Statistical Limitations on the Resolving Time of a Scintillation Counter*

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SCINTILLATION counters are being used increasingly for applications that require extremely short resolving times. In this letter, we discuss the limitations on resolving time that arise from fluctuations in the emission, transmission, and collection of scintillation photons. For this purpose we assume that following excitation of the scintillator by an energetic event, the photo-multiplier multiplies the primary photo-electrons without time spread, and that the resulting pulses are fed into a discriminator that gives a signal when it has accumulated a definite number, say Q, of pulses. We ask for the fluctuation in time of these signals. Except for the neglect of time spread in the photo-multiplier, this model contains the essential features of actual counting systems.

The average or expected number of photo-multiplier pulses between the initial excitation of the scintillator at zero time (allowing for a possible constant time delay in the photo-multiplier) and time t is $f(t)$, where $f(0)=0$, f is a continuous monotonically increasing function[1] of t, and df/dt is piecewise continuous. It is assumed that the probability that N pulses occur between 0 and t is given by the Poisson distribution:

$$P_N(t)=[f(t)]^N e^{-f(t)}/N!. \tag{1}$$

This means that either (a) a relatively small, randomly selected, sample of the emitted photons is converted into output pulses, or (b) the magnitude of the initial excitation is randomly distributed,[2] or both. Then the probability that the Qth pulse occurs between t and $t+dt$ is[3]

$$W_Q(t)dt=P_{Q-1}(t)[df(t)/dt]dt. \tag{2}$$

The total probability that the Qth pulse occurs between 0 and ∞ is

$$\int_0^\infty W_Q(t)dt=\frac{1}{(Q-1)!}\int_0^R f^{Q-1}e^{-f}df$$
$$=1-e^{-R}\left[1+R+\frac{R^2}{2!}+\cdots+\frac{R^{Q-1}}{(Q-1)!}\right]\equiv I_Q(R), \tag{3}$$

where $R\equiv f(\infty)$ is the average total number of primary photo-electrons. It is readily verified by comparison of Eqs. (1) and (3) that $I_Q(R)$ is the probability that Q or more pulses occur between 0 and ∞, as it should be. $I_Q(R)$ can be evaluated from Eq. (2), or found from tables of the incomplete gamma-function.[4]

The variance of the signal time is

$$v\equiv\langle t^2\rangle_{\text{Av}}-\bar{t}^2=\frac{1}{I_Q(R)}\int_0^\infty t^2W_Q(t)dt-\left[\frac{1}{I_Q(R)}\int_0^\infty tW_Q(t)dt\right]^2.$$

It may also be calculated from a generating function:[5]

$$G(x)=\int_0^\infty x^tW_Q(t)dt, \tag{4}$$
$$v=\left[\frac{1}{G}\left(\frac{\partial^2G}{\partial x^2}+\frac{\partial G}{\partial x}\right)-\left(\frac{1}{G}\frac{\partial G}{\partial x}\right)^2\right]_{x=1}.$$

If we define the inverse function to $f(t)$ as $t=g(f)$, then Eq. (4) becomes

$$G(x)=\frac{1}{(Q-1)!}\int_0^R x^{g(f)}f^{Q-1}e^{-f}df.$$

As a simple example,[6] we consider the case $f(t)=\lambda t$ (not applicable to scintillation counters), in which case $R=\infty$ and $g(f)=f/\lambda$:

$$G(x)=(1-\lambda^{-1}\ln x)^{-Q},\quad \bar{t}=(\partial G/\partial x)_{x=1}=Q/\lambda,$$
$$v=Q/\lambda^2=\bar{t}^2/Q. \tag{5}$$

For a scintillator, R must be finite, and $g(f)$ has a singularity at the point $f=R$. It is apparent, however, that the integral over f that gives $\langle t^n\rangle_{\text{Av}}$ diverges if and only if the nth moment of the probability function (2) diverges, so that any difficulties caused by the singularity can easily be traced.

