because the proton momentum is the experimental quantity most likely to be measured.

For the doublet case there will be two distinct matrices, corresponding to final states symmetric and antisymmetric in the neutron spins. These will be designated by ${}^2R_{in}{}^S$ and ${}^2R_{in}{}^A$, respectively. Following the same convention as above, the doublet cross section will then be

$$ {}^2\sigma_{in}d\mathbf{k}_pd\mathbf{k}_{12} =2\pi/\hbar v_0\{|(\mathbf{k}_{12},\mathbf{k}_p|{}^2r_{in}{}^S|\mathbf{k}_0,0)-(\mathbf{k}_{21},\mathbf{k}_p|{}^2r_{in}{}^S|\mathbf{k}_0,0)|^2 +|(\mathbf{k}_{12},\mathbf{k}_p|{}^2r_{in}{}^A|\mathbf{k}_0,0)+(\mathbf{k}_{21},\mathbf{k}_p|{}^2r_{in}{}^A|\mathbf{k}_0,0)|^2\} \times\delta(E_1+E_2+E_p-E_0)d\mathbf{k}_pd\mathbf{k}_{12}. \quad (16)$$

The average cross section for unpolarized scattering would be given by two-thirds of (15) plus one-third of (16), the respective statistical weights of the quartet and doublet states.

The recipe stated in Section III permits an unambiguous calculation of the three required inelastic matrices in terms of the four two-body matrices which describe n–p and n–n scattering in triplet and in singlet states. In all cases, the two-body matrices will be taken outside the momentum integrals and evaluated at the final observed values of the momenta. To save space, the following shorthand notation for the two-body matrices will be used:

$$\left(\mathbf{k}_{1p}|r_{np}{}^t|\frac{\mathbf{k}_0+\mathbf{k}_2}{2}\right)=r_{1p}{}^t,$$
$$\left(\mathbf{k}_{1p}|r_{np}{}^s|\frac{\mathbf{k}_0+\mathbf{k}_2}{2}\right)=r_{1p}{}^s,$$
$$\left(\mathbf{k}_{12}|r_{nn}{}^t|\frac{\mathbf{k}_0+\mathbf{k}_p}{2}\right)=r_{12}{}^t,$$
$$\left(\mathbf{k}_{12}|r_{nn}{}^s|\frac{\mathbf{k}_0+\mathbf{k}_p}{2}\right)=r_{12}{}^s. \quad (17)$$

The superscripts t and s refer to triplet and singlet two-body interactions. This notation is really unambiguous since only these combinations of initial and final momenta, except for interchange of $\mathbf{k}_1$ and $\mathbf{k}_2$ can ever occur.

As an example of the procedure of calculation, consider the simplest case, the quartet, which is totally symmetric in all three spins. Only the triplet two-body interactions enter and no spin changes are induced. Because of the criteria for taking account of final state interactions the terms generated by R_{np} must be considered separately from those generated by R_{nn}. The contribution of the n–p interaction to ${}^4R_{in}{}^S$ is as follows, for $k_{2p}>k_{21}$:

$$\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_p-\mathbf{k}_0)\int d\mathbf{k}_{21}'g_{\mathbf{k}_{21}}{}^{t*}(\mathbf{k}_{21}')$$
$$\times\left(\frac{\mathbf{k}_1-\mathbf{k}_p}{2}+\frac{\mathbf{k}_{21}'-\mathbf{k}_{21}}{2}|r_{np}{}^t|\frac{\mathbf{k}_0+\mathbf{k}_2}{2}-\frac{\mathbf{k}_{21}'-\mathbf{k}_{21}}{2}\right)$$
$$\times g_0\left(\frac{\mathbf{k}_0-\mathbf{k}_p}{2}+\mathbf{k}_{21}'\right). \quad (18)$$

This term corresponds to the case in which the incident neutron transfers most of its momentum to the proton so that in the final state the two neutrons are the most strongly interacting pair. The function $g_{\mathbf{k}_{12}}{}^t(\mathbf{k}_{12}')$ is the triplet relative wave function (not antisymmetrized) in momentum space of two neutrons with relative momentum $\mathbf{k}_{12}$.

