Distribution of Recoil Nucleus in Pair Production by Photons

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The angular and momentum distribution of the recoil nucleus in pair production by a photon is calculated covariantly by a method which utilizes the unitarity of the S matrix. The results are in disagreement with a recent experiment, particularly for small angles and high momentum transfers. The exact total cross section for pair creation is also given.

I. INTRODUCTION

RECENT attempts to measure the momentum and angular distributions of the recoil nucleus in pair production by energetic photons[1] have made a theoretical investigation of these effects desirable. There exists in the literature only an estimate of the momentum distribution.[2] An opportunity is also afforded of illustrating a method of calculation of somewhat more general applicability, which is based on the unitary character of the S matrix. A simplified and covariant calculation of the total pair production cross section and its asymptotic form (Bethe-Heitler formula[3]), is also carried through.

II. METHOD OF CALCULATION

In this discussion, the effect of the nucleus is represented by its static field

$$A_\mu(\mathbf{x})=\int \exp(-i\mathbf{q}\cdot\mathbf{x})A_\mu(\mathbf{q})d^3\mathbf{q}, \tag{1}$$

which is viewed as an external field. Higher radiative corrections are disregarded (see Appendix II), as is excitation of atomic states by the incident photon, and only the lowest term in the external field is computed (first Born approximation). The validity of some of these assumptions has been questioned,[4] but these effects are in any case not large.

The total cross section for pair production by a photon of energy k_0 (units in which $\hbar=c=1$ are used throughout) is then obtained in the form:

$$\sigma(k_0)=\int d^3\mathbf{q}A_\mu(\mathbf{q})A_\nu(-\mathbf{q})T_{\mu\nu}(\mathbf{q};k), \tag{2}$$

where $\mathbf{q}$ is the momentum transferred to the field and k is the energy momentum 4-vector of the photon, so that the integrand of (2) gives directly the differential cross section for obtaining a recoil nucleus of momentum $\mathbf{q}$.

To obtain $\sigma(k_0)$ we consider the scattering matrix, S, developed in a power series in the charge e:

$$S=1+eS_1+e^2S_2+\cdots. \tag{3}$$

If S^* is the Hermitian conjugate of S, the condition for the unitarity of S becomes

$$\begin{aligned}S^*S=1+e(S_1^*+S_1)+e^2(S_1^*S_1+S_2^*+S_2)\\+e^3(S_1^*S_2+S_2^*S_1+S_3^*+S_3)\\+e^4(S_1^*S_3+S_3^*S_1+S_2^*S_2+S_4^*+S_4)+\cdots=1,\end{aligned} \tag{4}$$

so that each of the expressions in parentheses must vanish. Here, apart from mass terms,

$$\begin{aligned}e^nS_n=\frac{(-i)^n}{n!}\int_{1,\cdots,n} P[j_{\mu1}(1),\cdots,j_{\mu n}(n)]\\ \times P[A_{\mu1}(1)+\phi_{\mu1}(1),\cdots,A_{\mu n}(n)+\phi_{\mu n}(n)],\end{aligned} \tag{5}$$

where $j_{\mu1}(1)=ie\bar{\psi}(x_1)\gamma_{\mu1}\psi(x_1)$, $\phi_{\mu1}(1)$ is the electromagnetic field operator at x_1, etc., and the integral is taken over all the indicated 4-spaces.[5]

Consider next the expectation value for a state with a single photon of momentum k_0 of the e^4 term of (4), limiting ourselves to terms quadratic in the external field. If the external field cannot create pairs, which is true of a static field, the S_1 terms give zero. Furthermore, if the initial state in S_2 is a one-photon state, the final state can only have an electron-positron pair. The $e^4S_2^*S_2$ term is then the sum of the squares of all matrix elements between the initial state and all possible final states containing a pair; that is, the total probability for the creation of any pair by the photon. The cross section $\sigma(k)$ is thus given by $\sigma(k_0)=(k_0|e^4S_2^*S_2|k_0)$, which by virtue of (4) becomes

$$\sigma(k_0)=e^4(k_0|-S_4^*-S_4|k_0)=-2Re(k_0|e^4S_4|k_0). \tag{6}$$

This form is much more convenient for computational purposes than is the conventional non-covariant summation over final states.[6] It is to be noted that the absence of internal photon lines in the matrix elements of interest here makes unnecessary any explicit mass subtraction.

Employing the usual prescription of Feynman and Dyson[5] to evaluate the S_4 matrix element and averaging over the polarizations of the photon, one obtains

$$\begin{aligned}T_{\mu\nu}(q,k)=+(e^4/8\pi k_0)Re[A_{\mu\nu}(q,k)+B_{\mu\nu}(q,k)\\+B_{\mu\nu}(-q,k)+A_{\mu\nu}(q,k)\\+B_{\mu\nu}(q-k)+B_{\mu\nu}(-q,-k)].\end{aligned} \tag{7}$$

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[1] G. E. Modesitt and H. W. Koch, Phys. Rev. **77**, 175 (1950).
[2] H. A. Bethe, Proc. Camb. Phil. Soc. **30**, 524 (1934).
[3] H. A. Bethe and W. Heitler, Proc. Roy. Soc. **146**, 83 (1934).
[4] J. Wheeler and W. Lamb, Phys. Rev. **55**, 858 (1939).
[5] F. J. Dyson, Phys. Rev. **75**, 486 (1949); **75**, 1736 (1949).
[6] Some other applications of Eqs. (4)–(6) are discussed briefly in Appendix I.

