Neutron Penetration and Slowing Down at Intermediate Distances through Medium and Heavy Nuclei*

H. A. BETHE,** L. TONKS, AND H. HURWITZ, JR.
Knolls Atomic Power Laboratory, Schenectady, New York

(Received February 2, 1950)

This paper reports new theoretical results and numerical calculations of neutron slowing down and penetration to fairly large distances from the source assuming constant mean free path and no absorption. The Placzek one-velocity problem which results from a Laplace transformation of the Boltzmann equation is solved approximately by keeping only a finite number of harmonic terms in the Laplace transform of the scattering function but not restricting the expansion of the neutron density. The Laplace inversion is carried out by the method of steepest descent. Numerical results are given for nuclei of mass infinity and 9, carried out to three orders of approximation, namely zeroth, first, and second. They show the expected transition behavior between age theory and the asymptotic exponential behavior. The accuracy of the method is high, being greater than that of the numerical work over the useful range of the calculations. Rough agreement with Wick's asymptotic results for $M=12$ is found.

I. INTRODUCTION

In many problems it is necessary to know the distribution of neutrons in a moderating material at large distances from a source. If the distance expressed in mean free paths is not small compared with $u'=(M/2)u$, where $u=\ln(E_0/E)$ the customary age theory breaks down, and improvements on this theory, such as the spherical harmonics method, are also inadequate. (In our notation E_0 is the energy of the neutrons as they are emitted from the source, E is the energy of the neutrons whose spatial distribution is to be found, and M is the mass number of the moderator. The quantity u' is a measure of the number of collisions required to reduce the neutron energy from E_0 to E.) On the other hand, "asymptotic" theories in which the neutrons are assumed to move very nearly in the forward direction are not valid until $z \gg u'^3$, which, even for moderate values of u' is far outside the practical range. It is the purpose of this paper to bridge the gap between "age theory" and the "asymptotic theory." The methods which we employ are similar to those which have been developed independently by Wick.[1] The numerical results which we shall present are for moderators of fairly large M (say $M \geqq 9$), whereas Wick and his collaborators have paid particular attention in their numerical work to the case of hydrogen ($M=1$).

We shall use the notation of Marshak[2] except where otherwise mentioned. All distances are measured in units of the mean free path, which is assumed to be independent of the velocity of the neutron. The slowing-down medium is assumed to be non-absorbing and infinite. We assume that the neutrons are emitted isotropically from a monoenergetic infinite plane source with unit strength per unit area located at the position $z=0$.

The theory we present here is a development of Placzek's "one-velocity theory" which is described by Marshak. Our formalism will give results accurate to 10 or 20 percent for *any* distance, provided only that the number of collisions u' is large. (For any practical problem with materials as heavy as Be or heavier, u' is at least about 20.) This is in contrast, for example, to the spherical harmonics method (Marshak, reference 2, p. 222), which breaks down at distances z of the order $u'^{\frac{3}{4}}$, or at attenuations of the order $\exp(-u'^{\frac{1}{2}})$ which is insufficient for many shielding problems. Placzek's extension of the spherical harmonics method (Marshak, Eq. (146a)) extends this to attenuations of the order $\exp(-u'^{\frac{2}{3}})$ which is often valuable; it represents a second approximation toward our theory.

II. TRANSPORT EQUATION. DEFINITIONS

We start from the transport equation (Marshak's Eq. (140)):

$$(1-iy\mu)\phi(y,\eta,\mu) = \int d\Omega'\phi(y,\eta,\mu')g(\eta,\mu_0)+1/(4\pi), \quad (1)$$

where ϕ is the Fourier-Laplace transform of the collision density (number of collisions per unit volume, unit time and unit u), that is,

$$\phi(y,\eta,\mu) = \int_{-\infty}^{\infty} e^{izy}dz \int_0^{\infty} du e^{-\eta u}\psi(z,u,\mu). \quad (2)$$

The notation is the same as Marshak's: z is the distance from the plane source in units of the mean free path,

* *Note added June 14, 1950:* This paper is based on the Atomic Energy Commission report, KAPL-256, which was declassified on January 16, 1950. Since its submission for publication it has come to the authors' attention that report A.E.R.E. T/R 523, April, 1950, of the Atomic Energy Research Establishment, Harwell, Berks., entitled "Calculation of the Functions Used for Determining Neutron Diffusion at Large Distances from the Source" by J. P. Price was issued. This report covers very much the same ground as the present paper but gives about half the number of values. It uses the fact that the function ψ can be expanded as a Taylor series in increasing powers of $1/M$, of which the first two terms are given. This is equivalent in our case to linear-in-$1/M$ interpolation between ψ_∞ and ψ_9. J. P. Price's results agree with those of the present authors when the sign of her $1/M$ term is reversed.

** Permanently at Cornell University, Ithaca, New York.

[1] G. C. Wick, Phys. Rev. 75, 738 (1949).

[2] R. E. Marshak, Rev. Mod. Phys. 19, 185 (1947).

$u=\ln(E_0/E)$, μ is the direction cosine of the motion of the neutron, y is the parameter of the Fourier transform and η is that of the Laplace transform. The function g in (1) is given by Marshak's (140b), (see also his definition of α, Eq. (17a)) *viz.*,

$$g(\eta,\mu)=\alpha F(\mu,\eta)=\int_0^\infty e^{-\eta u}f(\mu,u)du, \tag{3}$$

where $f(\mu, u)$ is the transfer function in Marshak's Eq. (7).

It is advantageous to discuss first the range of parameters η and y which will be of importance. It will be shown that by far the main contribution to the Fourier inverse of ϕ comes from the pole of ϕ, i.e. from that value of y for which the *homogeneous* Eq. (1) has a solution. This is the case for some purely imaginary value of y; we therefore set

$$iy=\nu. \tag{4}$$

Positive values of ν are relevant for positive z; ν will lie between 0 and 1, with small values of ν corresponding to age theory and ν near 1 corresponding to large attenuation.