There are several reasons for thinking that the matrix $r_{np}{}^t$ which appears under the integral sign, does not in fact depend appreciably on $\mathbf{k}_{12}'$. If $\mathbf{k}_{12}$ corresponds to a relative energy greater than about 10 Mev, $g_{\mathbf{k}_{12}}{}^t(\mathbf{k}_{12}')\approx\delta(\mathbf{k}_{12}'-\mathbf{k}_{12})$, so that the peak of the integrand occurs in a region in which the dependence of $r_{np}{}^t$ on $\mathbf{k}_{12}'$ vanishes. If k_{12} is small, this will usually be because both k_1 and k_2 are small. Energy and momentum conservation, together with the character of the function g_0, which is appreciable only for small arguments, make it very unlikely that as a consequence of an n–p collision the two neutrons should go off with high but equal momenta. This is also rather obvious physically, since neutron number two has no way to acquire a high velocity. Now if k_1 and k_2 are both small, then $\mathbf{k}_p$ must be large and will dominate the final argument of $r_{np}{}^t$. The initial argument is likewise dominated by $\mathbf{k}_0$, so that again $\mathbf{k}_{21}$ plays only a small role. A different argument may be based on the experimentally observed fact[2] that for large momentum transfers $r_{np}{}^t$ depends much more sensitively on the sum of initial and final relative momenta than on the difference. The reverse appears to be true for small momentum transfers. Taking the sum of initial and final relative momenta in (18) just cancels out the dependence on $\mathbf{k}_{21}'$; and it turns out that when $k_{2p}<k_{21}$ (which corresponds to small momentum transfers), taking the difference would cancel out the dependence on $\mathbf{k}_{2p}'$, the variable of integration in that case.

Similar arguments can be made for the terms generated by the n–n interaction, so that in general, the removal of the two-body scattering matrices from the integrals is well justified. In the special case of (18), this leads to

$$\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_p-\mathbf{k}_0)r_{1p}{}^t$$
$$\times\int d\mathbf{k}_{21}'g_{\mathbf{k}_{21}}{}^{t*}(\mathbf{k}_{21}')g_0\left(\frac{\mathbf{k}_0-\mathbf{k}_p}{2}+\mathbf{k}_{21}'\right). \quad (19)$$

Overlap integrals, such as that which appears in (19), will appear in every term of the complete matrix. It is therefore appropriate to introduce another abbreviation as follows:

$$I_{21}{}^t=\int d\mathbf{k}_{21}'g_{\mathbf{k}_{21}}{}^{t*}(\mathbf{k}_{11}')g_0(\mathbf{k}_{0p}+\mathbf{k}_{21}')$$
$$=\int d\mathbf{r}\,\varphi_{\mathbf{k}_{21}}{}^{t*}(\mathbf{r})e^{-i\mathbf{k}_{0p}\cdot\mathbf{r}}\varphi_0(\mathbf{r}), \quad (20)$$

where $\varphi_{\mathbf{k}_{21}}{}^t(\mathbf{r})$ is the relative wave function of the two neutrons in configuration space. The other integrals will be designated by $I_{21}{}^s$, $I_{p2}{}^t$, $I_{p2}{}^s$, $I_{p1}{}^t$, etc., where the subscript identifies the pair of particles whose final interaction is being accounted for and the superscript gives their spin. If the pair ij is being considered, then the momentum occurring in the exponential of the configuration space integral is always $\frac{1}{2}(\mathbf{k}_i+\mathbf{k}_j)$, so that additional indices are unnecessary.

It is clear, then, that the complete inelastic scattering matrix can be expressed as a sum of products of the type, $r_{1p}{}^tI_{21}{}^t$. The origin of each term will be indicated by the superscripts and subscripts. This particular one, for instance, comes from a collision in the triplet state between neutron number one and the proton, leaving neutron number one with the smaller final momentum and in a triplet state with respect to neutron number two.

Evaluation of this sum of products is merely a matter of bookkeeping and only the results will be given here.

$$(\mathbf{k}_{12}, \mathbf{k}_p|{}^4r_{in}{}^S|k_0, 0) = \begin{Bmatrix} r_{1p}{}^t & I_{21}{}^t \\ r_{1p}{}^t & I_{2p}{}^t \end{Bmatrix} + \begin{Bmatrix} r_{12}{}^t & I_{p2}{}^t \\ r_{12}{}^t & I_{p1}{}^t \end{Bmatrix}$$

$$(\mathbf{k}_{12}, \mathbf{k}_p|{}^2r_{in}{}^S|\mathbf{k}_0, 0)$$

$$= \begin{Bmatrix} -\tfrac{1}{4}(r_{1p}{}^t - 3r_{1p}{}^s)I_{21}{}^t \\ \tfrac{1}{8}(r_{1p}{}^t + 3r_{1p}{}^s)I_{2p}{}^t - \tfrac{3}{8}(r_{1p}{}^t - r_{1p}{}^s)I_{2p}{}^s \end{Bmatrix}$$

$$+ \begin{Bmatrix} \tfrac{1}{8}(r_{12}{}^t + 3r_{12}{}^s)I_{p2}{}^t + \tfrac{3}{8}(r_{12}{}^t - r_{12}{}^s)I_{p2}{}^s \\ \tfrac{1}{8}(r_{12}{}^t + 3r_{12}{}^s)I_{p1}{}^t + \tfrac{3}{8}(r_{12}{}^t - r_{12}{}^s)I_{p1}{}^s \end{Bmatrix}, \quad (21)$$

