The Production of High Energy Deuterons by Energetic Nucleons Bombarding Nuclei

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A calculation is given of the cross section for production of fast deuterons ($\gtrsim$50 Mev) by fast nucleons ($\gtrsim$100 Mev) bombarding heavy nuclei. A corrected Fermi model is used. The dependence on energy is given and also the momentum wave function of a nucleon in a heavy nucleus.

I. INTRODUCTION

THE production of fast deuterons in nuclear events involving energies of the order of 100 Mev has been shown to take place in cosmic-ray stars[1] and in the Berkeley cyclotron experiments.[2–4] The most advanced study has been made by York in bombarding C, Cu and Pb targets with 90-Mev neutrons.

A semi-empirical calculation of the process has been made by Chew and Goldberger[5] in which the wave function of the nucleon which will be picked up is taken from the experiments themselves; another treatment using the Wheeler alpha-particle model has been given elsewhere[6] for the case of light nuclei.

In this paper we give a treatment for heavy nuclei which is a continuation of the Chew and Goldberger work, in the sense that no data essential for the calculation are taken from the deuteron production experiments. That will enable us to draw some conclusions concerning the energy dependence of the process.

II. FORMULATION OF THE PROBLEM

As we are dealing with energies of the order of 100 Mev we shall use the Born approximation. Let the indices 0 and 1 refer to the incident neutron and the proton of the nucleus which will be picked up to produce the deuteron. Let $\mathbf{r}_i$ be the position vector of the ith nucleon in the center of mass system of the nucleus, which we shall assume to be heavy enough to be used as a convenient reference system; let $\mathbf{k}$ and $\mathbf{K}$ be the propagation vectors of the incident neutron and of the produced deuteron in that system. Then the amplitude for the production of a deuteron of momentum $\hbar\mathbf{K}$ is:

$$A=-(1/4\pi)\bigg[P\exp[i\mathbf{K}\cdot(\mathbf{r}_0+\mathbf{r}_1)/2]\psi_D(0,1)\psi_f(2,\cdots A)\times\sum_{n=1}^{A}V_{0n}\chi_0\exp(i\mathbf{k}\cdot\mathbf{r}_0)\psi_i(1,\cdots A)\bigg],\quad(1)$$

where χ_0 is the spin wave function of 0, ψ_D the total wave function of the deuteron, ψ_i and ψ_f those of the initial and final nuclei, V_{0n} the potential between nucleons 0 and n. As we use as a perturbation the interaction before the rearrangement collision we must antisymmetrize the final wave function, which is accomplished by the operator P. It is seen that the contribution of the terms $n\neq 1$ will be negligible if $K/2$ is large compared to the internal momenta of the nucleons of the nucleus, which is roughly the case at 90 Mev.

For the term $n=1$ the antisymmetrization gives a small correction, as k is large compared to the internal momenta. Finally, as the final wave function is symmetric with respect to nucleons 0 and 1 in both spin and space coordinates, the P_x and P_σ exchange operators which occur in V are equivalent to unity. The amplitude is then simply:

$$A=-(1/4\pi)\bigg[\exp[i\mathbf{K}\cdot(\mathbf{r}_0+\mathbf{r}_1)/2]\psi_D(0,1)\psi_f(2,\cdots A)\times V_{01}\chi_0\exp(i\mathbf{k}\cdot\mathbf{r}_0)\psi_i(1,\cdots A)\bigg].\quad(2)$$

Integrating over $\mathbf{r}_0-\mathbf{r}_1$ first we can separate the terms depending on the state of the nucleus from those depending on the nucleonic forces:

$$A=-(1/4\pi)\sum_{\text{spin}}\int\exp[i(\mathbf{k}-\mathbf{K}/2)\cdot(\mathbf{r}_0-\mathbf{r}_1)]\times\psi_D^*(0,1)V_{01}\chi_0\times\int\exp[i(\mathbf{k}-\mathbf{K})\cdot\mathbf{r}_1]\psi_f^*\psi_i.\quad(3)$$

The first factor is the Fourier component of $\psi_D V_{01}$, corresponding to the momentum change of the neutron. The second factor is the component of $\psi_f^*\psi_i$ corresponding to the total momentum change of the free particles (i.e., final deuteron momentum minus initial neutron momentum); if the ψ's were products, this would be exactly the momentum wave function of the proton, as was stated by Chew and Goldberger.