It turns out further that the relevant values of η are of order M. It is therefore not permissible to expand in powers of η as is done in the spherical harmonics method. Instead, we expand the function g in spherical harmonics as does Wick,[4] except that our $g_l(\eta)$ are $(2l+1)$ times his. Thus

$$g(\eta,u)=(1/4\pi)\sum g_l(\eta)P_l(u). \tag{5}$$

It may be noted that the evaluation of the g_l's becomes simple if the double integration on $f(\mu, u)$ is carried out first with respect to the angular distribution before the Laplace transformation is performed.

If in Wick's equations, modified by $2l+1$, the transformation

$$\eta=\tfrac{1}{2}Mx-1 \tag{6}$$

is made for the transform parameter, and the resulting functions be denoted by $g_l(x)$, then

$$g_0(x)=\left(\frac{M+1}{M}\right)^2\frac{1-[(M-1)/(M+1)]^{Mx}}{2x}, \tag{7a}$$

$$g_1(x)=\frac{3}{2}\left(\frac{M+1}{M}\right)^2\frac{x-1+(x+1)[(M-1)/(M+1)]^{Mx}}{x^2-1/M^2}, \tag{7b}$$

$$g_2(x)=\frac{5}{2}\left(\frac{M+1}{M}\right)^2\frac{x^2-3x+3-M^{-2}-[x^2+3x+3-M^{-2}][(M-1)/(M+1)]^{Mx}}{x(x^2-4/M^2)}, \tag{7c}$$

etc.

When M is large, then putting

$$G_l(x)=[M/(M+1)]^2g_l(x) \tag{8}$$

we find

$$G_0=(1/2x)[1-e^{-2x}], \tag{9a}$$

$$G_1=(3/2x^2)[x-1+(x+1)e^{-2x}], \tag{9b}$$

$$G_2=(5/2x^3)[3-3x+x^2-(3+3x+x^2)e^{-2x}]. \tag{9c}$$

The G_l are essentially Bessel functions[3] of ix.

For reasonable values of x (up to about unity), the G_l decrease with increasing l; for small x,

$$G_l\sim x^l/[1\cdot 3\cdot 5\cdots(2l-1)]. \tag{10}$$

Inserting (5) into (1), we get

$$(1-iy\mu)\phi(y,x,\mu)=\sum_l g_l(x)\phi_l(y,x)P_l(\mu)+1/(4\pi), \tag{11}$$

where

$$\phi_l(y,x)=\tfrac{1}{2}\int_{-1}^1 d\mu\phi(y,x,\mu)P_l(\mu). \tag{12}$$

The relation (2) can be rewritten, introducing (6) and inverting the Fourier transformation:

$$(2/M)\int_0^\infty du'\psi(z,u',u)e^{-u'(x-2/M)}=(1/2\pi)\int_{-\infty}^\infty dy\phi(y,x,\mu)e^{-iyz}, \tag{13}$$

where u' is approximately the average number of collisions and is given by

$$u'=Mu/2. \tag{14}$$

It should be noted that the factor in (14) is $M/2$, not $1/\xi$, which differs from it in the relative order $1/M$.

III. SOLUTION. RELATION BETWEEN ν AND x

As previously indicated, we shall first seek a solution of the homogeneous Eq. (11), postponing consideration of the complete equation until the next section. Such a solution exists for some real value of iy. We therefore consider the equation

$$(1-\nu\mu)\phi(\nu,x,\mu)=\sum_l g_l(x)\phi_l(\nu,x)P_l(\mu) \tag{15}$$

and try to determine the eigenvalue ν, for a given x.

[3] Gegenbauer's Integral. See Watson, *Bessel Functions*, p. 50.

[4] Reference 1, Eqs. (2) to (5).

The right-hand side of (15) is a rapidly converging series in l for any reasonable value of x (let us say, not much greater than unity), because the g_l are rapidly decreasing. (At this point, our assumption differs from that of Wick,[2] who considers the case in which many g_l's are about equal in magnitude.) Therefore it is permissible to consider only a small number of l's on the right-hand side of (11). Indeed, our numerical calculations which are reported below, show that in all cases of practical importance, the eigenvalue ν is given with sufficient accuracy by taking only the terms through $l=2$. Since g_1 contains the anisotropy of scattering due to the finite mass of the target nucleus, it is understandable that in all but the $M=\infty$ case both $l=0$ and $l=1$ terms must be included even for the smallest ν. In terms of age theory the omission of $l=1$ amounts to neglecting $\langle\cos\theta\rangle_{\mathrm{Av}}$ in the transport cross section in age theory.

The error incurred by omitting $l=2$, 3, etc. terms will be indicated in Section VII.

We have considered as of possible practical importance all elements from Be up ($M\geqslant 9$) and all values of u from 3 up so that $u'\geqslant 13$, and any attenuation up to $10^{17}\approx e^{40}$.

While the right-hand side of (11) is rapidly convergent, an expansion of ϕ itself in spherical harmonics of μ would converge exceedingly slowly. In fact, the most important dependence of ϕ on μ is given by the factor $1/(1-\nu, u)$ in which ν may be as large as 0.9 or even closer to unity (see Table III). The expansion of ϕ in spherical harmonics will therefore converge very slowly, and the spherical harmonics method as used by Marshak[2] cannot give a good approximation. In other words, φ_l does not decrease rapidly with l, but $g_l\varphi_l$ does.

With any finite number $N+1$ of terms on the right-hand side of (15), the dependence of ϕ on μ is determined:

$$\phi(\nu, x, \mu)=\sum_{l=0}^{N} g_l(x)\phi_l(\nu, x)P_l(\mu)/(1-\nu\mu). \qquad (16)$$

From this we obtain

$$\phi_l(\nu, x)=\sum_{n=0}^{N} g_n(x)A_{ln}(\nu)\phi_n(\nu, x), \qquad (17)$$

where

$$A_{ln}(\nu)=\tfrac{1}{2}\int_{-1}^{1}\frac{P_l(\mu)P_n(\mu)}{1-\nu\mu}d\mu. \qquad (18)$$

With the abbreviation

$$f(\nu)=\frac{1}{\nu}\tanh^{-1}\nu=\frac{1}{2\nu}\ln[(1+\nu)/(1-\nu)], \qquad (19)$$

the first A_{ln} are

$$\begin{aligned}
&A_{00}=f,\\
&A_{01}=(f-1)/\nu,\\
&A_{11}=(f-1)/\nu^2,\\
&A_{02}=(3/(2\nu^2))[f(1-\nu^2/3)-1],\\
&A_{12}=A_{02}/\nu,\\
&A_{22}=A_{02}[3/(2\nu^2)-\tfrac{1}{2}].
\end{aligned} \qquad (20)$$