When R is finite, $G(x)$ and v cannot be written in simple closed forms. However, in the physically useful case $R\gg1$, $R\gg Q$, Eq. (3) shows that we can put $I_Q(R)\cong1$, and simple asymptotic expressions can be obtained that involve only the behavior of $f(t)$ near $t=0$. Suppose that

$$f(t)=a_1t+a_2t^2+\cdots;$$

then

$$t=g(f)=b_1f+b_2f^2+\cdots,$$

where

$$b_1=1/a_1,\quad b_2=-a_2/a_1^3,\quad \cdots.$$

In general, a_1, a_2, $\cdots$ are proportional to R, so that $b_1\propto1/R$, $b_2\propto1/R^2$, etc., and the series that follow are expansions in powers of Q/R; these are convergent if the series for $g(f)$ converges inside the circle $f=R$, and are otherwise asymptotic. We find that

$$\bar{t}=[1/I_Q(R)][b_1QI_{Q+1}(R)+b_2Q(Q+1)I_{Q+2}(R)+\cdots],$$
$$v=[1/I_Q^2(R)]\{b_1^2[Q(Q+1)I_Q(R)I_{Q+2}(R)-Q^2I_{Q+1}^2(R)]$$
$$+2b_1b_2[Q(Q+1)(Q+2)I_Q(R)I_{Q+3}(R)$$
$$-Q^2(Q+1)I_{Q+1}(R)I_{Q+2}(R)]+\cdots\}.$$

With the approximation $I_Q(R)\cong1$, these become asymptotic series:

$$\bar{t}\cong b_1Q+b_2Q(Q+1)+\cdots=\frac{Q}{a_1}\left[1-\frac{a_2(Q+1)}{a_1^2}+\cdots\right],$$
$$v\cong b_1^2Q+4b_1b_2Q(Q+1)+\cdots=\frac{Q}{a_1^2}\left[1-\frac{4a_2(Q+1)}{a_1^2}+\cdots\right],$$
$$\frac{v}{\bar{t}^2}\cong\frac{1}{Q}\left[1-\frac{2a_2(Q+1)}{a_1^2}+\cdots\right];$$

the last of these agrees with Eq. (5) to lowest order, as expected.

For an exponentially decaying scintillator:

$$f(t)=R(1-e^{-\lambda t}),\quad a_1=R\lambda,\quad a_2=-\tfrac{1}{2}R\lambda^2,\quad \cdots.$$

In this case:

$$\bar{t}\cong(Q/R\lambda)\{1+[(Q+1)/2R]+\cdots\},$$
$$v\cong\frac{Q}{R^2\lambda^2}\left[1+\frac{2(Q+1)}{R}+\cdots\right]\cong\frac{\bar{t}^2}{Q}\left(1+\frac{Q+1}{R}+\cdots\right). \tag{6}$$

To see the order of magnitude of the r.m.s. signal time deviation in a practical case, we calculate $v^{\frac{1}{2}}$ when $Q=10$, $\lambda=5\times10^8$ sec.$^{-1}$, and $R=75$, which corresponds roughly to the excitation of a fast liquid scintillator[7] by a 1-Mev electron and the conversion of 1 percent of the emitted photons into photo-multiplier pulses, when the spread in photon collection time can be neglected. In this case, the first two terms of Eq. (6) give $v^{\frac{1}{2}}\cong10^{-10}$ sec., with an error of a few percent. With slower scintillators and less energetic events, $v^{\frac{1}{2}}$ can be substantially larger, of the order of 10^{-8} sec. in a typical case. The time $v^{\frac{1}{2}}$ represents a loss of resolution that is superposed on any losses owing to the electronic circuits.

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1 The case in which f is not monotonic, which might be of interest in connection with the time fluctuations in the disintegrations of radioactive chains, could also be treated by a straightforward extension of the formalism.

2 L. I. Schiff, Phys. Rev. **50**, 88 (1936).

3 C. H. Westcott, Proc. Roy. Soc. **A194**, 508 (1948).

4 K. Pearson, *Tables of the Incomplete Gamma Function* (H. M. Stationery Office, London, 1922).

5 For a discussion of generating functions, see for example F. Seitz and D. W. Mueller, Phys. Rev. **78**, 605 (1950).

6 W. C. Elmore, Nucleonics **6**, No. 1, 26 (1950), has considered this problem from a somewhat different point of view.

7 R. F. Post, Phys. Rev. **79**, 735 (1950).