TABLE II. Values of collision cross section per unit solid angle in barns.

$\theta=$	70°	80°	100°	110°	120°	140°	160°	180°	E (Mev)
(p-d)	0.115	0.095	0.060	(0.052)	0.055	0.105	0.230		5.25
(n-d)	0.125	0.088	0.050		0.055	0.105	0.200	0.235	5.5
(n-d)		0.095	0.050		0.055	0.130	0.210	0.275	4.5

that for $\theta=130°$ the difference between the values of σ at 5.5 and 4.5 Mev is sufficiently large to make reasonable an estimate of errors of the order of 10 percent in the n-d measurements. The outstanding differences at $\theta=160°$ may not be real, therefore.

On account of effects in the interior of the nucleus one may expect the phase shifts to be affected when a neutron is changed into a proton. An exact correspondence cannot be expected, therefore, even apart from the interference with Coulomb scattering. The author is indebted to Dr. Louis Rosen for helpful discussion of his data.

* Assisted by the joint program of the ONR and AEC.
† Released by declassification authorities following request of September 25, 1950.
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A Note on the Classical Spin-Wave Theory of Heller and Kramers*

MARTIN J. KLEIN AND ROBERT S. SMITH
Department of Physics, Case Institute of Technology, Cleveland, Ohio
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IN 1934 Heller and Kramers[1] obtained the Bloch energy levels[2] for a ferromagnet by starting with a classical theory of spin-waves and then quantizing this theory. Certain obscurities remained in the theory, however, as recognized by Heller and Kramers, and it is the purpose of this note to clarify these obscure points. We shall discuss particularly the physical and mathematical origin of the apparent zero-point energy, which HK had to omit, and the canonical nature of the variables used. Reference will be made to the quantal spin-wave theory as given by Holstein and Primakoff.[3]

The Hamiltonian for the simplest case, a linear chain of N atoms with nearest-neighbor interactions, is

$$\mathcal{H}=-2\beta H\sum_{l=1}^{N}S_{l,z}-2J\sum_{l=1}^{N}\mathbf{S}_l\cdot\mathbf{S}_{l+1}, \tag{1}$$

where $\mathbf{S}_l$ is the spin vector (operator in the quantal case) of the lth atom in units of $\hbar$, β the Bohr magneton, J the exchange integral, and H the z-directed magnetic field.

Considering $\mathbf{S}_l$ as a classical vector, HK observe that near saturation $S_{l,x}$ and $S_{l,y}$ are small and, therefore,

$$S_{l,z}=[\check{S}^2-(S_{l,x}^2+S_{l,y}^2)]^{\frac{1}{2}}\cong S[1-(S_{l,x}^2+S_{l,y}^2)/2S^2], \tag{2}$$

where $\check{S}$ is the magnitude of the spin vector. In a quantal treatment $\check{S}$ is to be replaced by $[S(S+1)]^{\frac{1}{2}}$, where S is now the maximum z component of the spin.

Using Eq. (2) causes the Hamiltonian to become

$$\mathcal{H}\cong\mathcal{H}_{\rm HK}(\check{S})=-2N\check{S}[\beta H+J\check{S}]+2J\sum_{l=1}^{N}\times\{(1+\alpha)(S_{l,x}^2+S_{l,y}^2)-(S_{l,x}S_{l+1,x}+S_{l,y}S_{l+1,y})\}, \tag{3}$$

where $\alpha=\beta H/2J\check{S}$.

We may compare Eq. (3) with the quantal Hamiltonian obtained[4] by HP after introduction of approximations appropriate near saturation. In the same notation:

$$\mathcal{H}_{\rm HP}=\mathcal{H}_{\rm HK}(S)+2J(1+\alpha)i\sum_{l=1}^{N}(S_{l,x}S_{l,y}-S_{l,y}S_{l,x}). \tag{4}$$

We notice that the quantal Hamiltonian has an additional sum of commutators, which vanish classically, and that it is S rather than $\check{S}$ which appears. We shall discuss this below.

