Some Remarks on Non-Local Field Theory†

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The S-matrix formalism introduced by Yukawa for non-local field theory is considered. For suitable types of non-local fields and interactions it is shown that the S-matrix is convergent through the second order of interaction. In the limiting case in which the non-local fields become local, it is found that the S-matrix formalism yields results inconsistent with the usual formalism unless a certain limiting process is introduced. The limiting process brings agreement between the two formalisms only through the second order of interaction; and the higher orders will, in general, disagree. Unfortunately, the limiting process also destroys the convergence of the S-matrix in the general case of non-local fields. These results suggest that the S-matrix formalism will need to be revised, but no definite recommendations for doing this are made here. An internal angular momentum operator for non-local fields is introduced; this operator aids in the decomposition of the field into irreducible parts of different spins.

I. INTRODUCTION

RECENTLY Yukawa considered the possibility of extending ordinary (local) field theory by introducing the concept of non-local fields which were supposed to represent elementary particles having a finite extension in space-time.[1] In local field theory we associate a field quantity with each space-time point x^μ ($x^1=x_1=x$, $x^2=x_2=y$, $x^3=x_3=z$, $x^4=-x_4=ct$); for example, a local scalar field is represented by $U(x)$. In non-local field theory, however, we consider the field to be a matrix element of a non-local operator, the rows and columns of the matrix being characterized by space-time coordinates; for example, a non-local scalar field would be written as $(x'|U|x'')$. A local field is then a special case of a non-local field in which the matrix is diagonal:

$$(x'|U|x'')=U(x')\delta(x'-x''), \tag{1}$$

where

$$\delta(x)=\delta(x^1)\delta(x^2)\delta(x^3)\delta(x^4).$$

The problem of introducing interactions between non-local fields seems to be much more difficult than in local field theory, since it does not seem to be possible to extend the concept of the Schrödinger wave functional and Schrödinger equation in any simple manner. Yukawa[2,3] suggested, however, that it might be possible to introduce interactions between fields by extending the formalism of the S-matrix to the case of non-local fields. To do this he defined a space-time average for an arbitrary operator A by

$$\{A\}\equiv\int\cdots\int(x'|A|x'')(dx')^4(dx'')^4, \tag{2}$$

where

$$(dx)^4=dx^1dx^2dx^3dx^4.$$

The proposed S-matrix then has the form

$$S=1+(i/\hbar c)\{L'\}+(i/\hbar c)^2\{L'D_+L'\}+(i/\hbar c)^3\{L'D_+L'D_+L'\}+\cdots, \tag{3}$$

where L' is an invariant Hermitian operator characterizing the interaction and is expressed as a sum of products of non-local and local field quantities. D_+ is an invariant displacement operator with the matrix elements

$$\begin{aligned}(x'|D_+|x'')&=D_+(x'-x'')\\&=1,\tfrac{1}{2},0,\end{aligned} \tag{4}$$

as $(x'-x'')$ is time-like in the future, space-like, or time-like in the past. As shown by Yukawa,[3] this S-matrix has the following properties: (i) it is relativistically invariant; (ii) it is unitary; and (iii) it is diagonal with respect to total energy and momentum.

The question of the convergence of the S-matrix will be considered in Sec. II. No general investigation will be attempted, but a simple example will be considered and it will be shown that for a suitable choice of L' the S-matrix is convergent through the second order of interaction. In Sec. III, the limiting case in which L' is made up of local field quantities will be discussed, and it will be shown that the present S-matrix does not yield the same results as the usual formulation of field theory. This difference can be eliminated in second order by introducing a certain limiting process to be applied to the operator D_+. Unfortunately, however, if this same limiting process is applied to the general case in which L' is composed of non-local fields, the S-matrix will no longer be convergent. Because of this difficulty the S-matrix (3) is not completely satisfactory, but, of course, it may still be possible to modify it in some way so that it will satisfy all the desired conditions.

Yukawa has discussed the problem of decomposing arbitrary non-local fields into irreducible parts, each of which is characterized by four constants referring to the mass, internal radius, internal angular momentum, and another quantity which has no immediate physical meaning.[3] These irreducible parts are eigenfunctions of an invariant internal angular momentum operator, and

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[1] H. Yukawa, Phys. Rev. **76**, 300 (1949); **76**, 1731 (1949); **77**, 219 (1950).
[2] H. Yukawa, Phys. Rev. **77**, 849 (1950).
[3] H. Yukawa, Phys. Rev., preceding paper.

the corresponding eigenvalues of this operator may be taken to be $\hbar^2 l(l+1)$. This internal angular momentum operator, along with operators for the components of internal angular momentum, will be derived and discussed in Sec. IV.