$A_{\mu\nu}$ and $B_{\mu\nu}$ arise respectively from the Feynman diagrams (a) and (b) in Fig. 1, and are given by:

$$A_{\mu\nu}(q,k)=\int d^4p\frac{Sp[\gamma_\mu(i\gamma\cdot p-m)\gamma_\sigma(i\gamma\cdot(p-k)-m)\gamma_\nu(i\gamma\cdot(p-k-q)-m)\gamma_\sigma(i\gamma\cdot(p-q)-m)]}{(p^2+m^2)((p-k)^2+m^2)((p-k-q)^2+m^2)((p-q)^2+m^2)}, \tag{8}$$

$$B_{\mu\nu}(q,k)=\int d^4p\frac{Sp[\gamma_\mu(i\gamma\cdot(p+q)-m)\gamma_\nu(i\gamma\cdot p-m)\gamma_\sigma(i\gamma\cdot(p+k)-m)\gamma_\sigma(i\gamma\cdot p-m)]}{[(p+q)^2+m^2](p^2+m^2)^2[(p+k)^2+m^2]}, \tag{9}$$

where, for instance, p stands for the 4-vector p_μ, pq for $p_\mu q_\mu=\mathbf{p}\cdot\mathbf{q}-p_0q_0$, etc.

Terms arising from diagrams such as (c) of Fig. 1 all involve a photon self-energy operator acting on a free photon state, and hence give zero. Even the formally divergent momentum integrals associated with such operators will not be encountered here, since only real terms are carried in (7).

III. EVALUATION OF $T_{\mu\nu}(q;k)$

In calculating $T_{\mu\nu}$, we make use of the gauge invariance of $\sigma(k_0)$. If a gauge transformation is performed on the $A_\mu(x)$, it is seen from (2) that for gauge invariance

$$q_\mu T_{\mu\nu}(q;k)=0. \tag{10}$$

$T_{\mu\nu}$ must be of the form

$$T_{\mu\nu}=+(e^4/8\pi k_0)Re[I_1\delta_{\mu\nu}+I_2k_\mu k_\nu+I_3q_\mu q_\nu+I_4(q_\mu k_\nu+q_\nu k_\mu)], \tag{11}$$

where the I's are invariant functions of q and k. Equation (10) then yields the conditions:

$$I_1+I_3q^2+I_4qk=0,\quad I_2qk+I_4q^2=0, \tag{12}$$

so that only two of the I's need be computed. (Actually, three were computed as a check.)

We write

$$I_i(q,k)=A_i(q,k)+B_i(q,k)+B_i(-q,k)+(k\to -k)\quad (i=1,2,3,4)\cdots, \tag{13}$$

where $A_1(q,k)$ is the contribution from $A_{\mu\nu}(q,k)$, etc., cf. (8), (9), and similarly for the remaining I's. In the calculation, imaginary parts will be dropped freely, by virtue of (11).

An evaluation of the spur in (9), and use of the first of the relations:[7]

$$\frac{1}{abc^2}=3!\int_0^1dx\int_0^x dy\frac{y}{[a+(b-a)x+(c-b)y]^4},$$
$$\frac{1}{abcd}=3!\int_0^1dx\int_0^x dy\int_0^y dz\times\frac{1}{[a+(b-a)x+(c-b)y+(d-c)z]^4} \tag{14}$$

then yields:

$$B_2=-48\int d^4p\int_0^1dx\int_0^x dy(x-y)y\frac{p^2(1+4(x-y)+2[(x-y)(Q^2+3m^2-2Qk)+Q^2+m^2]}{[p^2+m^2+q^2(1-x)-Q^2]^4}, \tag{15}$$

$$B_3=-48\int d^4p\int_0^1dx\int_0^x dy(1-x)y\frac{p^2(1-4x)-2x(Q^2+3m^2-2Qk)}{[p^2+m^2+q^2(1-x)-Q^2]^4}, \tag{16}$$

where $Q=q(1-x)+k(x-y)$. Then, making use of:[7]

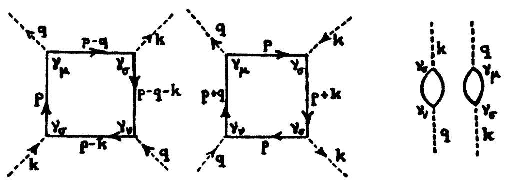
$$\int d^4p\frac{1}{[p^2+\lambda]^4}=\frac{\pi^2i}{6\lambda^2},\quad \int d^4p\frac{p^2}{[p^2+\lambda]^4}=\frac{\pi^2i}{3\lambda} \tag{17}$$

and the transformations $x-y=\eta$, $1-x=\xi$, it follows that

$$B_2=-16\pi^2i\int_0^1d\xi\int_0^{1-\xi}d\eta\cdot\eta(1-\xi-\eta)\times\left[\frac{1+4\eta}{m^2+q^2\xi-Q^2}+\frac{\eta(Q^2+3m^2-2qk\xi)+Q^2+m^2}{(m^2+q^2\xi-Q^2)^2}\right], \tag{18}$$

$$B_3=-16\pi^2i\int_0^1d\xi\int_0^{1-\xi}d\eta\cdot\xi(1-\xi-\eta)\times\left[\frac{4\xi-3}{m^2+q^2\xi-Q^2}-\frac{(1-\xi)(Q^2+3m^2-2qk\xi)}{(m^2+q^2\xi-Q^2)^2}\right], \tag{19}$$

with $Q=q\xi+k\eta$.



(a) (b) (c)

FIG. 1. Feynman diagrams for S_4.

[7] R. P. Feynman, Phys. Rev. 76, 769 (1949).