Equation (17) is the fundamental system of equations of our theory. It is a system of $N+1$ ordinary linear homogeneous equations for the $N+1$ unknowns ϕ_0, ϕ_1, $\cdots\phi_N$. The system is soluble only if the determinant of the coefficients vanishes, and this condition determines the eigenvalue ν for any given x. Thus, the condition is

$$\begin{vmatrix} g_0A_{00}-1 & g_1A_{01} & \cdots & g_NA_{0N}\\ g_0A_{01} & g_1A_{11}-1 & \cdots & g_NA_{1N}\\ \cdots & \cdots & \cdots & \cdots\\ g_0A_{0N} & g_1A_{1N} & \cdots & g_NA_{NN}-1\end{vmatrix}=0. \qquad (21)$$

Note that all g's are known functions of x, and the A's are known functions of ν.

In the "zero-order" approximation we have, for instance,

$$A_{00}(\nu)\equiv f(\nu)\equiv\frac{1}{2\nu}\ln\frac{1+\nu}{1-\nu}=\frac{1}{g_0}$$
$$=\left(\frac{M}{M+1}\right)^2\frac{2x}{1-[(M-1)/(M+1)]M^x} \qquad (22)$$

as in the equation succeeding Marshak's Eq. (149). From this equation it is easy to determine ν as a function of x, or, conversely, x as a function of ν.

It is not, in general, permissible to expand Eq. (21) in powers of ν or x or both. Such an expansion is unsatisfactory especially because the power series for $f(\nu)$ is very slowly convergent except for very small ν. *If* an expansion is made, and M is assumed large, equating the lowest powers of ν and x gives

$$\nu=(3x)^{\frac{1}{2}} \qquad (23)$$

which corresponds to the age theory. Further results of expansion will be given at the end of Section V.

The "first-order" approximation, in which g_0 and g_1 are taken into account, gives for ν the relation

$$f(\nu)=\frac{1}{g_0}-\frac{f(\nu)-1}{\nu^2}\left(\frac{1}{g_0}-1\right)g_1. \qquad (24)$$

It is convenient to determine an approximate value of ν from (22), then insert this into the right-hand side of (24), determine a more accurate value of ν, etc. Equation (24) is equivalent to the equation below (151) in Marshak's paper.

In principle, the relation between ν and x can be obtained to any desired accuracy by numerical calculation. This will be done in Sections IV and V for infinite mass of the slowing atoms and for $M=9$; the results for any "practical" value of M can then be estimated by interpolation. Once the relation $\nu(x)$ is known and tabulated, the neutron density can be obtained without difficulty, as will be shown in the following.

IV. FOURIER INVERSION

To calculate the Fourier transform (13) of ϕ for positive z, the path of integration can be deformed from the real axis into the negative imaginary plane. The function ϕ is regular in this plane except for a pole at $y=-i\nu$, corresponding to the solution of the homogeneous equation found in the last section, and a branch line extending from $-i$ to $-i\infty$ along the imaginary axis. As will be justified below (see also Marshak's remarks after Eq. (149)), the contribution of the branch line is unimportant. Therefore, the integral on the right of (13) will be $-2\pi i$ times the residue at the pole.

Thus we need the solution of the inhomogeneous Eq. (11) for $iy=\nu'$ in the neighborhood of the eigenvalue ν. We have then in the "Nth approximation"

$$\phi(\nu', x, \mu)=\sum_{l=0}^{N} g_l(x)\phi_l'\nu', x)\frac{P_l(\mu)}{1-\nu'\mu}+\frac{1}{4\pi(1-\nu'\mu)}. \quad (25)$$

Multiplying by P_l and integrating we find

$$\phi_l(\nu', x)=\sum_{n=0}^{N} g_n\phi_n A_{ln}+A_{0l}/(4\pi). \quad (26)$$

While it is possible to obtain all of the ϕ's, we are interested mostly in the total neutron intensity moving in all directions, which is given by

$$\int \phi(y, x, \mu)d\Omega=4\pi\phi_0(y, x). \quad (27)$$

The "residue" of this quantity near the pole ν can be obtained in the same numerical process as the eigenvalue ν itself, as follows:

Introducing the quantity

$$\varphi=g_0\phi_0+1/(4\pi), \quad (28)$$

and dividing Eqs. (26) for $l\neq 0$ by this quantity, one gets

$$(\phi_l/\varphi)-\sum_{n=1}^{N} g_n(x)A_{ln}(\nu')(\phi_n/\varphi)=A_{0l}(\nu'). \quad (29)$$

For any x and ν' these equations can be solved for the unknowns ϕ_l/φ for $l=1, 2, \cdots N$. The result can then be inserted into the equation for $l=0$. If this equation is also divided by φ and (26) used, we get

$$-\frac{1}{4\pi g_0\varphi}=A_{00}(\nu')+\sum_{l=1}^{N} g_l(x)A_{0l}(\nu')(\phi_l/\varphi)-1/g_0 \quad (30)$$
$$\equiv\Lambda(x, \nu').$$

Since the ϕ_l/φ can be determined as functions of x and ν' from (29), the middle member of (30) is a calculable function of x and ν' for which the abbreviation Λ is introduced. For general values of x and ν', (30) determines the remaining unknown φ. However, the *eigenvalue* ν of ν' is determined by the condition that the homogeneous equation (1) has a solution, so that φ is infinite and

$$\Lambda(x, \nu)=0. \quad (31)$$

Thus this can replace Eq. (21) in the actual calculation of $x(\nu)$.

In the process of determining x numerically from Eq. (31) one also obtains automatically the derivative

$$\lambda=-\left(\frac{\partial}{\partial x}\Lambda(x, \nu)\right) \quad (32a)$$

which will be useful in the numerical calculation. In the analysis we need

$$\Lambda'=[(\partial/\partial\nu')\Lambda(x, \nu')]_x. \quad (32b)$$

It is easily seen that Λ' and λ are positive.