II. SELF-ENERGY CALCULATION

It is of course impossible to make a general investigation of the convergence of the S-matrix, so that we shall content ourselves with the consideration of a particularly simple example. Although the term "self-energy" may not have a meaning in the same sense here as in local field theory, we shall carry out the calculation which in the limit of local field theory would be called a self-energy calculation. For this purpose we introduce two non-local fields, U and V; U is a real field and V is a complex field. We assume the interaction between them to be given by

$$L' = gV^*UV, \tag{5}$$

where V^* is the Hermitian conjugate of V. Using the notation

$$X = \tfrac{1}{2}(x'+x''), \quad r = (x'-x''), \tag{6}$$

and

$$ab = a_\mu b^\mu = a_1b^1 + a_2b^2 + a_3b^3 + a_4b^4, \quad a^2 = a_\mu a^\mu, \tag{7}$$

the matrix form of these fields is

$$(x'|U|x'') = (\hbar/(2\pi)^3c)^{\frac{1}{2}}(\kappa_u/2\pi\lambda_u) \times \int\cdots\int \delta(q^2+\kappa_u^2)\delta(r^2-\lambda_u^2) \times \delta(rq)u(q)\exp(iqX)(dq)^4, \tag{8}$$

with

$$u^*(q) = u(-q), \tag{9}$$

and

$$\begin{aligned}(x'|V|x'') &= [\hbar/(2\pi)^3c]^{\frac{1}{2}}(\kappa_v/2\pi\lambda_v) \times \int\cdots\int \delta(k^2+\kappa_v^2)\delta(r^2-\lambda_v^2) \times \delta(rk)v(k)\exp(ikX)(dk)^4,\\ (x'|V^*|x'') &= [\hbar/(2\pi)^3c]^{\frac{1}{2}}(\kappa_v/2\pi\lambda_v) \times \int\cdots\int \delta(k^2+\kappa_v^2)\delta(r^2-\lambda_v^2) \times \delta(rk)v^*(k)\exp(ikX)(dk)^4\end{aligned} \tag{10}$$

These non-local fields are less general than those introduced by Yukawa in that the expansion coefficients $u(k)$ and $v(k)$ are not functions of r; this corresponds to using just the spherical harmonic of order $l=0$ in his expansion in terms of spherical harmonics. In the rest system $(k_1=k_2=k_3=0)$, we see that the internal structure is given by $r^4=0$, $r_1^2+r_2^2+r_3^2=\lambda^2$, and that there is no dependence on angle.

By analogy with local field theory, we take the non-vanishing commutation relations to be

$$\delta(q^2+\kappa_u^2)[u^*(q), u(q')] = -(q^4/|q^4|)\delta(q-q'), \tag{11}$$

$$\delta(k^2+\kappa_v^2)[v^*(k), v(k')] = -(k^4/|k^4|)\delta(k-k'), \tag{12}$$

where

$$[A,B] \equiv AB-BA. \tag{13}$$

These commutation relations lead to the following definitions for occupation, creation, and annihilation operators:

U-type particles: $q_0=q^4=(\mathbf{q}^2+\kappa_u^2)^{\frac{1}{2}}$ ($\mathbf{q}$ is the space vector part of q and $\mathbf{q}^2=\mathbf{q}\cdot\mathbf{q}$)
- Occupation operator: $M(\mathbf{q}) = (1/2q_0)u^*(\mathbf{q},q_0)u(\mathbf{q},q_0)$
- Creation operator: $(1/2q_0)^{\frac{1}{2}}u^*(\mathbf{q},q_0)$
- Annihilation operator: $(1/2q_0)^{\frac{1}{2}}u(\mathbf{q},q_0)$

V_+-type particles: $k_0=k^4=(\mathbf{k}^2+\kappa_v^2)^{\frac{1}{2}}$
- Occupation operator: $N^+(\mathbf{k}) = (1/2k_0)v^*(\mathbf{k},k_0)v(\mathbf{k},k_0)$
- Creation operator: $(1/2k_0)^{\frac{1}{2}}v^*(\mathbf{k},k_0)$
- Annihilation operator: $(1/2k_0)^{\frac{1}{2}}v(\mathbf{k},k_0)$