Consider first B_3: integrating (19) with respect to η, and dropping some purely imaginary terms, one obtains:

$$B_3=-\frac{16\pi^2 i}{\beta^2}\int_0^1 d\xi\frac{1}{\xi}[(3\xi-2)\{m^2+4r^2\xi(1-\xi)\}$$
$$+4(1-\xi)(m^2+r^2\xi)]\{\ln[m^2+4r^2\xi(1-\xi)]$$
$$-\ln[m^2+q^2\xi(1-\xi)]\}, \quad (20)$$

where

$$\beta=2qk \quad \text{and} \quad 4r^2=(q-k)^2. \quad (20a)$$

Since $q^2\geqslant 0$ for a static field, the term in

$$\ln[m^2+q^2\xi(1-\xi)]$$

may be dropped if the logarithm is chosen to be real at $\xi=0$. Equation (20) will then be purely imaginary unless

$$r^2<-m^2, \quad (21)$$

in which case $m^2+4r^2\xi(1-\xi)$ has two zeros in the range of integration, at

$$\xi_1=\tfrac{1}{2}(1-\omega), \quad \xi_2=\tfrac{1}{2}(1+\omega), \quad [\omega=(1+m^2/r^2)^{\frac{1}{2}}]. \quad (22)$$

Since m^2 must be thought to contain a small negative imaginary part, the path of integration for (20) is that of Fig. 2. For this branch of the logarithm we have

$$\begin{aligned}\ln[m^2+4r^2\xi(1-\xi)]\\ &=\ln|m^2+4r^2\xi(1-\xi)| && \text{from } 0 \text{ to } \xi_1\\ &\ln|m^2+4r^2\xi(1-\xi)|-i\pi && \text{from } \xi_1 \text{ to } \xi_2\\ &\ln|m^2+4r^2\xi(1-\xi)| && \text{from } \xi_2 \text{ to } 1.\end{aligned}$$

Then, if $f(\xi)$ is regular along the path,

$$Im\int_0^1 (d\xi/\xi)f(\xi)\ln[m^2+4r^2\xi(1-\xi)]$$
$$=-i\pi\int_{\xi_1}^{\xi_2} d\xi(1/\xi)f(\xi). \quad (23)$$

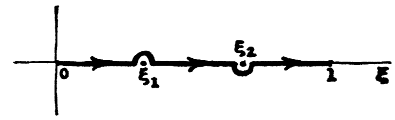


FIG. 2. Path of integration for B_3.

Then

$$B_3=-\frac{16\pi^2 i}{\beta^2}(-i\pi)\int_{\xi_1}^{\xi_2}\frac{d\xi}{\xi}[2m^2+\xi(-4r^2-m^2)$$
$$+\xi^2(16r^2)+\xi^3(-12r^2)] \quad (24)$$
$$=\frac{32\pi^3 m^2}{\beta^2}\left[\ln\frac{1-\omega}{1+\omega}+\omega\right].$$

In a completely analogous manner, one obtains from (18), after considerable calculation:

$$B_2=\frac{16\pi^3}{\beta^4}\left\{q^2(12m^2q^2-8m^2\beta-\beta^2)\ln\left(\frac{1-\omega}{1+\omega}\right)\right.$$
$$+\omega[\beta^3+2\beta^2(m^2-q^2)$$
$$\left.+2\beta q^2(-8m^2+q^2)+2q^4(8m^2-q^2)]\right\}. \quad (25)$$

Next, using the second of the relations (14) in Eq. (8), replacing $\int_0^1 dx\int_0^x dy\int_0^y dz\cdots$ by $\int_0^1 dy\int_0^y dz\int_y^1 dx\cdots$, and making the transformation of variables

$$z=(1-\xi)y-\zeta y, \quad x=y(1-\zeta)+\xi(1-y),$$

so that

$$\int_0^y dz\int_y^1 dx\cdots \text{ becomes } y\left[\int_0^{1-y}d\zeta\int_{\zeta y/(1-y)}^{1-\zeta}d\xi\cdots\right.$$
$$\left.+\int_{-(1-y)}^0 d\zeta\int_{-\zeta}^{1+\zeta y/(1-y)}d\xi\cdots\right],$$

we obtain after some algebra (putting $-\zeta$ for ζ in the second part of this integral, and then $\zeta=(1-y)t$):

$$A_{\mu\nu}(q,k)=6\int d^4p\int_0^1 dyy(1-y)\int_0^1 dt\left\{\int_{yt}^{1-(1-y)t}d\xi\frac{Sp[\ \ ]}{[(p-Q)^2+m^2+q^2y(1-y)+2qky(1-y)t]^4}\right.$$
$$\left.+\int_{(1-y)t}^{1-yt}d\xi\frac{Sp[\ \ ]}{[(p-Q)^2+m^2+q^2y(1-y)-2qky(1-y)t]^4}\right\}. \quad (26)$$

Here, $Q=k\xi+qy$, and the spur is that appearing in (8). Now, evaluating the spur, putting $(1-y)$ for y in the second term of (26), and putting

$$\begin{aligned}A_2(q,k)&=A_2'(q,k)+A_2'(-q,k),\\ A_3(q,k)&=A_3'(q,k)+A_3'(-q,k)\end{aligned} \quad (27)$$

we obtain

$$A_2'(q,k)=-96\int d^4p\int_0^1 dyy(1-y)\int_0^1 dt\int_{(1-y)t}^{1-yt}d\xi\frac{-(p^2/2)-m^2+q^2y(1-y)-(2m^2+q^2)\xi(1-\xi)}{[p^2+m^2+q^2y(1-y)-2qky(1-y)t]^4},$$

$$A_3'(q,k)=-48\int d^4p\int_0^1 dyy(1-y)\int_0^1 dt\int_{(1-y)t}^{1-yt}d\xi\frac{p^2+2m^2(1-2y(1-y))}{[p^2+m^2+q^2y(1-y)-2qky(1-y)t]^4}. \quad (28)$$

Using (17) and performing first the ξ-integration, we obtain, using arguments similar to those used above in obtaining B_3,

$$A_3'=-\frac{16\pi^3}{\beta^2}[q^2-\beta+2m^2]\omega,$$

$$A_2'=\frac{16\pi^3}{\beta^4}\bigg\{2(2m^2+q^2)q^2(\beta-q^2+2m^2)\ln\frac{1-\omega}{1+\omega}$$
$$-\omega[\beta^2+\beta^2(-q^2+2m^2)+(2m^2+q^2)6q^2(q^2-\beta)]\bigg\}. \quad (29)$$