III. USE OF FERMI MODEL

If we make use of the statistical Fermi model, then restricting ourselves to the case of heavy nuclei, an opposing case to our previous treatment,[6] we can write the initial wave function of the nucleus as:

$$\psi_i(1,\cdots A)=v^{-\frac{1}{2}}\chi_1\exp(i\mathbf{p}\cdot\mathbf{r}_1)\phi_i(2,\cdots A)$$

* Attache de Recherches du C.N.R.S.
[1] Hoang, Jauneau, and Morellet, *Cosmic Radiation* (Butterworth's Scientific Publications Ltd., London, 1949).
[2] K. Brueckner *et al.*, Phys. Rev. **75**, 1274 (1949).
[3] H. F. York, Phys. Rev. **75**, 1467 (1949) and thesis, University of California, Radiation Laboratory, 1949.
[4] Cuer, Morand, and van Rossum, Comptes Rendus **228**, 481 (1949).
[5] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950).
[6] J. Heidmann, Phil. Mag. **41**, 444 (1950).

where χ_1 is the spin function of 1, $\mathbf{p}$ its propagation vector, v the volume of the nucleus.

Integration over nucleons $2\cdots A$ will give unity if these nucleons remain in the original states and integration over proton 1 will also give unity if $\mathbf{K}-\mathbf{k}\sim\mathbf{p}$. Since in a Fermi distribution p has a definite upper limit L, the differential cross section will be different from zero only for $|\mathbf{K}-\mathbf{k}|<L$. In reality, of course, momentum changes greater than L will occur; in fact, if the initial energy of the neutron is high, these are the only ones that *can* occur for energetic reasons. In this case, then, it is necessary to go beyond the Fermi model.

IV. CORRECTION TO FERMI MODEL

The correction we shall make consists in the introduction of an interaction between proton 1 and another nucleon 2 of the nucleus. We shall then write ψ_i as:

$$\psi_i(1,\cdots A)=v^{-\frac{1}{2}}\chi_{12}\exp(2i\mathbf{P}\cdot\mathbf{r})\psi_p(\boldsymbol{\varrho})\phi_i(3,\cdots A),$$

with

$$\mathbf{r}=(\mathbf{r}_1+\mathbf{r}_2)/2,\quad \boldsymbol{\varrho}=\mathbf{r}_1-\mathbf{r}_2.$$

In the same way we shall write the final wave function as:

$$\psi_f(2,\cdots A)=v^{-\frac{1}{2}}\chi_2\exp(i\mathbf{p}'\cdot\mathbf{r}_2)\phi_f(3,\cdots A).$$

The antisymmetrization of that final wave function will give but a small correction (exchange term).

Integration over $\mathbf{r}$ gives $\mathbf{k}-(\mathbf{K}-2\mathbf{P})\sim\mathbf{p}'$ and a factor v, and integration over nucleons $3\cdots A$ gives unity. Then (3) reduces to:

$$A=-(1/4\pi)\int\exp[i(\mathbf{k}-\mathbf{K}/2)\cdot(\mathbf{r}_0-\mathbf{r}_1)]\times\psi_D{}^*(\text{space})V_{01}\times S\times G$$

with

$$G=\int\exp(i\mathbf{q}\cdot\boldsymbol{\varrho})\psi_p(\boldsymbol{\varrho})d^3\boldsymbol{\varrho},\quad \mathbf{q}=\mathbf{k}-\mathbf{K}+\mathbf{P}.\tag{4}$$

The spin factor is $S=(T_{01}\chi_2,\chi_0\chi_{12})$ where T_{01} is a triplet state of the final deuteron.

V. CHOICE OF ψ_p

The interaction we introduce between nucleons 1 and 2 will be pictured by the choice we shall make for ψ_p. The form of ψ_p will depend upon the nature and state of nucleon 2. If that nucleon is a proton and if we assume a Serber interaction, nucleons 1 and 2 must be in a singlet state, which will occur $w_1=Z/4$ times; if nucleon 2 is a neutron we shall have an even triplet state with $w_2=3(A-Z)/8$ or an even singlet state with $w_3=(A-Z)/8$. These three possible states will be denoted later by the index i, which, however, will be omitted in unambiguous cases.

S can be calculated; if we are interested in the formation of a deuteron of any spin by an incident neutron of undefined spin it is found that in the three cases outlined above S^2 is equal to 3/4.