An orthogonal transformation is now made by HK to make H a sum of squares. The new variables P_λ, Q_λ after renormalization are determined by

$$S_{l,x}=(S)^{\frac{1}{2}}\sum_{\lambda=0}^{N-1}a_{l\lambda}P_\lambda;\quad S_{l,y}=(S)^{\frac{1}{2}}\sum_{\lambda=0}^{N-1}a_{l\lambda}Q_\lambda, \tag{5a}$$

where

$$a_{l\lambda}=(2/N)^{\frac{1}{2}}\cos[(2\pi\lambda/N)+(\pi\lambda/2N)]\quad(\lambda\neq0)$$
$$a_{l0}=(1/N)^{\frac{1}{2}}. \tag{5b}$$

The variables P_λ, Q_λ are obviously wave-like in nature.

Quantization of the HK theory is now possible, since the classical Poisson bracket is

$$\{P_\lambda,Q_\lambda\}_{\rm P.B.}=(\delta_{\lambda\lambda'}/S)\{S_{l,x},S_{l,y}\}_{\rm P.B.}=\delta_{\lambda\lambda'}S_{l,z}/S\cong\delta_{\lambda\lambda'} \tag{6}$$

from Eq. (5) and with $S_{l,z}/S$ approximately equal to unity, corresponding to conditions near saturation. Hence, $P_\lambda^2+Q_\lambda^2$ has eigenvalues $2(n_\lambda+\frac{1}{2})$ with $n_\lambda=0,1,2,\cdots$.

The energy levels for $\mathcal{H}_{\rm HK}$ are:

$$E_{\rm HK}(n_\lambda,\check{S})=-2N\check{S}[\beta H+J\check{S}]+4JS\sum_{\lambda=0}^{N-1}\left(1+\alpha-\cos\frac{2\pi\lambda}{N}\right)(n_\lambda+\tfrac{1}{2}), \tag{7}$$

a result differing from the quantal energy levels in the appearance of $\check{S}$ and the additional $\frac{1}{2}$ in the factor $(n_\lambda+\frac{1}{2})$ as will now be shown. For the additional terms in Eq. (4) can readily be calculated using the HP approximations[5] to be

$$S_{l,x}S_{l,y}-S_{l,y}S_{l,x}=iS. \tag{8}$$

(Equation (8) indicates that the same approximation is made in the quantal theory as in Eq. (6); i.e., $S_{l,z}$ is replaced by S.) It follows that, using the same orthogonal transformation and quantizing the result, we obtain

$$E_{\rm HP}(n_\lambda)=E_{\rm HK}(n_\lambda,S)-2JS\sum_\lambda(1+\alpha)=E_{\rm HK}(n_\lambda,S)-2JS\sum_\lambda\left(1+\alpha-\cos\frac{2\pi\lambda}{N}\right), \tag{9}$$

which verifies the statement made above.

The physical origin of the extra $\frac{1}{2}$ is not hard to see. It arises from Eq. (2): in the HK theory the distinction between S and $[S(S+1)]^{\frac{1}{2}}$ is not maintained, so that, in effect, the expression $S-S_z$, the deviation of the z component of the spin from its maximum value, is replaced by $[S(S+1)]^{\frac{1}{2}}-S_z$. Now the former quantity has, rigorously, integral values, while the latter is approximately $S-S_z+\frac{1}{2}$ (expanding $[S(S+1)]^{\frac{1}{2}}$). The $\frac{1}{2}$ comes from the fact that, quantum-mechanically, the spin vector never lies along the z axis and $S_x^2+S_y^2$ is never rigorously zero. In the formal theory this is expressed by the appearance of the commutator of S_x and S_y as mentioned above.

* Supported in part by the ONR.
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See reference 3, Eqs. (3) to (6) and following discussion.

The Scattering Lengths of the Deuteron and p-d Scattering*

M. M. GORDON**
Washington University, St. Louis, Missouri
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AN analysis of the S-wave scattering of nucleons by deuterons has been carried out by means of a spin-dependent, "effective" two-body model of the nucleon-deuteron interaction.[1] This analysis was begun by considering the deuteron as a single structureless particle so that the actual nucleon-deuteron interaction could be replaced by an "effective" two-body interaction.