Collecting the results (24), (25), and (29), we have in view of (27) and (13)

$$I_2=\frac{64\pi^3q^2}{\beta^4}\bigg\{\ln\frac{1-\omega}{1+\omega}[2(4m^4+6m^2q^2-q^4)$$
$$+2(-2m^2+q^2)\beta-\beta^2]$$
$$+\omega[4(q^2-\beta)(m^2-2q^2)-\beta^2]\bigg\}, \quad (30)$$

$$I_3=\frac{64\pi^3}{\beta^2}\left[2m^2\ln\frac{1-\omega}{1+\omega}+\omega(\beta-q^2)\right], \quad (31)$$

if (21) is satisfied, and otherwise

$$I_2=I_3=0. \quad (32)$$

Then, from (12)

$$I_4=-(\beta/2q^2)I_2;\quad I_1=-q^2I_3+(\beta^2/4q^2)I_2. \quad (33)$$

Equations (30)–(33), substituted in (11), then complete the evaluation of $T_{\mu\nu}$. The distribution of the momentum vector $\mathbf{q}$ of the recoil nucleus, which is essentially $T_{\mu\nu}$, is thus given exactly and fairly concisely in terms of elementary functions by this method.

As a check on the correctness of the expression for $T_{\mu\nu}$, it is very easy to obtain from it the cross section for the production of pairs by two photons.[8] For this purpose, the momentum q must also be taken to belong to a photon (so that $q^2=0$) and the A_μ in (2) must be replaced by the creation operators ϕ_μ. Averaging over polarizations, the total probability for pair creation by the two photons is then proportional to

$$\frac{1}{2q_0}\cdot\frac{1}{2}T_{\mu\mu}(q,k)=\frac{e^4}{16\pi k_0q_0}(2I_1+kqI_4)\bigg|_{q^2=0}$$
$$=\frac{e^4}{64\pi q_0k_0}\left(\frac{\beta^2}{q^2}I_2\right)\bigg|_{q^2=0}$$

or

$$\sigma=\frac{\pi^2e^4}{\beta^2}\bigg\{\ln\frac{1-\omega}{1+\omega}(8m^4-4m^2\beta-\beta^2)-\omega\beta(\beta+4m^2)\bigg\},$$

which is the correct result.

[8] G. Breit and J. A. Wheeler, Phys. Rev. **46**, 1087 (1934).

IV. THE MOMENTUM DISTRIBUTION OF THE RECOIL NUCLEUS

For a screened Coulomb field, $A_\mu(\mathbf{q})$ as computed from (1) is:

$$\mathbf{A}(\mathbf{q})=0;\quad A_0(\mathbf{q})=\frac{Ze}{(2\pi)^3}\frac{1}{\mathbf{q}^2}[1-F(\mathbf{q}^2)]. \quad (34)$$

Then, since $T_{00}=-T_{44}=-I_1+I_2k_0^2$ from (11) and (33), we have from (2)

$$\sigma(k_0)=\frac{-Ze^6}{4k_0(2\pi)^7}\int d^3\mathbf{q}\frac{(1-F(\mathbf{q}^2))^2}{\mathbf{q}^4}(I_1-I_2k_0^2) \quad (35)$$

where

$$I_1-I_2k_0^2=\frac{16\pi^3}{\beta^2}\bigg[-4q^2\bigg\{2m^2\ln\frac{1-\omega}{1+\omega}+\omega(\beta-q^2)\bigg\}$$
$$-\frac{\beta^2-4q^2k_0^2}{\beta^2}\bigg\{\omega[\beta^2-4(q^2-\beta)(m^2-2q^2)]$$
$$+\ln\frac{1-\omega}{1+\omega}[\beta^2+2(\beta-q^2)(2m^2-q^2)-8m^2(q^2+m^2)]\bigg\}\bigg] \quad (36)$$

if $r^2<-m^2$, and is zero otherwise.

Next, let us make the transformation of variables

$$2mQ=|\mathbf{q}|;\quad \mu=[k_0^2-|\mathbf{q}-\mathbf{k}|^2]/4m^2,\quad \alpha=k_0/2m. \quad (37)$$

Then $\int d^3\mathbf{q}$ over the region $r^2<-m^2$ or $|\mathbf{q}-\mathbf{k}|^2<k^2-4m^2$ becomes

$$\int_{\alpha-(\alpha^2-1)^{\frac{1}{2}}}^{\alpha+(\alpha^2-1)^{\frac{1}{2}}}dQ\int_1^{2\alpha Q-Q^2}d\mu\pi Q\cdot 4m^2/\alpha$$

and

$$\sigma(k_0)=\int_{\alpha-(\alpha^2-1)^{\frac{1}{2}}}^{\alpha+(\alpha^2-1)^{\frac{1}{2}}}dQP(Q,k_0). \quad (38)$$

$P(Q,k_0)$ is the distribution of the momentum Q of the recoil nucleus, in units of $2m$, and is given by

$$P(Q,k_0)=(Z^2e^6/32\pi^3k_0^2)[1-F(Q^2)]^2I(Q,\alpha)/Q^3,$$
$$I(Q,\alpha)=\{J_0+(1-2Q^2)J_1+(2Q^4-Q^2-4Q^2\alpha^2-\tfrac{1}{2})J_3$$
$$+2Q^2\alpha^2(1+6Q^2-4Q^4)J_4$$
$$+I_0+(1-4Q^2)I_1+[(4Q^2-1)Q^2-4Q^2\alpha^2]I_2$$
$$+4Q^2\alpha^2(8Q^2-1)I_3+4Q^4\alpha^2(1-8Q^2)I_4\}, \quad (39)$$

where

$$I_n=\int_1^y\frac{d\mu}{(Q^2+\mu)^n}\omega,\quad J_n=\int_1^y\frac{d\mu}{(Q^2+\mu)^n}\ln\frac{1-\omega}{1+\omega},$$
$$\omega=(1-1/\mu)^{\frac{1}{2}},\quad y=2\alpha Q-Q^2. \quad (40)$$