The spatial part of ψ_p will then be an even state of small energy $(\hbar p)^2/M$, which we shall assume to be an S state. The wave function can be written approximately, following Hulthén, as

$$\rho\psi_p=B\alpha[\sin(p\rho+\delta)/\sin\delta-e^{-\mu\rho}]/(\alpha^2+p^2)^{\frac{1}{2}}\tag{5a}$$

where α^{-1} is the scattering length, μ^{-1} the range of the Yukawa potential and δ the phase shift given by:[7]

$$\cot\delta\sim-\alpha/p.$$

B is a normalization factor; as μ^{-1} is small compared with the radius of the nucleus, B will be determined by the condition that, for ρ of the order of the radius of the nucleus R and $\delta\rightarrow 0$, ψ_p is the S-part of a plane wave. We have then:

from (5a):

$$\psi_p\rightarrow B\alpha\sin p\rho/p\rho$$

from the normalized plane wave:

$$v^{-\frac{1}{2}}\exp(i\mathbf{p}\cdot\boldsymbol{\varrho})\rightarrow v^{-\frac{1}{2}}j_0(p\rho)$$

hence:

$$B\alpha=v^{-\frac{1}{2}}.$$

For $\rho\gtrsim R$ we must have $\psi_p=0$. This will be achieved essentially by putting in (4) a convergence factor $\exp(-\beta\rho)$ with $\beta\sim 6^{\frac{1}{2}}/R$. Integrating over the angular variables in (4) we get:

$$G=\frac{4\pi B\alpha}{q(\alpha^2+p^2)^{\frac{1}{2}}}\int_0^\infty\sin q\rho\left[\frac{\sin(p\rho+\delta)}{\sin\delta}-e^{-\mu\rho}\right]e^{-\beta\rho}d\rho$$

hence:

$$G=\frac{4\pi B\alpha}{(\alpha^2+p^2)^{\frac{1}{2}}}\left[-\frac{1}{q^2+(\mu+\beta)^2}+\frac{\beta^2+q^2-\alpha\beta-p^2}{(\beta^2+q^2+p^2)^2-4q^2p^2}\right]$$

of which a good approximation is, for $|\mathbf{q}-\mathbf{p}|\gg\beta$ and $\mu\gg\beta$:

$$G=\frac{4\pi B\alpha}{(\alpha^2+p^2)^{\frac{1}{2}}}\left(-\frac{1}{q^2+\mu^2}+\frac{1}{q^2-p^2}\right).\tag{5}$$

The first term of (3) is simply the Fourier transform of $\psi_D V_{01}$. Assuming a Yukawa potential and an approximate (Hulthén) wave function:

$$\psi_D(\rho)=A_D(e^{-\alpha_1\rho}-e^{-\alpha_2\rho})/\rho,\quad V_{01}(\rho)=Ve^{-\mu\rho}/\mu\rho$$

this term is:

$$F(\mathbf{l})=\frac{4\pi A_D V}{\mu l}\left(\tan^{-1}\frac{l}{\alpha_1+\mu}-\tan^{-1}\frac{l}{\alpha_2+\mu}\right),\quad \mathbf{l}=\mathbf{k}-\mathbf{K}/2.$$

VI. CROSS SECTION

The differential cross section for the production of a deuteron of momentum $\hbar\mathbf{K}$ while nucleons 1 and 2 are

[7] H. A. Bethe, Phys. Rev. **76**, 38 (1949).

in the initial state described by $\mathbf{P}$ and $\mathbf{p}$ is:

$$\sigma_{\mathbf{P}\mathbf{p}}=(K/2k)(2M_{\text{red}}/4\pi\hbar^2)^2\cdot S^2\cdot F^2\cdot \textstyle\sum_i Zw_iG_i^2$$

where the index i refers to the three cases outlined in the preceding paragraph.

In order to get the cross section without any regard to the initial state of nucleons 1 and 2, we have to integrate G_i^2 with respect to $\mathbf{P}$ and $\mathbf{p}$ with a weighting factor given by the probability of getting given values of $\mathbf{P}$ and $\mathbf{p}$ and taking into account the limitations given by the energy conservation relation.