$$(\mathbf{k}_{12}, \mathbf{k}_p|{}^2r_{in}{}^A|\mathbf{k}_0, 0)$$

$$= \begin{Bmatrix} \sqrt{3}/4(r_{1p}{}^t + r_{1p}{}^s)I_{21}{}^s \\ \sqrt{3}/8(r_{1p}{}^t + 3r_{1p}{}^s)I_{2p}{}^t + \sqrt{3}/8(r_{1p}{}^t - r_{1p}{}^s)I_{2p}{}^s \end{Bmatrix}$$

$$+ \begin{Bmatrix} \sqrt{3}/8(r_{12}{}^t + 3r_{12}{}^s)I_{p2}{}^t - \sqrt{3}/8(r_{12}{}^t - r_{12}{}^s)I_{p2}{}^s \\ \sqrt{3}/8(r_{12}{}^t + 3r_{12}{}^s)I_{p1}{}^t - \sqrt{3}/8(r_{12}{}^t - r_{12}{}^s)I_{p1}{}^s \end{Bmatrix}.$$

Of the two alternatives within each bracket, that one is to be chosen which takes account of the strongest interaction between a final pair of particles. It can be verified quickly that in case both final interactions are negligible, the two alternatives are equal. For example if $\varphi_{\mathbf{k}_{21}}{}^t(\mathbf{r}) \approx (1/2\pi)^{\frac{3}{2}} e^{i\mathbf{k}_{21}\cdot\mathbf{r}}$, then

$$I_{21}{}^t \approx (1/2\pi)^{\frac{3}{2}} \int d\mathbf{r} e^{-i(\mathbf{k}_2 - \mathbf{k}_1/2)\cdot\mathbf{r}} e^{-i(\mathbf{k}_2 + \mathbf{k}_1/2)\cdot\mathbf{r}} \varphi_0(\mathbf{r})$$

$$= (1/2\pi)^{\frac{3}{2}} \int d\mathbf{r} e^{-\mathbf{k}_2\cdot\mathbf{r}} \varphi_0(\mathbf{r}) = g_0(\mathbf{k}_2).$$

$I_{21}{}^s$, $I_{21}{}^t$, $I_{2p}{}^s$ all approach this same limit, while $I_{p2}{}^t$, $I_{p2}{}^s$, $I_{p1}{}^t$, $I_{p1}{}^s$ all approach the limit $g_0(\mathbf{k}_p)$. The formulas (21) simplify in this limit to

$$(\mathbf{k}_{12}, \mathbf{k}_p|{}^4r_{in}{}^S|\mathbf{k}_0, 0) \approx r_{1p}{}^t g_0(\mathbf{k}_2) + r_{12}{}^t g_0(\mathbf{k}_p),$$

$$(\mathbf{k}_{12}, \mathbf{k}_p|{}^2r_{in}{}^S|\mathbf{k}_0, 0) \approx -\tfrac{1}{4}(r_{1p}{}^t - 3r_{1p}{}^s)g_0(\mathbf{k}_2) + \tfrac{1}{2}r_{12}{}^t g_0(\mathbf{k}_p), \quad (21a)$$

$$(\mathbf{k}_{12}, \mathbf{k}_p|{}^2r_{in}{}^A|\mathbf{k}_0, 0) \approx \sqrt{3}/4(r_{1p}{}^t + r_{1p}{}^s)g_0(\mathbf{k}_2) + \sqrt{3}/2 r_{12}{}^s g_0(\mathbf{k}_p).$$

V. SUMMARY

A formula (21) has been derived for the complete n–d inelastic scattering matrix in the case of central forces only. Properly antisymmetrized with respect to the neutron variables, this matrix leads to the corresponding cross section through formulas (15) and (16). The impulse approximation which is the basis of (21) is believed to be better than the usual Born approximation, depending on the large radius of the deuteron, the high incident velocity of the neutron, and the short range of nuclear forces, but not on the weakness of nuclear forces. Appearing in (21) are two kinds of quantities, each of which refers to a two-body problem. The first kind are n–p and n–n scattering matrices, evaluated for the initial and final momenta which occur in the three-body problem. The second kind are overlap integrals between final continuum wave functions and the initial deuteron function.

If the theory of nuclear forces were on a firm basis, as it may be someday, it would be a straightforward procedure to solve the required number of two-body problems and substitute into (21). At the present time, it is perhaps more sensible to adopt an empirical approach in terms of experimental values for the two-body scattering matrices. In the second part of this report, which will follow when experiments at Berkeley are complete and have been analyzed, this author will attempt such an approach. In particular, an effort will be made to derive information about the high energy n–n interaction from the experimental observations.

It might seem that the overlap integrals which occur in (21) would preclude the empirical approach, but this is not so. The asymptotic form of these integrals for high final relative momenta depends only on the deuteron wave function, as seen in (21a): and at low relative energies the perturbation is important only in the S part of the final wave function. Low energy S-state interactions are very reliably described by the empirical theory of the effective range,[9] so no essential difficulty need be encountered there. Trouble could come from interference terms in the n–d cross section, and one of the chief tasks of the second part of this report will be to show that these terms are actually not likely to be large for inelastic scattering.

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