For ν' near ν we find from Eqs. (28) and (30), since the last term of the former is negligible near the pole,

$$4\pi\phi_0=4\pi\varphi/g_0=-1/(g_0^2(\nu'-\nu)\Lambda') \quad (33)$$

and the Fourier inversion of (13) gives

$$L(z, x)=(2/M)\int_0^\infty du'\psi_0(z, u')e^{-u'(x-2/M)}$$
$$=1/(2\pi)\int_{-\infty}^{\infty} dy 4\pi\phi_0(y, x)e^{-iyz}=e^{-\nu z}/g_0^2\Lambda', \quad (34)$$

where ψ_0 is the total collision density integrated over all directions of neutron motion,

$$\psi_0(z, u')=\int d\Omega\psi(z, u', \mu). \quad (35)$$

The parameters ν and x in (34) are connected by the relation derived in Section III.

Equation (34) is exact (insofar as the relation between ν and x is correct) except for the neglect of the integral along the branch line from $y=-i$ to $-i\infty$. In the "zero order" approximation, this integral is given explicitly in Marshak's Eq. (149). Except for a numerical factor depending on x which is of order unity (in general smaller), this integral is proportonal to e^{-z}. Its ratio to (41) is therefore,

$$\text{branch/pole}\sim e^{-z(1-\nu)}. \quad (36)$$

For moderately small ν, this is of order e^{-z}; we can therefore neglect the branch integral at distances large compared to a mean free path, which are the only ones of interest to us. As was pointed out by Marshak, the branch integral represents the direct effect of the source which is of no interest for our purposes. One might be afraid that $(1-\nu)z$ is no longer large when ν is nearly unity, which is the case for large attenuation. However, as we shall see in Section VI, $(1-\nu)z$ is in this case about $u'/4$ which we have assumed to be large. Therefore in spite of the smallness of $1-\nu$, the product $(1-\nu)z$ stays large, and the contribution of the branch integral is always negligible.

This conclusion remains true in spite of the fact that the numerical factor of the pole contribution (especially the factor $1/\Lambda'$) is small for ν close to unity. This only gives a factor of order $1/z$, which is unimportant compared to the *exponential* factor $(\exp(-u'/4)$.

For the numerical work it is not convenient to keep x fixed and to calculate the function Λ for various values of ν', but it is preferable to calculate Λ as a function of x, for fixed ν. Equation (31) for the eigenvalue $\nu(x)$ then implies, using Eqs. (32a) and (32b), that

$$\Lambda' \equiv (\partial\Lambda/\partial\nu)_x = -(\partial\Lambda/\partial x)(dx/d\nu) \equiv \lambda dx/d\nu. \quad (37)$$

The derivative $\partial\Lambda/\partial x$ must be evaluated at the eigenvalue of x as defined by (31). The quantity λ is then a function of ν only, but through $\nu(x)$ it can also be regarded as a function of x.

Another convenient modification is the introduction of the slowing-down density,

$$q = \xi\psi_0. \quad (38)$$

Using this and (37), we get from (34)

$$L(z, x) = 2/(M\xi)\int_0^\infty du' q(z, u') e^{-u'(x-2/M)} = (d\nu/dx)(1/g_0^2\lambda)e^{-\nu z}. \quad (39)$$

The slowing-down density has the advantage that it remains finite for $M\to\infty$, whereas ψ_0 would become infinite (proportional to M). In the limit $M\to\infty$, the factor $2/(M\xi)$ in (39) tends to unity.

TABLE I. Functions of ν, Eqs. (20).

ν	$A_{00}(\nu)$	A_{01}	A_{11}	A_{02}	A_{12}	A_{22}
0	1.0000000	0	0.3333	0	0	
0.05	1.0008346	0.01669	0.3338	0.00054	0.01080	0.32373
0.10	1.003354	0.03354	0.3354	0.00140	0.01400	0.20930
0.15	1.007603	0.05069	0.3379	0.00306	0.02040	0.20247
0.20	1.013663	0.06830	0.3415	0.00554	0.02772	0.20498
0.25	1.021651	0.08660	0.3464	0.00881	0.03524	0.20704
0.30	1.031732	0.1058	0.3527	0.01300	0.04333	0.2102
0.35	1.044125	0.1261	0.3603	0.01824	0.05211	0.2142
0.40	1.0591	0.1478	0.3695	0.02452	0.06130	0.2176
0.45	1.0771	0.1713	0.3807	0.03256	0.07236	0.2249
0.50	1.0986	0.1972	0.3944	0.04230	0.08460	0.2327
0.55	1.1244	0.2260	0.4109	0.05466	0.09938	0.2437
0.575	1.1390	0.2417	0.4204	0.06113	0.10628	0.2467
0.60	1.1552	0.2587	0.4312	0.06908	0.11513	0.2533
0.625	1.1730	0.2768	0.4429	0.07784	0.12454	0.2600
0.65	1.1928	0.2966	0.4563	0.08808	0.13554	0.2687
0.675	1.2146	0.3179	0.4710	0.09919	0.14695	0.2770
0.70	1.2390	0.3414	0.4877	0.11213	0.16018	0.2872
0.725	1.2663	0.3673	0.5066	0.12705	0.17524	0.2990
0.75	1.2973	0.3964	0.5285	0.14416	0.19221	0.3124
0.775	1.3326	0.4292	0.5538	0.16433	0.21204	0.3282
0.80	1.3733	0.4666	0.5833	0.18827	0.23534	0.3471
0.825	1.4210	0.5103	0.6185	0.21732	0.26342	0.3703
0.85	1.4778	0.5621	0.6613	0.25307	0.29773	0.3989
0.875	1.5475	0.6257	0.7151	0.29891	0.34161	0.4362
0.90	1.6358	0.7064	0.7849	0.35950	0.39944	0.4860
0.925	1.7542	0.8154	0.8815	0.44509	0.48118	0.5577
0.95	1.9282	0.9771	1.0285	0.57861	0.60906	0.6724

V. LAPLACE INVERSION; MAIN FORMULA

Having calculated $L(z, x)$ in Eq. (39) we now wish to obtain its Laplace inverse $q(z, u')$.