VII. ENERGY CONSERVATION RELATION

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the propagation vectors of nucleons 1 and 2 in the initial state when the distance between them is large compared to μ^{-1}; we have the relations:

$$\mathbf{P}=(\mathbf{p}_1+\mathbf{p}_2)/2,\quad \mathbf{p}=(\mathbf{p}_1-\mathbf{p}_2)/2.$$

Assuming that the energy of a bound nucleons is given by its potential energy (assumed to be -29 Mev), plus the kinetic energy of a free particle of the given momentum, the energy conservation relation is:

$$-k^2-p_1^2-p_2^2+\frac{K^2}{2}+\frac{(29-2)\text{Mev}}{20.7\times10^{-26}}+(\mathbf{k}-\mathbf{K}+\mathbf{p}_1+\mathbf{p}_2)^2=0.$$

Here 2 Mev is the binding energy of the deuteron, and 20.7×10^{-26} converts Mev into cm^{-2}. From this relation it is deduced that $\mathbf{K}$ lies on a sphere of center $\mathbf{C}$ and radius R given by:

$$\mathbf{C}=2(\mathbf{k}+2\mathbf{P})/3$$
$$R^2=4[(\mathbf{k}-\mathbf{P})^2+3p^2]/9-2\times27\text{ Mev}/3\times20.7\times10^{-26}$$

with the restriction:

$$K^2/2<k^2+(2-8)\text{ Mev}/20.7\times10^{-26}$$

which means that the residual nucleus cannot have less energy than its ground state.

VIII. DISTRIBUTION OF P AND p

We shall calculate this distribution from the distributions of $\mathbf{p}_1$ and $\mathbf{p}_2$, which we shall assume to be Fermi distributions; as G depends on $\mathbf{p}$ only through its modulus, we can integrate immediately over the angular parameters of $\mathbf{p}$ and get:

$$\mathcal{P}(\mathbf{P},p)d^3\mathbf{P}dp=P^2dPd\Omega_Pdp\times72p\mathcal{H}/4\pi L^6$$

with

$$\mathcal{H}=p \quad\text{if}\quad p<L-P,$$
$$\mathcal{H}=(L^2-P^2-p^2)/2P\quad\text{if}\quad L-P<p<(L^2-P^2)^{\frac{1}{2}}$$

where L is the maximum of the Fermi distribution. $\mathcal{P}$ is normalized such that

$$\int\mathcal{P}d^3\mathbf{P}dp=1.$$

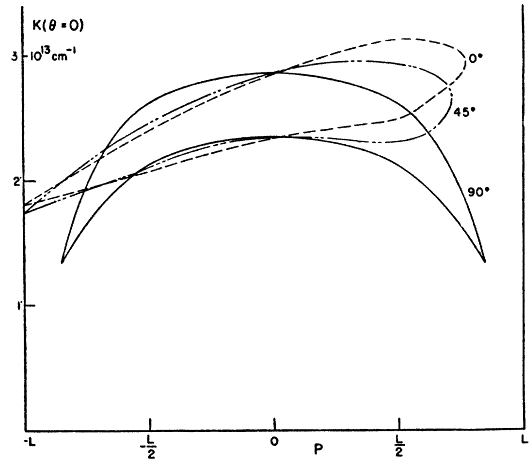

Fig. 1. Energy conservation relation.

The relation between $\mathbf{P}$ and $\mathbf{p}$ for a given $\mathbf{K}$ due to the energy conservation relation is very intricate, except in the asymptotic case $k\to\infty$. But as G^2 is not a too-rapidly varying function of K we can write (note the normalization of $\mathcal{P}$):

$$\left(\int G^2\mathcal{P}d^3\mathbf{P}dp\right)_K\text{fixed}\sim\left(\int\mathcal{P}d^3\mathbf{P}dp\right)_K\text{fixed}\times\left(\int G^2\mathcal{P}d^3\mathbf{P}dp\right)\text{all space.}$$

From this equation we shall calculate the energy and angular distribution of the deuterons. The first term of the right-hand side is simply the distribution of $\mathbf{K}$ obtained from purely geometrical considerations. It can be estimated graphically (Fig. 1). For example, when the angle θ between $\mathbf{k}$ and $\mathbf{K}$ is zero and the angle between $\mathbf{k}$ and $\mathbf{P}$ is 45°, the modulus K of $\mathbf{K}$ must lie inside the contour labelled 45° in Fig. 1. From it, one can read the maximum and minimum value of K as a function of P, with p varying between 0 and $(L^2-P^2)^{\frac{1}{2}}$. Figure 1 also gives curves for an angle between $\mathbf{k}$ and $\mathbf{P}$ equal to 0 and 90°.

From the inspection of Fig. 1 and from the fact that the dependence of G on K will give greater probability to small values of K, we shall suppose that for any angle θ the energy distribution of the emerging deuterons (i.e., the distribution of K^2) is uniform between 40 and 80 Mev with zero probability outside that range. If an accurate calculation on the basis of Fig. 1 gives different results, it will only be necessary to correct the final result but not the calculation in the next section.