With the exception of J_1, the integrals I_n and J_n are all elementary and for $n>1$ can be computed from I_1

and J_1 by differentiations with respect to the parameter Q^2. J_1 can be expressed, in terms of the functions[9]

$$R(t)=\int_0^t \ln(1+x)/x dx, \tag{41}$$

as

$$J_1=-R(1/Z\lambda)-R(\lambda/Z)+\pi^2/6+\tfrac{1}{2}(\ln\lambda)^2 +\tfrac{1}{2}(\ln Z)^2-(\ln Z)(\ln 8\alpha Q); \tag{42}$$

$$Z=[(y-1)^{\frac{1}{2}}+y^{\frac{1}{2}}]^2, \quad \lambda=[Q+(Q^2+1)^{\frac{1}{2}}]^2.$$

Then, after a lengthy calculation, one obtains:

$$\begin{aligned} I(Q,\alpha)=&(1-2Q^2)J_1+\left(1-4Q^2-8Q\alpha+\frac{4Q^2-1}{3\alpha Q}\right)\\ &\times\ln[(y)^{\frac{1}{2}}+(y-1)^{\frac{1}{2}}]+\left(3+\frac{2\alpha}{3Q}+\frac{2Q^2-1}{3\alpha Q}\right)(y(y-1))^{\frac{1}{2}}\\ &+\left\{-2(1+Q^2)+\frac{2\alpha^2}{3}\left(-4+\frac{1}{Q^2}\right)\right\}\frac{1}{(1+1/Q^2)^{\frac{1}{2}}}\\ &\times\ln\left\{\frac{(1+1/Q^2)^{\frac{1}{2}}-(1-1/y)^{\frac{1}{2}}}{(1+1/Q^2)^{\frac{1}{2}}+(1-1/y)^{\frac{1}{2}}}\right\}. \end{aligned} \tag{43}$$

As a rather stringent test of (39), (38) was integrated asymptotically and found to give precisely the Bethe-Heitler formula.[10] Care must be taken because certain terms in (43) very nearly cancel over a large region. In fact, in the region where $y=2\alpha Q-Q^2\approx 2\alpha Q$ (i.e., $Q\ll\alpha$), a much more useful expression is:

$$\begin{aligned} I(Q,\alpha)\approx&(1-2Q^2)J_1+\frac{1}{2}\left(1-4y-\frac{2}{3y}\right)\ln Z\\ &+\frac{y^2(1-1/y)^{\frac{1}{2}}}{3}\left[11-\frac{13}{3}\left(1-\frac{1}{y}\right)-2\left(1-\frac{1}{y}\right)^2\right] \end{aligned} \tag{44}$$

and if $Q\ll 1$,

$$(1-2Q^2)J_1\approx -2R(1/Z)-(\ln Z)(\ln 4y) +\tfrac{1}{6}\pi^2+\tfrac{1}{2}(\ln Z)^2. \tag{45}$$

In the latter region, it is therefore seen that $I(Q,\alpha)$ is a function only of $y\approx 2\alpha Q$.

In Fig. 3, the momentum distribution $P(Q, k_0)$ is plotted for several values of k_0 of possible experimental interest. The general behavior of the curves does not differ appreciably from that estimated by Bethe (reference 2, p. 537). The effect of screening may be taken into account by inserting appropriate values for

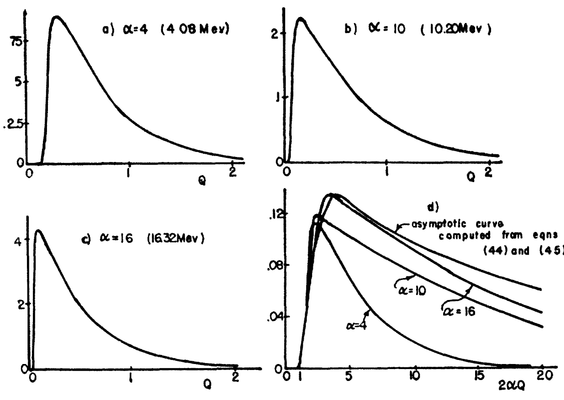


FIG. 3. Momentum distribution of recoil nucleus for several photon energies. Curves (a), (b), and (c): Ordinate: $P(Q,k_0)/(Z^2e^6/32\pi^3m^2)$. Abscissa: Q = momentum/$2m$. Curve (d): Ordinate: $P(Q, k_0)/(2\alpha Z^2e^6/32\pi^3m^2)$. Abscissa: $2\alpha Q$.

F in (39). It is easily verified, however, that for all Z and all energies considered in Fig. 3, the effect of screening is completely negligible.

V. ANGULAR DISTRIBUTION

Since, for the energies of interest here, screening may be omitted, F in Eq. (34) can be taken to be zero. We introduce in Eqs. (37) and (36) the variable Q of Eq. (37) and

$$\eta=\alpha\cos\theta=\alpha\mathbf{q}\cdot\mathbf{k}/qk_0, \tag{46}$$

θ being the angle the recoil nucleus makes with the direction of the incident photon. It then follows that

$$\begin{aligned} \sigma(k_0)=&\frac{Z^2e^6}{16\pi^3k_0^2}\int_1^\alpha d\eta\int_{\eta-(\eta^2-1)^{\frac{1}{2}}}^{\eta+(\eta^2-1)^{\frac{1}{2}}}\frac{dQ}{Q^2}\\ &\times\left\{\ln\frac{1-\omega}{1+\omega}\left[\left(1-\frac{\alpha^2}{\eta^2}\right)\left[1-\frac{1}{4\eta^2}+\frac{1}{2\eta Q}\right.\right.\right.\\ &\left.\left.-\frac{1}{8Q^2\eta^2}-\frac{Q}{\eta}+\frac{Q^2}{2\eta^2}\right]+\frac{\alpha^2}{2\eta^4}\right]\\ &+\omega\left[\left(1-\frac{\alpha^2}{\eta^2}\right)\left(1-\frac{1}{4\eta^2}+\frac{1}{2\eta Q}\right)\right.\\ &\left.\left.+\frac{1}{\eta^2}\left(1-\frac{2\alpha^2}{\eta^2}\right)(-2Q\eta+Q^2)\right]\right\}, \end{aligned} \tag{47}$$

where $\omega=[1-1/(2Q\eta-Q^2)]^{\frac{1}{2}}$.