According to the general method for this we get from (39):

$$q(z, u') = \frac{1}{2\pi i}\frac{M\xi}{2}e^{-2u'/M}\int_{-i\infty}^{+i\infty} dx L(z, x) e^{u'x}$$

$$= -\frac{M\xi}{2}e^{-2u'/M}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{d\nu}{dx}\frac{1}{g_0^2\lambda}e^{-\nu z+u'x}dx. \quad (40)$$

In this expression, ν, g_0, and λ should, at this point, be regarded as functions of x, determined by the theory of the preceding sections. Equation (40) is exact except for the neglect of the branch-line integral [see Eq. (36)].

To evaluate (40) we remember that we are interested only in large values of u' and/or z. In this case, the exponential varies far more rapidly with x than do any of the other factors. The exponential has a saddle point at a certain value of x determined by the condition

$$z(d\nu/dx)_{x_0} = u'. \quad (41)$$

Since ν is a given function of x, this determines x_0 (or ν_0,

TABLE II. Functions of x for $M=\infty$, Eqs. (9).

x	$G_0^{-1}(x)$	$G_1(x)$	$G_2(x)$
0	1.000000	0	0
0.05	1.050833	0.047573	0.0015776
0.10	1.103331	0.090574	0.0060365
0.15	1.157489	0.129397	0.012931
0.20	1.213298	0.164402	0.021897
0.25	1.270747	0.195920	0.032595
0.30	1.329822	0.224267	0.044735
0.35	1.390504	0.249674	0.058055
0.40	1.4528	0.272447	0.07232
0.45	1.5166	0.292793	0.08733
0.50	1.5820	0.310914	0.10291
0.55	1.6489	0.327025	0.11889
0.60	1.7172	0.341292	0.13514
0.65	1.7870	0.353887	0.15153
0.70	1.8582	0.364943	0.16797
0.75	1.9308	0.374608	0.18437
0.80	2.0048	0.383004	0.20063
0.85	2.0800	0.390238	0.21671
0.90	2.1565	0.396422	0.23254
0.95	2.2342	0.401651	0.24807
1.00	2.3130	0.406010	0.26327
1.05	2.3931	0.409530	0.27810
1.10	2.4741	0.412420	0.29252
1.15	2.5563	0.414621	0.30652
1.20	2.6394	0.416230	0.32010
1.25	2.7236	0.417303	0.33323
1.30	2.8086	0.417896	0.34589
1.35	2.8945	0.418053	0.35819
1.40	2.9813	0.417814	0.36986
1.45	3.0689	0.417223	0.38116
1.50	3.1572	0.416313	0.39197
1.55	3.2462	0.415115	0.40233
1.60	3.3360	0.413661	0.41228
1.65	3.4264	0.411979	0.42169
1.70	3.5174	0.410090	0.43074
1.75	3.6090	0.408022	0.43933
1.80	3.7011	0.405790	0.44749
1.85	3.7938	0.403416	0.45527

i.e. the corresponding value of ν) for any given ratio z/u'.

In Eq. (40) the factors multiplying the exponential are slowly variable functions of x and may be replaced by their values at x_0 (or ν_0). For the exponential we need its value at ν_0 which we write in the form

$$e^{-u'\varphi(\nu_0)}, \tag{42}$$

where

$$\varphi(\nu)=\nu dx/d\nu - x. \tag{43}$$

In addition, we need the second derivative of the exponent,

$$\frac{\partial^2}{\partial x^2}(u'x-\nu z)=-z\frac{\partial^2\nu}{\partial x^2}=u'\left(\frac{d\nu}{dx}\right)^2\frac{\partial^2 x}{\partial \nu^2}, \tag{44}$$

where Eq. (41) has been used in the elimination of z. This is easily seen to be positive so that the steepest descent from the saddle point is for purely imaginary changes of ν or x. The slowing-down density is then easily calculated to be

$$q(z, u')=\frac{K_M(\nu)}{\surd u'}e^{-u'\psi_M(\nu)} \tag{45}$$

with

$$K_M(\nu)=\frac{M\xi}{2g_0{}^2\lambda}\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{(d^2x/d\nu^2)^{\frac{1}{2}}} \tag{46A}$$

and

$$\psi_M(\nu)=\varphi(\nu)+2/M, \tag{46B}$$

where all functions of ν must be determined at the value of ν defined by (41), i.e., by

$$(dx/d\nu)=z/u'. \tag{41'}$$

TABLE III. Functions in slowing-down density Eq. (45) for $M=\infty$.

ν	x	$\frac{dx}{d\nu}=\frac{z}{u'}$	ψ	$\psi/(z/u')$	$-\frac{\partial\Lambda}{\partial x}=\lambda$	$\frac{d^2x}{d\nu^2}$	K_∞
0	0						
		0.0167	0.0002097	0.0126	(1.001)	0.667	0.4886
0.05	0.0008346						
		0.0504	0.001897	0.0376	(1.003)	0.679	0.4859
0.10	0.0033535						
		0.085	0.005366	0.0631	(1.0035)	0.705	0.4797
0.15	0.0076029						
		0.1213	0.010814	0.0892	(1.004)	0.744	0.4713
0.20	0.0136637						
		0.1600	0.018552	0.1160	(1.0045)	0.802	0.4594
0.25	0.0216537						
		0.2017	0.029056	0.1441	(1.006)	0.881	0.4454
0.30	0.0317402						
		0.2484	0.04301	0.1731	(1.008)	0.983	0.4302
0.35	0.0441573						
		0.3008	0.06147	0.2044	(1.0125)	1.1215	0.4124
0.40	0.05920						
		0.3610	0.08562	0.2372	(1.0175)	1.302	0.3938
0.45	0.07725						
		0.4346	0.11883	0.2734	(1.022)	1.550	0.3737
0.50	0.09898						
		0.5248	0.16404	0.3126	(1.027)	1.899	0.3513
0.55	0.1252				1.03		
		0.6320	0.22318	0.3531	(1.035)	2.413	0.3264
0.60	0.1568						
		0.774	0.3095	0.3999	(1.045)	3.311	0.2947
		0.732	0.2826	0.3861	(1.042)	3.04	0.3029
0.625	0.1751						
		0.816	0.3352	0.4108	(1.048)	3.614	0.2878
0.650	0.1955						
		0.916	0.4002	0.4369	(1.055)	4.35	0.2711
0.675	0.2184						
		1.036	0.4813	0.4646	(1.062)	5.31	0.2544
0.70	0.2443						
		1.184	0.5850	0.4941	(1.07)	6.52	0.2398
0.725	0.2739						
		1.374	0.7229	0.5261	(1.075)	8.068	0.2269
0.75	0.3080				1.08		
		1.596	0.8898	0.5575	(1.085)	10.32	
0.775	0.3479						
		1.892	1.1195	0.5917	(1.095)	14.14	
0.80	0.3938						
0.825	0.4506	2.53	1.63	0.6443	(1.11)		
0.850	0.5217	3.23	2.23	0.6904	(1.12)		
0.875	0.6154	4.36	3.20	0.7339	(1.14)		
0.900	0.7460	7.26	5.79	0.7975	1.16		
0.925	1.0260	16.27	14.02	0.8617			
0.950	1.7582						