IX. CALCULATION OF $\Sigma Zw_i\int G^2{}_i\mathcal{P}d^3\mathbf{P}dp$

This is a function of $\mathbf{Q}=\mathbf{K}-\mathbf{k}$ which we set equal to $N(\mathbf{Q})$. We shall distinguish the three following cases:

(a). Q *small.* As stated above we can neglect the correction to the Fermi model. G is then:

$$G=\int_{\text{nucleus}}\exp[i(\mathbf{k}-\mathbf{K})\cdot\mathbf{r}_1]v^{-\frac{1}{2}}\exp(i\mathbf{p}\cdot\mathbf{r}_1)d^3\mathbf{r}_1$$

hence:

$$N(\mathbf{Q})=v$$

and:

$$\mathbf{p}\sim\mathbf{K}-\mathbf{k},$$

which means that the initial propagation vector of the proton must be $\mathbf{K}-\mathbf{k}$. S^2 is still equal to 3/4.

(b). *Q large.* We can write $\mathbf{q}\sim\mathbf{Q}$, and G^2 is equal to:

$$G^2=(4\pi B\alpha)^2(p^2+\mu^2)^2/(\alpha^2+p^2)Q^8.$$

Integrating first over P we get:

$$\int\mathcal{P}d^3\mathbf{P}=12p^2(L-p)^2(2L+p)/L^6.$$

Then:

$$N(\mathbf{Q})=\sum_i 12(4\pi B\alpha)^2 I_i Z w_i/L^6Q^8$$

with:

$$I=\int_0^L p^2(\alpha^2+p^2)^{-1}(L-p)^2(2L+p)(p^2+\mu^2)^2dp$$

hence:

$$I=(\mu^2-\alpha^2)^2\left[\frac{\alpha^2}{2}(3L^2+\alpha^2)\ln\frac{\alpha^2+L^2}{\alpha^2}-2\alpha L^3\tan^{-1}\frac{L}{\alpha}+\frac{3}{4}L^4-\frac{\alpha^2L^2}{2}\right]+\frac{L^8}{40}+\frac{L^6}{12}(2\mu^2-\alpha^2)$$

where the index i is omitted.

(c). *Q intermediate.* We shall restrict ourselves to the case in which G can be written in the form (5); i.e., $|\mathbf{Q}-\mathbf{L}|>\beta$. The calculation is simpler if we calculate first the distribution of the modulus of $\mathbf{Q}$; from it we shall get N by dividing the result by $4\pi Q^2$; we can do that as G^2 is proportional to the probability for the propagation vector of $\rho/2$ to be $\mathbf{q}$. We have then:

$$N(\mathbf{Q})=\sum_i\frac{Zw_i}{4\pi Q^2}\int_{q=Q-P}^{\infty}\int\int\frac{-2\pi q^2Q}{Pq}G_i^2\mathcal{P}dqdpd^3\mathbf{P},$$

i.e.,

$$N(\mathbf{Q})=-\sum_i 18(4\pi B\alpha)^2J_iZw_i/L^6Q$$

with, omitting again the index i:

$$J=\int\left\{\frac{Pp\mathcal{H}}{\alpha^2+p^2}\left[\frac{1}{X^2+\mu^2}+\frac{1}{X^2-p^2}+\frac{2}{\mu^2+p^2}\ln\left|\frac{X^2-p^2}{X^2+\mu^2}\right|\right]-\left[\frac{1}{2X(X-p)}+\frac{2}{X^2+\mu^2}\ln|X-p|\right]\frac{PX\mathcal{H}_X}{\alpha^2+X^2}\right\}dpdP$$
$$+\int\frac{PX\mathcal{H}_X}{\alpha^2+X^2}\left[\frac{1}{2X(X-p)}+\frac{2}{X^2+\mu^2}\ln|X-p|\right]dpdP,$$

where $X=Q-P$ and $\mathcal{H}_X$ means that in $\mathcal{H}$ we write $p=X$. In J the integration over q and the angular parameters of $\mathbf{P}$ has been performed. The second integral of J cancels the second term of the first integral; we introduced it in order to make the first integral calculable graphically by eliminating the poles occurring for $p=X$.