Introducing the variable $\mu=2\eta Q-Q^2$, (47) becomes

$$\sigma(k_0)=(Z^2e^6/16\pi^3k_0^2)\int_1^\alpha d\eta I(\eta;\alpha),$$

[9] See W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 172. ($R(t)$ is there called $F(t)$). $R(x)$ is equal to the function $-\xi(1, 21-x)$ tabulated by K. Mitchell, Phil. Mag. **40**, 351 (1949).

[10] W. Heitler, reference 6, p. 200, Eq. (15).

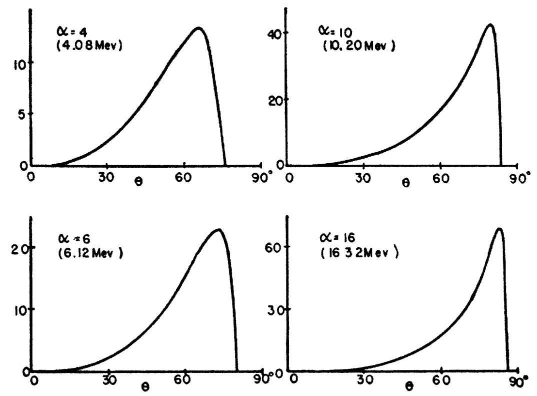


FIG. 4. Angular distribution of recoil nucleus for several photon energies. Ordinate: $P(\theta, k_0)/(Z^2e^6/64\pi^3m^2)$. Abscissa: Angle of scattering θ in degrees.

with

$$I(\eta;\alpha)=\int_1^{\eta^2} d\mu \frac{1}{(\eta^2-\mu)^{\frac{1}{2}}}\Bigg[\omega\Bigg\{\left(1-\frac{\alpha^2}{\eta^2}\right)\times\left(\frac{2\eta^2}{\mu^3}+\frac{2(\eta^2-1)}{\mu^2}+\frac{(1/4\eta^2)-1}{\mu}\right)+\left(1-\frac{2\alpha^2}{\eta^2}\right)\left(-\frac{2}{\mu}+\frac{1}{\eta^2}\right)\Bigg\}+\ln\frac{1-\omega}{1+\omega}\Bigg\{\frac{\alpha^2}{\eta^2\mu^2}-\frac{\alpha^2}{2\eta^4\mu}+\left(1-\frac{\alpha^2}{\eta^2}\right)\left(+\frac{1}{2\eta^2}+\frac{1-8\eta^2}{4\eta^2\mu}+\frac{(2\eta^2-2-(1/8\eta^2))}{\mu^2}+\frac{2\eta^2+1}{\mu^3}-\frac{\eta^2}{\mu^4}\right)\Bigg\}\Bigg]$$

with
$$\omega=(1-1/\mu)^{\frac{1}{2}}. \tag{48}$$

We may then write

$$\sigma(k_0)=\int_0^{\cos^{-1}1/\alpha} d\theta P(\theta, k_0),$$

where

$$P(\theta, k_0)=(Z^2e^6\alpha/16\pi^3k_0^2)I(\eta;\alpha)\sin\theta \tag{49}$$

is the angular distribution of the recoil nucleus.

Integrating the term in $\ln[(1-\omega)/(1+\omega)]$ in (48) by parts and combining with the remaining terms, we obtain:

$$I(\eta, \alpha)=I_1(\eta, \alpha)+I_2(\eta, \alpha), \tag{50}$$

where

$$I_1(\eta, \alpha)=\frac{2}{\eta}\left[\left(\frac{37}{9}-\frac{14}{9}\eta^2\right)E-\left(\frac{11}{9}+\frac{4}{3\eta^2}\right)F\right]+\frac{2\alpha^2}{\eta^3}\left[\left(-\frac{73}{9}+\frac{14}{9}\eta^2\right)E+\left(\frac{29}{9}+\frac{10}{3\eta^2}\right)F\right], \tag{51}$$

and F and E are the complete elliptic integrals of the first and second kind, respectively, of modulus $(1-1/\eta^2)^{\frac{1}{2}}$:

$$F=\int_0^{\pi/2}\frac{d\phi}{\left[1-\left(1-\frac{1}{\eta^2}\right)\sin^2\phi\right]^{\frac{1}{2}}}=F\left[\left(1-\frac{1}{\eta^2}\right)^{\frac{1}{2}}\right],$$

$$E=\int_0^{\pi/2}d\phi\left[1-\left(1-\frac{1}{\eta^2}\right)\sin^2\phi\right]^{\frac{1}{2}}=E\left[\left(1-\frac{1}{\eta^2}\right)^{\frac{1}{2}}\right].$$

I_2 is not expressible in such simple form, and is:

$$I_2=\left(1-\frac{\alpha^2}{\eta^2}\right)\frac{1}{\eta}\int_1^{\eta^2}\frac{d\mu}{[\mu(\mu-1)]^{\frac{1}{2}}}\ln\frac{\eta-(\eta^2-\mu)^{\frac{1}{2}}}{\eta+(\eta^2-\mu)^{\frac{1}{2}}}=\left(1-\frac{\alpha^2}{\eta^2}\right)\frac{4}{\eta}F_1(\eta), \tag{52}$$

where

$$F_l(\eta)=\int_1^{\eta}\frac{d\xi}{\xi^l}F\left[\left(1-\frac{1}{\xi^2}\right)^{\frac{1}{2}}\right]. \tag{53}$$

Series expansions for the integral in (52) are easily obtainable, though numerical integrations proved more convenient for the actual calculations used to construct the graphs in Fig. 4.

