TABLE II. Values of collision cross section per unit solid angle in barns.

$\theta =$	70°	80°	100°	110°	120°	140°	160°	180°	E (Mev)
(p-d)	0.115	0.095	0.060	(0.052)	0.055	0.105	0.230		5.25
(n-d)	0.125	0.088	0.050		0.055	0.105	0.200	0.235	5.5
(n-d)		0.095	0.050		0.055	0.130	0.210	0.275	4.5

that for $\theta = 130°$ the difference between the values of σ at 5.5 and 4.5 Mev is sufficiently large to make reasonable an estimate of errors of the order of 10 percent in the n-d measurements. The outstanding differences at $\theta = 160°$ may not be real, therefore.

On account of effects in the interior of the nucleus one may expect the phase shifts to be affected when a neutron is changed into a proton. An exact correspondence cannot be expected, therefore, even apart from the interference with Coulomb scattering. The author is indebted to Dr. Louis Rosen for helpful discussion of his data.

* Assisted by the joint program of the ONR and AEC.
† Released by declassification authorities following request of September 25, 1950.
[1] E. Wantuch, Phys. Rev. **79**, 729 (1950).
[2] J. C. Allred and L. Rosen, Phys. Rev. **79**, 227 (1950).

A Note on the Classical Spin-Wave Theory of Heller and Kramers*

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IN 1934 Heller and Kramers[1] obtained the Bloch energy levels[2] for a ferromagnet by starting with a classical theory of spin-waves and then quantizing this theory. Certain obscurities remained in the theory, however, as recognized by Heller and Kramers, and it is the purpose of this note to clarify these obscure points. We shall discuss particularly the physical and mathematical origin of the apparent zero-point energy, which HK had to omit, and the canonical nature of the variables used. Reference will be made to the quantal spin-wave theory as given by Holstein and Primakoff.[3]

The Hamiltonian for the simplest case, a linear chain of N atoms with nearest-neighbor interactions, is

$$\mathcal{H} = -2\beta H \sum_{l=1}^{N} S_{l,z} - 2J \sum_{l=1}^{N} \mathbf{S}_l \cdot \mathbf{S}_{l+1}, \qquad (1)$$

where $\mathbf{S}_l$ is the spin vector (operator in the quantal case) of the lth atom in units of $\hbar$, β the Bohr magneton, J the exchange integral, and H the z-directed magnetic field.

Considering $\mathbf{S}_l$ as a classical vector, HK observe that near saturation $S_{l,x}$ and $S_{l,y}$ are small and, therefore,

$$S_{l,z} = [\breve{S}^2 - (S_{l,x}^2 + S_{l,y}^2)]^{\frac{1}{2}} \cong S[1 - (S_{l,x}^2 + S_{l,y}^2)/2S^2], \qquad (2)$$

where $\breve{S}$ is the magnitude of the spin vector. In a quantal treatment $\breve{S}$ is to be replaced by $[S(S+1)]^{\frac{1}{2}}$, where S is now the maximum z component of the spin.

Using Eq. (2) causes the Hamiltonian to become

$$\mathcal{H} \cong \mathcal{H}_{\text{HK}}(\breve{S}) = -2N\breve{S}[\beta H + J\breve{S}] + 2J \sum_{l=1}^{N} \times \{(1+\alpha)(S_{l,x}^2 + S_{l,y}^2) - (S_{l,x}S_{l+1,x} + S_{l,y}S_{l+1,y})\}, \qquad (3)$$

where $\alpha = \beta H / 2J\breve{S}$.

We may compare Eq. (3) with the quantal Hamiltonian obtained[4] by HP after introduction of approximations appropriate near saturation. In the same notation:

$$\mathcal{H}_{\text{HP}} = \mathcal{H}_{\text{HK}}(S) + 2J(1+\alpha) i \sum_{l=1}^{N} (S_{l,x}S_{l,y} - S_{l,y}S_{l,x}). \qquad (4)$$

We notice that the quantal Hamiltonian has an additional sum of commutators, which vanish classically, and that it is S rather than $\breve{S}$ which appears. We shall discuss this below.

An orthogonal transformation is now made by HK to make H a sum of squares. The new variables P_λ, Q_λ after renormalization are determined by

$$S_{l,x} = (S)^{\frac{1}{2}} \sum_{\lambda=0}^{N-1} a_{l\lambda} P_\lambda; \quad S_{l,y} = (S)^{\frac{1}{2}} \sum_{\lambda=0}^{N-1} a_{l\lambda} Q_\lambda, \qquad (5a)$$

where

$$a_{l\lambda} = (2/N)^{\frac{1}{2}} \cos[(2\pi\lambda/N) + (\pi\lambda/2N)] \quad (\lambda \neq 0)$$
$$a_{l0} = (1/N)^{\frac{1}{2}}. \qquad (5b)$$

The variables P_λ, Q_λ are obviously wave-like in nature.