The integration over p in the second integral of J gives:

$$\int_0^L\left\{\frac{-P\mathcal{H}_X}{2(\alpha^2+X^2)}\ln\left|\frac{X-(L^2-P^2)^{\frac{1}{2}}}{X}\right|+\frac{2PX\mathcal{H}_X}{(\alpha^2+X^2)(\mu^2+X^2)}(L-P)[\ln|L-P|-1]\right\}dP$$

where the subsequent integration over P will be performed graphically.

In case (c) we have carried out the calculation only for one value of Q, *viz.*, 1.3×10^{13} cm^{-1}; the value thus found can be joined to the results of cases (a) and (b), for instance by a Fermi distribution at suitably chosen temperature.

I_i and J_i depend upon the triplet or singlet character of the initial state of nucleons 1 and 2 only through the scattering lengths α_i^{-1}; but as the dependence is not very sharp we shall perform the calculation only for the triplet case; then we can separate the summation over w_i and, with $Z=A/2$, we get:

$$\sum_i w_i=3A/8.$$

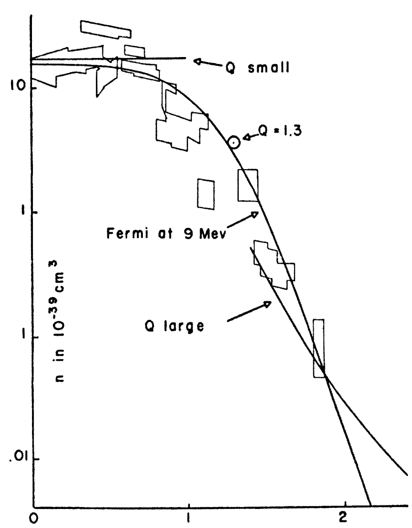

FIG. 2. Momentum distribution of a nucleon inside a nucleus (the odd contours are the result of the superposition of quadrangles representing York's data).

X. NUMERICAL RESULTS

We shall first deal with $N(\mathbf{Q})$ which is important as it is proportional to the momentum distribution of a nucleon inside a nucleus.

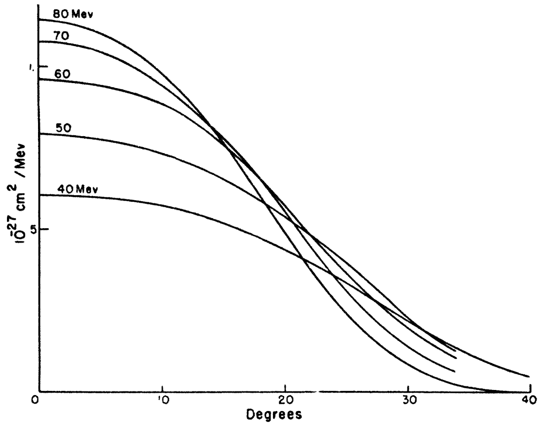

FIG. 3. Differential cross section per Mev and per nucleon for production of deuterons by 90-Mev nucleons.

The numerical values taken are:

$$L=1.0\times 10^{13}\ \mathrm{cm}^{-1}\qquad \mu=0.847\times 10^{13}\ \mathrm{cm}^{-1}$$
$$v=17\cdot A\times 10^{-39}\ \mathrm{cm}^3,\quad \alpha^{-1}=5.39\times 10^{-13}\ \mathrm{cm}.$$

Then for Q small we have:

$$n\equiv N(\mathbf{Q})/A=17\times 10^{-39}\ \mathrm{cm}^3,$$
$$\text{for } Q=1.3\times 10^{13}\ \mathrm{cm}^{-1}:\quad n=3.6\times 10^{-39}\ \mathrm{cm}^3,$$
$$\text{for } Q \text{ large}:\quad n=7.6\times 10^{65}/Q^8\times 10^{-39}\ \mathrm{cm}^3.$$

The n value for $Q=1.3$ is likely to be too high because $|\mathbf{Q}-\mathbf{L}|>\beta$ is only just satisfied, therefore we shall take $n=3.0$ instead. The temperature of the Fermi distribution joining these data is found to be 9 Mev, which is, within our uncertainties, in agreement with Watanabe's calculation.[8] The result of the calculation is shown in Fig. 2. Note that a Fermi distribution is used only for purpose of easy interpolation.