The ^2S and ^4S effective two-body neutron-deuteron interactions were represented by square-well central potentials having depths V_2 and V_4, respectively, and the same range r_0. It was then assumed that[2]

$$r_0=(5.0\pm2.0)\times10^{-12}\text{ cm}. \quad (1)$$

The analysis proceeded with the deduction of a theorem to the effect that the binding energy of the ^2S or ^4S ground state of H^3, calculated on the basis of such an effective two-body model, is not greater than the actual binding energy of a neutron to a deuteron in forming that state. The existence of H^3 in a ^2S ground state of known binding energy and with no bound excited states consequently places a restriction on the maximum values of V_2 and V_4 for a given value of r_0.

The value of V_x ($x=2$ or 4) can now be calculated *uniquely*, for a given value of r_0, from the corresponding scattering length of the deuteron a_x ($x=2$ or 4) by means of the relationship:

$$\tan(K_xr_0)/K_xr_0=1-a_x/r_0, \quad (2)$$

where

$$K_x^2=4MV_x/\hbar^2,$$

provided the above-mentioned restrictions on V_2 and V_4 are considered. At the time this work was undertaken, the experimental results on the scattering of slow neutrons from deuterons provided only two fairly definite conclusions regarding the values of a_2 and a_4;[3-5] namely: a_2, $a_4>0$; and Eq. (5) below. From the fact that $a_4>0$, it was then concluded from (2) that V_4 cannot be greater than zero, and consequently,

$$0<a_4<r_0. \quad (3)$$

From the fact that $a_2>0$, it was deduced from (2) that, for any given value of r_0 in the range of (1),

$$a_2\geqq(a_2)_{\min}\geqq0.4\times10^{-12}\text{ cm}. \quad (4)$$

Corresponding to four different values of r_0 in the range of (1), sixteen sets of values for V_2 and V_4 were then calculated from Eq. (2) by using values of (a_2, a_4) consistent with (3), (4), and the experimental value of

$$\sigma_D=(4\pi/3)(a_2^2+2a_4^2)=3.44\text{ barns}, \quad (5)$$

which was assumed to be correct. It proved convenient to combine (5) with (2), and consider a_2 and r_0 as the independent variables for the functions V_2 and V_4. On the assumption that the specifically nuclear *n-n* and *p-p* interactions are the same, these sets of potentials were then used to calculate the angular distributions obtained in *p-d* scattering at 250 and 275 kev,[6] and the 90° (center-of-mass system) cross sections obtained for *p-d* scattering from 1.5 to 3.0 Mev.[7] It was found (as was to be expected) that the fit of the calculated values to the experimental data was relatively insensitive to the value of r_0 within the range of (1) (or even to the assumption of two different values of r_0, in that range, for the ^2S and ^4S effective interactions). It was found, however, that a good fit to the data could be obtained only for

$$a_2\approx0.8\times10^{-12}\text{ cm}, \quad a_4\approx0.3\times10^{-12}\text{ cm}, \quad (6)$$

a_4 being obtained from a_2 via (5). These values, moreover, provided this fit over the entire energy range of the *p-d* data considered.

This work was carried out before the accurate results obtained at Chalk River for the scattering of slow neutrons from D_2 became available.[8] These latter results, together with (3), (4), and (5), give

$$a_2=(0.826\pm0.012)\times10^{-12}\text{ cm}, \quad a_4=(0.26\pm0.02)\times10^{-12}\text{ cm}, \quad (7)$$

which is in excellent agreement with (6) above, deduced from the analysis of *p-d* scattering. This agreement implies that an effective two-body model of the nucleon-deuteron interaction is indeed a practical means of correlating the low energy nucleon-deuteron scattering data, and, that the hypothesis of the equality of the *n-n* and *p-p* interactions is corroborated under these conditions.

A detailed report on the above work will be submitted for publication shortly.