Quantization of the HK theory is now possible, since the classical Poisson bracket is

$$\{P_\lambda, Q_{\lambda'}\}_{\text{P.B.}} = (\delta_{\lambda\lambda'}/S)\{S_{l,x}, S_{l,y}\}_{\text{P.B.}} = \delta_{\lambda\lambda'} S_{l,z}/S \cong \delta_{\lambda\lambda'} \qquad (6)$$

from Eq. (5) and with $S_{l,z}/S$ approximately equal to unity, corresponding to conditions near saturation. Hence, $P_\lambda^2 + Q_\lambda^2$ has eigenvalues $2(n_\lambda + \frac{1}{2})$ with $n_\lambda = 0, 1, 2, \cdots$.

The energy levels for $\mathcal{H}_{\text{HK}}$ are:

$$E_{\text{HK}}(n_\lambda, \breve{S}) = -2N\breve{S}[\beta H + J\breve{S}] + 4JS \sum_{\lambda=0}^{N-1} \left(1 + \alpha - \cos\frac{2\pi\lambda}{N}\right)(n_\lambda + \tfrac{1}{2}), \qquad (7)$$

a result differing from the quantal energy levels in the appearance of $\breve{S}$ and the additional $\frac{1}{2}$ in the factor $(n_\lambda + \frac{1}{2})$ as will now be shown. For the additional terms in Eq. (4) can readily be calculated using the HP approximations[5] to be

$$S_{l,x}S_{l,y} - S_{l,y}S_{l,x} = iS. \qquad (8)$$

(Equation (8) indicates that the same approximation is made in the quantal theory as in Eq. (6); i.e., $S_{l,z}$ is replaced by S.) It follows that, using the same orthogonal transformation and quantizing the result, we obtain

$$E_{\text{HP}}(n_\lambda) = E_{\text{HK}}(n_\lambda, S) - 2JS \sum_\lambda (1+\alpha)$$
$$= E_{\text{HK}}(n_\lambda, S) - 2JS \sum_\lambda \left(1 + \alpha - \cos\frac{2\pi\lambda}{N}\right), \qquad (9)$$

which verifies the statement made above.

The physical origin of the extra $\frac{1}{2}$ is not hard to see. It arises from Eq. (2): in the HK theory the distinction between S and $[S(S+1)]^{\frac{1}{2}}$ is not maintained, so that, in effect, the expression $S - S_z$, the deviation of the z component of the spin from its maximum value, is replaced by $[S(S+1)]^{\frac{1}{2}} - S_z$. Now the former quantity has, rigorously, integral values, while the latter is approximately $S - S_z + \frac{1}{2}$ (expanding $[S(S+1)]^{\frac{1}{2}}$). The $\frac{1}{2}$ comes from the fact that, quantum-mechanically, the spin vector never lies along the z axis and $S_x^2 + S_y^2$ is never rigorously zero. In the formal theory this is expressed by the appearance of the commutator of S_x and S_y as mentioned above.

* Supported in part by the ONR.
[1] G. Heller and H. A. Kramers, Proc. Amst. Acad. Sci. **37**, 378 (1934), referred to as HK.
[2] See A. Sommerfeld and H. Bethe, *Handbuch der Physik*, Vol. 24, Part 2, p. 601.
[3] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940), referred to as HP. See also D. Polder, Phil. Mag. **40**, 99 (1949).
[4] Reference 3, Eq. (7).
[5] $S_{l,x} + iS_{l,y} \cong (2S)^{\frac{1}{2}} a_l, \quad S_{l,x} - iS_{l,y} \cong (2S)^{\frac{1}{2}} a_l^*,$
$S_{l,z} = S - a_l^* a_l, \quad a_l a_{l'}^* - a_{l'}^* a_l = \delta_{ll'}.$
See reference 3, Eqs. (3) to (6) and following discussion.

The Scattering Lengths of the Deuteron and p-d Scattering*

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AN analysis of the S-wave scattering of nucleons by deuterons has been carried out by means of a spin-dependent, "effective" two-body model of the nucleon-deuteron interaction.[1] This analysis was begun by considering the deuteron as a single structureless particle so that the actual nucleon-deuteron interaction could be replaced by an "effective" two-body interaction.