It may be of interest to note that using (51) and (52) the integration indicated in (48) can be carried out in terms of complete elliptic integrals, F_1, and a simple integral over F_1. One obtains:

$$\sigma(k_0)=\bar{\phi}(1/\alpha^2)(W_1+W_2);\quad \bar{\phi}=(Z^2e^6/64\pi^3m^2), \tag{54}$$

where

$$W_1=\frac{2}{9}[37E_1-14E_{-1}-11F_1-12F_3+\alpha^2(-73E_3+14E_1+29F_3+30F_5)], \tag{55}$$

$$W_2=4\int_1^{\alpha}\frac{F_1(\eta)}{\eta}d\eta+2F_1(\alpha)+2F\left[\left(1-\frac{1}{\alpha^2}\right)^{\frac{1}{2}}\right]-2\alpha^2E\left[\left(1-\frac{1}{\alpha^2}\right)^{\frac{1}{2}}\right]. \tag{55}$$

W_1 can be further simplified[11] to

$$W_1=\frac{2}{9}\left\{-18F_1(\alpha)+\left(-\frac{64}{3}+\frac{82\alpha^2}{3}\right)E\left[\left(1-\frac{1}{\alpha^2}\right)^{\frac{1}{2}}\right]+\left(\frac{98}{3}+\frac{2}{\alpha^2}+14\alpha^2\right)F\left[\left(1-\frac{1}{\alpha^2}\right)^{\frac{1}{2}}\right]\right\}. \quad (56)$$

(55) and (56) then yield the exact pair production total cross section:

$$\sigma(k_0)=\bar{\phi}\frac{1}{\alpha^2}\left[4\int_1^\alpha \frac{F_1(\eta)}{\eta}d\eta+2F_1(\alpha)+\frac{2}{27}\left\{-(64+109\alpha^2)E\left[\left(1-\frac{1}{\alpha^2}\right)^{\frac{1}{2}}\right]+\left(125+\frac{6}{\alpha^2}+42\alpha^2\right)K\left(1-\frac{1}{\alpha^2}\right)^{\frac{1}{2}}\right\}\right]. \quad (57)$$

Using the following identities:

$$F(k)=(1+k_1)F(k_1),$$

where $k_1=[1-(1-k^2)^{\frac{1}{2}}]/[1+(1-k^2)^{\frac{1}{2}}]$

$$E(k)=[2/(1+k_1)]E(k_1)-(1-k_1)F(k_1).$$

(57) can be shown to be equivalent to a formula previously obtained by Racah.[12] This provides a very severe check on all the preceding algebra.

VI. COMPARISON WITH EXPERIMENT

The curves in Figs. 3 and 4 can be compared with the experimental data given by Modesitt and Koch.[1] Since these data give only relative intensities, the comparison is somewhat ambiguous. If the maxima of the experimental and theoretical curves are adjusted to agree approximately, the experimental curves are found to be too low for smaller angles and too high for high momentum transfers; the disagreement for the momentum distribution being particularly sharp.

APPENDIX I. OTHER APPLICATIONS OF THE UNITARITY CONDITION

An interesting application of Eqs. (4)–(6) seems to us to be the basis of an investigation by J. A. Wheeler and J. S. Toll,[13] the idea being to compute the imaginary part of S_4 from the real part, which is determined from (6) by $S_2{}^*S_2$. We shall illustrate the method for our case, and show also that the necessary assumptions are justified.

For this, we write the diagonal element of the gauge-invariant S_4 by the usual methods in the form (2):

$$S_4=\int d^3\mathbf{q}A_\mu(q)A_\nu(-q)T_{\mu\nu}(q,k).$$

For $T_{\mu\nu}$ we again have a decomposition of the type (11), (12), the invariants I_j now including real and imaginary parts. By expressing these invariants in the variables $q^2\geqslant 0$ and r^2 (Eq. (20a)), eliminating $\beta=q^2-4r^2$, and letting $r=-\zeta$ be a complex variable, one easily sees from (18), (19), and (28) that for fixed q^2, all the invariants I_j are analytic functions of ζ, with singularities only on the real axis.

It then follows that the real part of all the invariants is a regular potential function in either half-plane. Therefore, the values on the real axis define the real part of these invariants on one half-plane, and therefore also the imaginary part to within an additive constant. The constant is determined by the condition that the imaginary parts vanish for large real ζ. The formal connection reads, for real ζ:

$$Im[I(\zeta)]=\frac{1}{\pi i}P\int_{-\infty}^{\infty}\frac{Re[I(\zeta')]}{\zeta'-\zeta}d\zeta',$$

where $Re[I(\zeta)]=0$ for $\zeta<m^2$ (Eq. (21)). Physically, this means that we can deduce the *forward* scattering of light by a spherically symmetric static field (this restriction resulting from our summation over the polarizations of the light quantum) from the tensor $T_{\mu\nu}$ of (2), since this is the effect described by S_4.[14]

Of course, this procedure is widely generalizable. First, the restriction of (4) to diagonal matrix elements may be dropped, so that the full matrix S_4 may be computed from the second-order matrix S_2. Moreover, one can probably apply the same method to the calculation of radiative corrections. We do not believe, however, that the method is of great help for computation purposes, but it is certainly of interest in showing the interconnection between various parts of the S matrix.

APPENDIX II. ORDER OF MAGNITUDE OF RADIATIVE CORRECTIONS

To obtain an estimate of the order of magnitude of the radiative corrections to the total cross section for pair creation in a Coulomb field, we use the Weizsäcker-Williams method,[15] which is valid asymptotically for extremely high energies.

We transform to the Lorentz system in which the incoming photon, of energy k_0 in the rest system of the nucleus, has an energy m. As we assume $k_0\gg m$, the velocity of this system will be close to unity and

$$\xi=1/(1-\beta^2)^{\frac{1}{2}}\cong k_0/2m\gg 1.$$

The field of the fast-moving nucleus of charge Ze is decomposed into a suitable distribution of photons. The number of equivalent photons with an impact parameter S and an energy between ν and $\nu+d\nu$ reads

$$p(S,\nu)d\nu=(4Z^2e^2/\pi)\cdot\nu d\nu|K_1(S\nu/\xi)|^2/\xi^2,$$

where $K_1(Z)$ is that of Magnus and Oberhettinger.[16] The total cross section is therefore given by

$$\sigma(k_0)=2\pi\int_0^\infty d\nu\int_{S_{\min}}^\infty SdSp(S,\nu)Q(\nu,m),$$

where $Q(\nu,m)$ denotes the total cross section for pair-creation by the head-on collision of two photons with energies ν and m. ($Q(\nu,m)$ is given in lowest order by the Breit-Wheeler formula, see Section III.) $S_{\min}$ denotes a smallest impact parameter for the production of the electron pair, and will be of order $1/m$.