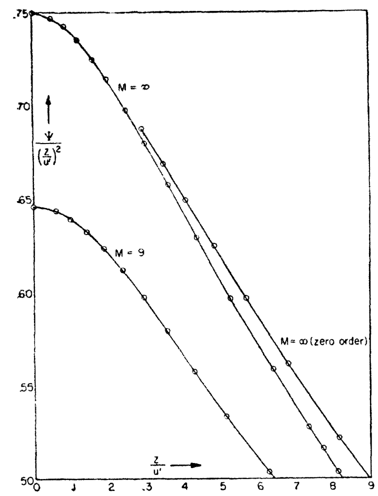

FIG. 1. Slowing-down density exponent in Eq. (47) for $M=\infty$ and $M=9$ for small z/u'.

Equation (45), with (43), (46), and (41′), is the principal result of our theory. In using it, one first has to determine x as a function of ν by solving (21) or (31). In doing this, one also finds λ [Eq. (32a)]. Having the relation $x(\nu)$, one also has $dx/d\nu$ and ψ [Eqs. (43) and (46B)] as functions of ν, as well as $d^2x/d\nu^2$ and g_0. Equation (41) then yields z/u' as a function of ν, or, conversely, ν as a function of the ratio z/u' which embodies the conditions of the experiment. Thus ψ and K are also determined as functions of z/u', and finally q is found from (45) as a function of z and u'; hence q can be expressed in the form:

$$q(z,u')=\frac{K_M(z/u')}{\sqrt{u'}}\exp[-u'\psi_M(z/u')]. \tag{47}$$

The exponent, which is the most important part of the expression, can also be written as

$$-z[\psi_M/(z/u')]. \tag{47′}$$

It can be shown that in the case of infinite M, ψ is determined correctly to order $(z/u')^6$ by taking only g_0 and g_1 into account. We find:

$$\psi=\frac{3}{4}\left(\frac{z}{u'}\right)^2-\frac{81}{80}\left(\frac{z}{u'}\right)^4+\frac{20613}{5600}\left(\frac{z}{u'}\right)^6. \tag{48}$$

For finite M, the coefficients of $(z/u')^4$ and $(z/u')^6$ are affected slightly by g_2 but not by g_3 which is zero for $\eta=0$. For Be($M=9$) and C($M=12$), ψ is given to order $(z/u')^6$ by the following expressions:

$$\text{Be:}\quad \psi=0.6456(z/u')^2-0.6972(z/u')^4+2.046(z/u')^6, \tag{49}$$

$$\text{C:}\quad \psi=0.6705(z/u')^2-0.7659(z/u')^4+2.372(z/u')^6. \tag{50}$$

Neglecting g_2, one obtains instead

$$\text{Be:}\quad \psi=0.6456(z/u')^2-0.6964(z/u')^4+2.048(z/u')^6, \tag{49′}$$

$$\text{C:}\quad \psi=0.6705(z/u')^2-0.7654(z/u')^4+2.374(z/u')^6. \tag{50′}$$

The effect of g_2 is thus very small. The values of ψ obtained by the numerical procedure described below agree with Eq. (49) to within 2 percent up to $(z/u')=0.35$. The "exponential" theory of Placzek[5] is equivalent to using only the first two terms in the expansion of ψ in $(z/u')^2$. For $M=1$, using Wick[1]

$$\text{H:}\quad \psi=\frac{1}{4}\left(\frac{z}{u}\right)^2-\frac{11}{240}\left(\frac{z}{u}\right)^4+\frac{73}{4032}\left(\frac{z}{u}\right)^6.$$

The power series for K_∞ and K_9 have been worked out to $(z/u')^4$. They are:

$$K_\infty=[3/(4\pi)]^{\frac{1}{2}}[1-(14/5)(z/u')^2+19.20(z/u')^4\cdots],$$
$$K_9=0.4533[1-2.225(z/u')^2+12.39(z/u')^4\cdots]. \tag{51}$$

The age theory approximation is equivalent to using only the first terms in the expansions of ψ and K,

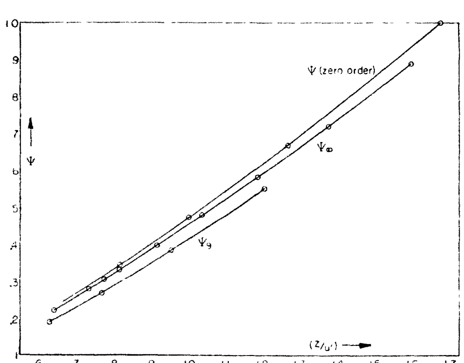

FIG. 2. Slowing-down density exponent in Eq. (47) for $M=\infty$ and $M=9$ for larger z/u'.

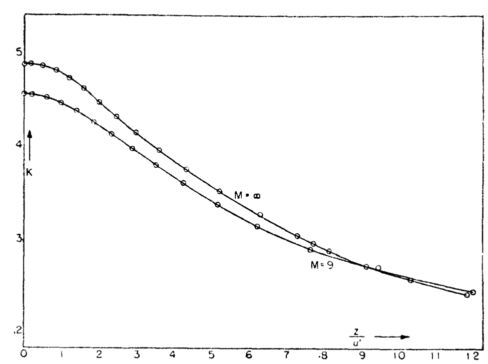

FIG. 3. Slowing-down density coefficient in Eq. (47) for $M=\infty$ and $M=9$.