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** Present address: Physics Department, University of Florida, Gainesville, Florida.
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Asymptotic Solutions of Ordinary Linear Differential Equations of the Second Order

ISAO IMAI
Department of Physics, Faculty of Science, University of Tokyo, Tokyo, Japan
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IN a recent paper[1] Kuhn has applied the WKB method to the calculation of the cohesive energies of monovalent metals. His analysis is based on a new procedure proposed by the writer.[2] In consideration of the possibility that this method may be capable of wide applicability, it will be justifiable to present it here in a more explicit form.

We consider the asymptotic solutions of the second-order linear differential equation

$$(d^2\Phi/dx^2)+a^2P(x)\Phi=0 \quad (\text{for } a\to\infty), \quad (1)$$

where

$$P(x)=a_1x+a_2x^2+\cdots \quad (a_1\neq0).$$

We make the transformations

$$\begin{aligned} z&=\int_0^x[P(y)]^{1/2}dy,\\ Q&=P^{-1/4}d^2P^{1/4}/dz^2=-P^{-3/4}d^2P^{-1/4}/dx^2\\ &=-[(5/36)z^{-2}+\lambda z^{-2/3}+\lambda_0+\lambda_1z^{2/3}+\cdots], \quad (2)\\ Q_1(z)&=\int^z Q(y)dy. \end{aligned}$$

It can be shown that the two independent solutions Φ_1, Φ_2, can be expressed in the form

$$\Phi_{1,2}=P^{-1/4}\Psi_{1,2} \quad (3)$$

with

$$\Psi_{1,2}=\exp[\pm iaz\pm(i2a)^{-1}Q_1(z)+(4a^2)^{-1}Q(z)]+O(a^{-3}) \quad (4)$$

for $z\neq0$, and

$$\Psi_{1,2}=(\tfrac{1}{2}\pi a)^{1/2}i^{\pm5/6}\exp\Big\{\mp(i2a)^{-1}\times\int_0^\infty[Q+(5/36z^2)]dz\Big\}\cdot z^{1/6}\zeta H_{1/3}^{(1,2)}(\eta)+O(a^{-3}) \quad (5)$$

for $z=O(a^{-1})$.

Here

$$\zeta=[\xi-\tfrac{1}{3}\lambda_1\kappa^{-2}\xi^2]^{1/2}, \quad \eta=\kappa[\xi+\tfrac{1}{5}\lambda_1\kappa^{-2}\xi^2]^{3/2}, \quad (6)$$
$$\xi=z^{2/3}+\lambda\kappa^{-2}, \quad \kappa^2=a^2+\lambda_0,$$

and the symbol H stands for the Hankel functions.

As an example, the asymptotic formula for the Hankel functions for the "transition region" can be written in the form

$$H_a^{(1,2)}(a\sec\theta)=i^{\pm1/3}\cot^{1/2}\theta\cdot z^{1/6}\zeta H_{1/3}^{(1,2)}(\eta)+O(a^{-\eta/2}), \quad (7)$$

where

$$z=\tan\theta-\theta, \quad \lambda=-3^{1/3}/105, \quad \lambda_0=2/75, \quad \lambda_1=-(69/13475)3^{2/3}.$$

This is to be compared with Langer's formula,[3] which can be obtained from the above by setting $\lambda=\lambda_0=\lambda_1=0$, giving

$$H_a^{(1,2)}(a\sec\theta)=i^{\pm1/3}\cot^{1/2}\theta\cdot z^{1/2}\cdot H_{1/3}^{(1,2)}(az)+O(a^{-5/3})$$

and Watson's formula[4]

$$H_a^{(1,2)}(a\sec\theta)=i^{\pm1/3}3^{-1/2}\tan\theta\cdot H_{1/3}^{(1,2)}(\tfrac{1}{3}a\tan^3\theta)+O(a^{-1}).$$

One of the merits of the new formula is that it permits the easy computation of the zeros of Φ (for example, of $J_a(x)$), from those of the Bessel functions of order 1/3.

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