We can make a comparison with York's results. In his thesis York gives the cross section σ for production of deuterons ejected at given angles and of energy lying in given intervals. From these values we can plot σ/F^2 as a function of Q. That is done on Fig. 2 in arbitrary units from York's data on C. As a matter of fact, our calculations should not be too good for such a light nucleus as C, but we cannot compare them with York's results on Cu and Pb, first because the latter experiments are not so complete, and second because the internal scatterings of the incident neutron play a very important role in such heavy nuclei.[9] Even if our curve of Fig. 2 was rectangle-like it would be possible to account for a spread σ/F^2 through these internal scatterings.

XI. CROSS SECTION FOR 90-MEV NEUTRONS

The differential cross section per Mev and per nucleon is:

$$\sigma/A=\frac{K}{2k}\cdot\left(\frac{M}{\pi\hbar^2}\right)^2\cdot\frac{3}{4}\cdot F^2(\mathbf{k}-\mathbf{K}/2)\cdot n(\mathbf{K}-\mathbf{k})\cdot\frac{1}{40}.$$

[8] S. Watanabe, Zeits. f. Physik **113**, 482 (1939).

[9] M. L. Goldberger, Phys. Rev. **74**, 1269 (1948).

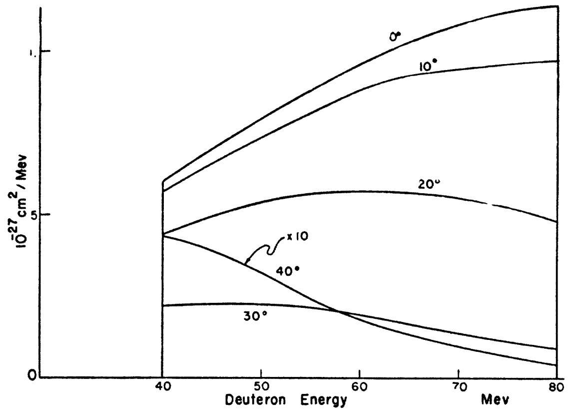

FIG. 4. Spectrum at various angles of the deuterons produced by 90-Mev nucleons.

The numerical values involved in F are taken as:[10]

$$A_D^2=0.0613\times 10^{13}\ \mathrm{cm}^{-1}$$
$$\alpha_1=0.231\ \times 10^{13}\ \mathrm{cm}^{-1}\quad V=67.8\ \mathrm{Mev}.$$
$$\alpha_2=1.55\ \ \times 10^{13}\ \mathrm{cm}^{-1}$$

The differential cross section per Mev and per nucleon is shown in Fig. 3. The corresponding spectrum of the deuterons at various angles is shown in Fig. 4. It seems that our calculation accounts in a natural way, without the introduction of internal scattering, for the shift of the most probable deuteron energy to lower energies at larger angles which was observed by York. This shift is due to the fact that for large angle the values of K permitted by the energy conservation relation are such that $|\mathbf{K}-\mathbf{k}|$ can take the value L. By graphical integration we find that the total cross section is:

$$\sigma_T=1.94\times A\times 10^{-26}\ \mathrm{cm}^2.$$

This is a very large cross section; it is equal to r_0^2 where r_0 is the radius of the sphere containing one nucleon. This explains qualitatively the experimental result that the cross section of the pick-up process is considerable, a result which was surprising to most physicists because of the small binding energy of the deuteron. For a comparison with experiment, we should take into account that the deuteron can easily disintegrate again while escaping from the nucleus. We shall assume that the layer of the nucleus effective in the pick-up process has a thickness of r_0 (measured in the direction of motion of the deuteron and on the escape side of the nucleus only). The effective volume of the nucleus is then $\pi r_0(r_0A^{\frac{1}{3}})^2$ as compared to the total nuclear volume $(4\pi/3)r_0^3A$. This means the effective number of nucleons is about $\frac{3}{4}A^{\frac{2}{3}}$ instead of A. For C, this yields about 4, giving a theoretical cross section of about 8×10^{-26} cm^2. The observations of York give 2.6×10^{-26} cm$^2\pm 25$ percent: our theoretical value is therefore about three times too high. An error in this direction is not unsatisfactory because (1) the "mean

[10] H. A. Bethe and C. Longmire, Phys. Rev. **77**, 647 (1950).

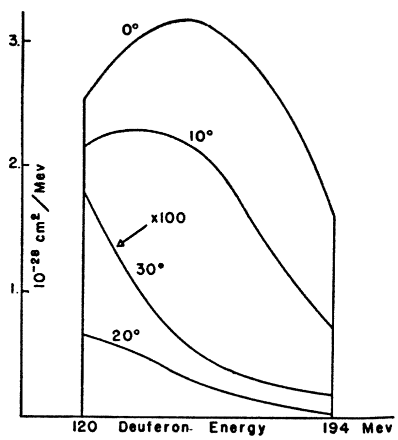

FIG. 5. Spectrum at various angles of the deuterons produced by 200-Mev nucleons.