[5] G. Placzek, "*The Spatial Distribution of Neutrons Slowed Down by Elastic Collisions.*" Declassified Report A-25 (1941). The author has given explicit formulas for the first two terms in ψ for any M.

specifically

$$\psi=\tfrac{3}{4}\cdot\tfrac{1}{2}M\xi[1-\tfrac{2}{3}M]\cdot(z/u')^2, \tag{52}$$

$$K=\left[\frac{4\pi}{\tfrac{3}{2}\xi M(1-2/3M)}\right]^{\frac{1}{2}}. \tag{53}$$

VI. CALCULATIONS FOR NUCLEI OF INFINITE MASS, $M=\infty$

Calculations have been made in accordance with the procedure outlined below Eq. (47). They were carried through the second order approximation. ν was used as the independent, x as the dependent variable.

Table I contains a tabulation of the necessary A's, and Table II covers the G's.

Equation (21) with $N=2$ was expanded to give Eq. (31) in the form

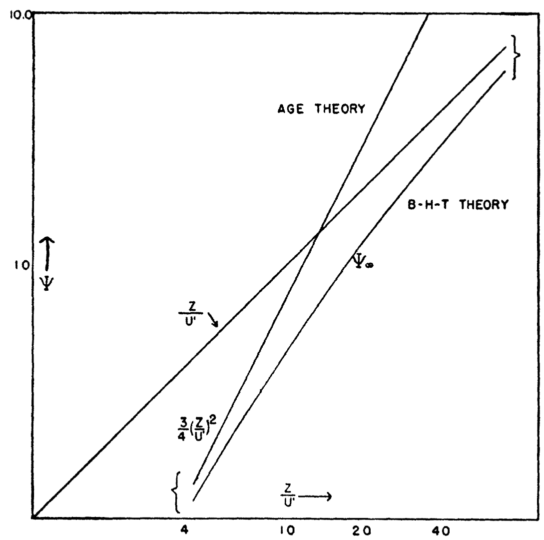

FIG. 4. Asymptotic relations of $q\simeq e^{-u'\psi}$ for large M.

$$\Lambda(x,\nu')=-\frac{1}{g_0(x)}+A_{00}(\nu')+\frac{\dfrac{A_{01}^2}{g_1^{-1}-A_{11}}+\dfrac{A_{02}^2}{g_2^{-1}-A_{22}}+\dfrac{2A_{01}A_{02}A_{12}}{(g_1^{-1}-A_{11})(g_2^{-1}-A_{22})}}{1-\dfrac{A_{12}^2}{(g_1^{-1}-A_{11})(g_2^{-1}-A_{22})}}. \tag{54}$$

The zero-order approximation is obtained by retaining g_0 and A_{00} only. A pair of these values, x and ν, from the zero-order calculations were used as x and ν' in the second and third terms of the right member and an improved g_0^{-1} found, since $\Lambda(x,\nu)=0$. From this a better value of x was found and this iterative process was continued until no further change in x occurred. Since each approximate value of x gave a non-vanishing value of Λ, this process furnished $\lambda\equiv-\partial\Lambda/\partial x$. The first two columns of Table III tabulate $x(\nu)$ in the second-order approximation.

TABLE IV. Functions of x for $M=9$, Eqs. (7).

x	$g_{09}^{-1}(x)$	$g_{19}(x)$	$g_{29}(x)$
0.20	0.979472	0.20433	0.01003
0.25	1.026030	0.243449	0.01600
0.30	1.073910	0.278597	
0.35	1.123097	0.310127	
0.40	1.173575	0.338347	
0.45	1.225325	0.363551	0.05102
0.50	1.278324	0.386002	0.06083
0.55	1.332549	0.405931	0.07099
0.575	1.360114		0.07608
0.60	1.387976	0.423575	0.08123
0.625	1.416131		0.08638
0.65	1.444577	0.439132	0.09155
0.675	1.473309		0.09672
0.70	1.502325	0.452786	0.10189
0.725	1.531619		0.10705
0.75	1.561189	0.464708	0.11218

From a table such as III we had to calculate ψ in the exponent of e in the slowing-down density, Eq. (45). The variable part of ψ is φ, Eqs. (43) and (46B), and it is desirable to have this as a function of $dx/d\nu$. This can be done by well-known methods using differences. For much of the work we used a method that gave $\nu dx/d\nu-\nu$ as a function of $dx/d\nu$ more accurately than usual methods but this led to later unfortunate computational complications because it required forsaking the established tabular points which are equally spaced in ν. The interlining of the results of this calculation in

TABLE V. Functions in slowing-down density Eq. (45) for $M=9$.

ν	x	$-\dfrac{\partial\Lambda}{\partial x}$	$\dfrac{dx}{d\nu}=\dfrac{z}{u'}$	ψ	$\psi/(z/u')$	$\dfrac{d^2x}{d\nu^2}$	K_9
0	0.22222	0.9298					0.4533
			0.0194	0.00024	0.0124	0.776	0.4532
0.05	0.22319	0.930					
			0.0586	0.00221	0.0377	0.792	0.4499
0.10	0.22612	0.931					
			0.0988	0.00624	0.0632	0.820	0.4444
0.15	0.23106	0.933					
			0.1412	0.01262	0.0894	0.876	0.4342
0.20	0.23812	0.934					
			0.1868	0.02176	0.1165	0.948	0.4231
0.25	0.24746	0.936					
			0.2364	0.03419	0.1446	1.036	0.4118
0.30	0.25928	0.938					
			0.2914	0.05073	0.1741	1.168	0.3961
0.35	0.27385	0.942	0.3211	0.06076	0.1892		
			0.3544			1.356	0.3775
0.40	0.29157	0.945	0.3895	0.08645	0.2220		
			0.4284			1.600	0.3552
0.45	0.31299	0.951	0.4694	0.12044	0.2566		
			0.5168			1.968	0.3349
0.50	0.33883	0.956	0.5691	0.16792	0.2951		
			0.6260			2.476	0.3123
0.55	0.37031	0.963	0.6930	0.23305	0.3363		
			0.766			3.200	0.2884
0.60	0.4087	0.970	0.8508	0.3240	0.3808		
			0.950			4.316	0.2690
0.65	0.4564	0.978	1.069	0.4607	0.4309		
			1.204			6.160	0.2425
0.70	0.5172	0.988	1.385	0.6743	0.487		
0.75	0.5978	1.003					
0.80	0.7109						

columns 3, 4, 5, etc. of Table III recognizes that values calculated in this way apply to points intermediate between the tabulated ν and x values.