[11] Jahnke-Emde, *Tables of Functions* (Dover Publications, New York, 1945), pp. 73 ff.

[12] G. Racah, Nuovo Cimento 13, 69 (1936).

[13] (To be published.) We are indebted to Professor Wheeler for showing us a preliminary draft, and for several discussions.

[14] Professor Wheeler informs us that Mr. Toll is applying their method to this effect.

[15] W. Heitler, reference 6, p. 263 ff. E. J. Williams, Kgl. Dansk Vid. Selsk. Math.-Fys. Medd. 13, 4 (1935).

[16] Magnus and Oberhettinger, *Special Functions of Math. Phys.* (Chelsea Publishing Company, New York, 1949), p. 29.

The integration over S is straightforward and yields

$$\sigma(k_0)=-\frac{16Z^2e^2}{k^2}m^2\cdot S_{\min}{}^2\int_0^\infty \nu d\nu Q(\nu,m)\{K_1{}^2(\zeta_m)-K_0(\zeta_m)K_2(\zeta_m)\},$$

where $\zeta_m=S_{\min}\cdot 2m\nu/k_0\cong 2\nu/k_0$.

If $Q(\nu,m)$ is to contain radiative correction $\sim e^2$, one must include the possibility for the emission of a photon, besides the pair. The dependence of the cross section on the energy of this photon is uninteresting, and even a classical effect when the photon has a low energy. $Q(\nu,m)$ should therefore be the cross section for pair creation including possible emission of a photon. As $Q(\nu,m)$ must fall off for high energies ν (as is generally found for such cross sections in quantum electrodynamics), the main contributions to $\sigma(k_0)$ will come from small values of ν/m. We can therefore expand the bracket $\{\ \}$ with respect to ζ_m, obtaining

$$\begin{aligned}\sigma(k_0)&=8Z^2e^2\int\frac{d\nu}{\nu}\left(\ln\frac{k}{2m}-\ln\frac{\nu}{2}-\ln\frac{\gamma}{2}-\frac{1}{2}\right)Q(\nu,m)\\&=A\ln(k_0/2m)+B,\end{aligned}$$

where

$$A=8Z^2e^2\int Q(\nu,m)d\nu/\nu,$$

$$B=8Z^2e^2\int\frac{d\nu}{\nu}\left(\ln\frac{\nu}{m}+\ln\frac{\gamma}{2}+\frac{1}{2}\right)Q(\nu,m),$$

$$\ln\gamma=0.577\cdots.$$

The only change caused by the radiative corrections will therefore consist in a change of A and B of the order of 1/137, which is quite negligible.

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The Inelastic Scattering of High Energy Neutrons by Deuterons According to the Impulse Approximation

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The high energy $n-d$ inelastic scattering problem is formulated in such a way that the amplitudes for $n-n$ and $n-p$ scattering at the same energy appear explicitly. The formulation depends on: (1) the large radius of the deuteron compared with the range of nuclear forces and high energy scattering amplitudes; (2) the high velocity of the incident neutron compared to the zero point motion of the deuteron. The method does not depend on the weakness of nuclear forces compared to the kinetic energy of the neutron and is therefore not equivalent to the Born approximation.

I. INTRODUCTION

IN a previous paper[1] the author discussed the elastic scattering of 90- to 350-Mev neutrons by deuterons and pointed out that because elastic scattering may well represent a considerable fraction of the total n–d cross section, it is dangerous to assume that the latter is equal to the sum of the free n–n and n–p cross sections. This is because binding and interference effects play a large role in elastic scattering. They do not play so important a part in the inelastic scattering, however, and it is possible that experiments which concentrate on the latter rather than on the total n–d cross section may yet reveal the magnitude of the n–n interaction at high energies. Because of the similar properties of mirror nuclei it appears certain that n–n and p–p forces are equal for low relative energies. So many surprises have appeared in high energy scattering experiments, however,[2] that it is desirable, if possible, to measure the high energy n–n interaction independently.

It is the purpose of this first of two papers to formulate the n–d inelastic scattering problem in such a manner that from its measurement one can attempt to deduce the value of the free neutron-neutron cross section. The actual attempted deduction from Berkeley experiments will be carried out in the second paper. A secondary feature of this first part is the demonstration of a phenomenological approach to high energy nuclear reactions in light nuclei, using as a basis the experimentally measured values of nucleon-nucleon cross sections. It is hoped that this general method may eventually be extended to nuclei more complex than the deuteron.

The fundamental assumptions will be twofold: (1) the "collision" time in high energy n–d scattering is so short compared to the period of the deuteron that the change in the wave function of the latter during the collision can be described by an "impulse" approximation; (2) the deuteron has such a diffuse structure compared to the range of nuclear forces that the wave function of the incident neutron at one of the two scattering centers within the deuteron is not appreciably perturbed by the presence of the other center. Outgoing waves from both centers are present and must be added together, but individually they will be assumed to be the same as would be produced by a single neutron or proton. In this way the three-body problem is reduced to a superposition of the two-body problems.

It should be noted that these assumptions are not completely equivalent to the Born approximation,

[1] G. F. Chew, Phys. Rev. **74**, 809 (1948).
[2] Hadley, Kelly, Leith, Segrè, Wiegand, and York, Phys. Rev. **75**, 351 (1949); O. Chamberlain and C. Wiegand, Phys. Rev. **79**, 81 (1950).