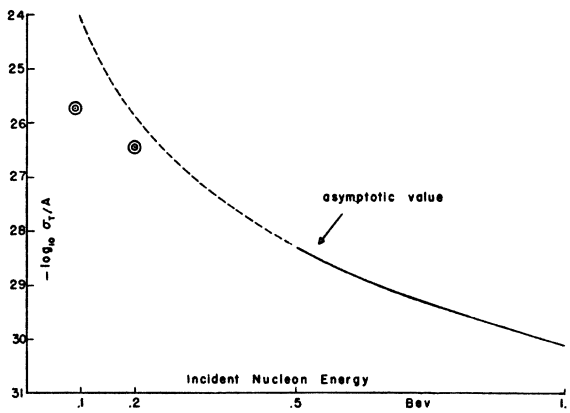

FIG. 6. Total cross section for production of deuterons as a function of the incident nucleon energy.

free path" of the escaping deuteron may be less than r_0, and (2) there are many competing processes for which our theory will also give large cross sections. The Born approximation customarily over-estimates cross sections. For heavier nuclei, there will be a further reduction because the incident neutron may be absorbed before it reaches the far side of the nucleus; the cross section should therefore increase less than $A^{\frac{2}{3}}$, in agreement with observation.

XII. CROSS SECTION FOR 200-MEV NEUTRONS

A similar procedure to that illustrated on Fig. 1 shows that the energy of the deuterons produced has a distribution nearly uniform in the range 120–194 Mev. Similarly to Fig. 4, we get a deuteron spectrum as shown in Fig. 5. By graphical integration the total cross section is:

$$\sigma_T = 0.37 \times A \times 10^{-26}\ \text{cm}^2.$$

The fact that the values of the total cross section calculated for 90- and for 200-Mev neutrons are harmoniously disposed compared to the asymptotic value calculated in the next paragraph (Fig. 6) shows that we were right enough in neglecting the terms $n \neq 1$ in (1) for 90-Mev neutrons.

XIII. DEPENDENCE OF THE CROSS SECTION ON ENERGY

For this purpose we shall use asymptotic forms for G, C, F, and R; then this calculation will be valid for incident neutron energy larger than about 500 Mev.

F becomes proportional to l^{-2} where $l = |\mathbf{k} - \frac{1}{2}\mathbf{K}|$, and we get:

$$\sigma_{as} = \frac{K}{2k} \cdot \left(\frac{M}{\pi\hbar^2}\right)^2 \cdot \frac{3}{4} \cdot \left(\frac{4\pi A_D V}{\mu}\right)^2 \cdot (\alpha_1 - \alpha_2)^2 \cdot \frac{12(4\pi B\alpha)^2}{L^6} \cdot I \cdot \frac{3A}{8} \cdot Z \cdot l^{-4} Q^{-8}.$$

We have, moreover:

$$Q^2 \sim l^2 \sim k^2(5 - 4\cos 2\theta)/9$$

hence:

$$\sigma_{as} = \frac{K}{2k} \cdots Z \cdot 3^{12}/k^{12}(5 - 4\cos 2\theta)^6.$$

Thus we get the remarkable result that the angular distribution of the deuterons becomes independent of the energy of the incoming neutron and is characterized by a half-half width equal to 7°.

The spectrum of the deuteron becomes a sharp line at $8E/9$, E being the energy of the incident neutron.

The excitation given to the nucleus becomes $E/9$, in the form of recoil of the nucleon 2, which if $E/9$ is large enough will be ejected in the forward direction.

The total cross section is proportional to the inverse of the sixth power of the energy of the incident neutron, as in the corresponding case of the capture of an electron by an ion He^+ traversing a gas of H atoms.[11*] The explicit result in our case is:

$$\sigma_{as \cdot T} = 7.7 \times A \times 10^{-25}(100/E_{\text{Mev}})^6\ \text{cm}^2.$$

which is shown in Fig. 6.

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[11] Brinkmann and Kramers, K. Wet. Amst. **33**, 973 (1930).

* Note added in proof: see explanation of that analogy given by Levinger and Bethe, Phys. Rev. **78**, 115 (1950) in paragraph IV.