As ν approaches unity there is difficulty because $dx/d\nu$ and x both increase indefinitely there. But there we used

$$-x^2\frac{d}{d\nu}\left(\frac{\nu}{x}\right)=dx/d\nu-x=\varphi, \tag{55}$$

whence

$$dx/d\nu=(\varphi+x)/\nu \tag{56}$$

and

$$d^2x/d\nu^2=\frac{1}{\nu}\frac{d\varphi}{d\nu}. \tag{57}$$

All of the above material appears in Table III. Plots essentially of ψ against z/u' appear in Figs. 1, 2, and 5. The first is of $\psi/(z/u')^2$ to make the comparison with age theory which gives a constant for this quantity. The second and third are appropriate to Eqs. (47) and (47′), respectively.

Next, $K_\infty(\nu)$ had to be calculated. The calculation of $x(\nu)$ led quite naturally to a value for λ and the other quantities presented no difficulties. $K_\infty(\nu)$ is tabulated in column 8 of Table III and is plotted in Fig. 3.

We now had everything necessary to evaluate Eq. (45):

$$q(z,u')=\frac{K_\infty(z/u')}{\sqrt{u'}}\exp[-u'\psi_\infty(z/u')]$$

for M very large. Here the functional dependence on the physical quantities z and u', instead of ν, is indicated.

Figure 4 compares the present results with age theory on the one hand and with the asymptotic form e^{-z} for large z/u' on the other. The comparison is based on the exponential factor only. The agreement with age theory at low z/u' is apparent, and at $z/u'\simeq 7$ the approach to e^{-z} is evident.

VII. CALCULATION FOR NUCLEUS OF MODERATE MASS, $M=9$

The difference between this and the previous case is that the functions $g_0(x)$, g_1, g_2 of Eqs. (7) were used in place of the functions G_0 of Eqs. (9). In these functions only the region $x\geq 2/M=2/9$ is of interest [see Eq. (6)] since $\eta\geq 0$. For purposes of calculation, however, the g's were tabulated from $x=0.20$. Values of g_0^{-1}, g_1, g_2 are given in Table IV for $M=9$ and are accordingly designated as g_{09}^{-1}, g_{19}, g_{29}, respectively.

The calculations of $x(\nu)$, ψ, etc. were carried out just as for $M=\infty$, noting that here $2/M$ must be added to φ to get ψ and that $M\xi/2=1.080$. The results are tabulated in Table V and in Figs. 1, 2, 3, and 5.

It was of immediate concern to know what the accuracy of the present approximation is. Using Eq. (54) it was possible to segregate (approximately) the various orders of approximation. In the calculations

$$0=\Lambda=-2.3542+1.7542+0.4557+0.1443 \quad \text{for } M=\infty,\ \nu=0.925,$$

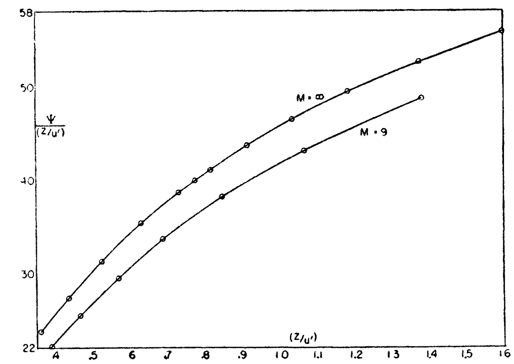

FIG. 5. Slowing-down density exponent as in Eq. (47′) for $M=\infty$ and $M=9$ for larger z/u'.

$$0=\Lambda=-1.5206+1.4210+0.0934+0.0062 \quad \text{for } M=\infty,\ \nu=0.825,$$

$$0=\Lambda=-1.5150+1.3733+0.1351+0.0066 \quad \text{for } M=9,\ \nu=0.80,$$

where, in turn, the two zero-order terms, the first- and the second-order appear. The first expression applies well beyond the range of z/u' for which we have made accurate calculations, yet the convergence is not very bad. In the second expression the corrections amount, successively, to 6 percent and 0.4 percent, thus predicting an error of well under 0.1 percent. The consequent error in x is less than 4×10^{-4} since $\lambda\sim 1$. A little analysis shows that, for the same $dx/d\nu$, ψ would decrease somewhat less than x increases, that is by less than 0.025 percent. The third expression indicates about the same accuracy as the second although we are here outside the tabulated range of ψ.

Another matter of interest concerns the error caused by using ψ_∞ instead of ψ_9 for Be. The comparison must, of course, be made at the same z (relative distance) and the same u' (relative slowing down). From 1 Mev. to thermal in Be, $u'=79$. If we choose $z=79$, a distance of about 130 cm in Be, the difference in $\psi/(z/u')$ from Fig. 5 is 0.04, the ψ_∞ giving too low a value of q by the factor

$$e^{-0.04\times 79}=0.043.$$

Finally, we can make a comparison of asymptotic results obtained by Marshall, a student of G. C. Wick,[6] for $M=12$ with our results. Marshall's results can be expressed as

$$\psi_{12}=z/u'-0.587(z/u')^{\frac{1}{3}}+0.0417.$$

From $z/u'=1$ to 16 both ψ_{12} and $\psi_{12}-z/u'$ agree within better than 10 percent with our ψ_∞ results, and this seems to be better than is to be expected.

We wish to acknowledge the services of Mrs. Dorothy Guy and Miss Bernice Warr in performing the calculations.

[6] By letter communication. Wick (see reference 1) has shown that the behavior of ψ for large z/u' can be studied by means of a Schroedinger-type differential equation for the transform of the neutron angular distribution. This approach was suggested independently by J. A. Wheeler.