V_--type particles: $k_0=k^4=-(\mathbf{k}^2+\kappa_v^2)^{\frac{1}{2}}$
- Occupation operator: $N^-(\mathbf{k}) = (1/2k_0)v(\mathbf{k},k_0)v^*(\mathbf{k},k_0)$
- Creation operator: $(1/2k_0)^{\frac{1}{2}}v(\mathbf{k},k_0)$
- Annihilation operator: $(1/2k_0)^{\frac{1}{2}}v^*(\mathbf{k},k_0)$

In addition we have the following vacuum expectation values which can be derived from the commutation rules and the creation and annihilation properties of the operators:

$$\delta(q^2+\kappa_u^2)\langle u(q)u^*(q')\rangle_0 = \delta(q-q'), \quad q_0>0; \tag{14}$$

$$\delta(k^2+\kappa_v^2)\langle v(k)v^*(k')\rangle_0 = \delta(k-k'), \quad k_0>0; \tag{15}$$

$$\delta(k^2+\kappa_v^2)\langle v^*(k)v(k')\rangle_0 = \delta(k-k'), \quad k_0<0. \tag{16}$$

We shall consider the second-order contribution of Eq. (5) to the S-matrix. Higher orders of interaction will not be discussed because of the difficulty of calculation and because the results of Sec. III indicate that Eq. (3) is probably not correct in higher orders. The second-order contribution can be written as

$$S^{(2)} = (ig/\hbar c)^2\{V^*UVD_+V^*UV\}. \tag{17}$$

For simplicity, we assume that

$$\{V^*UV\} = 0. \tag{18}$$

This is always valid for $\kappa_u < 2\kappa_v$, since then it is never possible to satisfy the energy-momentum conservation laws derived from Eq. (18). For the same reason it is also valid (regardless of the ratio κ_u/κ_v) when the initial or final state consists of a single V-type particle. Assuming Eq. (18), we may simplify Eq. (17) by replacing D_+ by $\frac{1}{2}D$, where D is defined by

$$D(x) = 1, 0, -1 \tag{19}$$

as x is time-like in the future, space-like, or time-like in the past. This replacement does not alter $S^{(2)}$ because the omitted part can be written as

$$\tfrac{1}{2}(ig/\hbar c)^2\{V^*UVEV^*UV\} = \tfrac{1}{2}(ig/\hbar c)^2\{V^*UV\}^2 = 0. \tag{20}$$

$S^{(2)}$ can now be written out in the following form:

$$S^{(2)}=\tfrac{1}{2}(ig/\hbar c)^2\int\cdots\int\{(x'+r_1+r_2|V^*|x'+r_2)$$
$$\times(x'+r_2|U|x')(x'|V|x'-r_3)$$
$$\times(x'-r_3|D|x''+r_4)(x''+r_4|V^*|x'')$$
$$\times(x''|U|x''-r_5)(x''-r_5|V|x''-r_5-r_6)\}$$
$$\times(dx')^4(dx'')^4\prod_{i=1}^{6}(dr_i)^4. \quad (21)$$

In Eq. (21) the variables of integration have been chosen to facilitate integration over the internal coordinates r; the remaining integrations over x then correspond to the usual local field theory except that they contain certain convergence factors introduced by the integration over the internal structure. To carry out the integrations we use the Fourier integral representations (8) and (10). In addition we need the Fourier expansion of $D(x)$ which is shown in the Appendix to be

$$D(x)=(-i/\pi^2)P\int\cdots\int\delta'(p^2)(p^4/|p^4|)\times\exp(ipx)(dp)^4, \quad (22)$$

where δ' is the derivative of the δ-function with respect to its argument and P represents a principal value which must be taken when performing the integral. In carrying out the integrations over r, we find the following simple integrals:

For r_1 and r_6,

$$(\kappa_v/2\pi\lambda_v)\int\cdots\int\delta(r^2-\lambda_v^2)\delta(rk)(dr)^4=1; \quad (23)$$

For r_2 and r_5,

$$F(kq)=(\kappa_u/2\pi\lambda_u)\int\cdots\int\delta(r^2-\lambda_u^2)\delta(rq)\exp(ikr)(dr)^4$$
$$=(\sin\lambda_u Q)/\lambda_u Q, \quad (24)$$

where

$$\begin{aligned}Q&=[k^2-(kq)^2/q^2]^{\frac{1}{2}}\\&=[(kq/\kappa_u)^2-\kappa_v^2]^{\frac{1}{2}};\end{aligned} \quad (25)$$

For r_3 and r_4,

$$G(k'p)=(\kappa_v/2\pi\lambda_v)\int\cdots\int\delta(r^2-\lambda_v^2)\delta(rk')\times\exp(ipr)(dr)^4$$
$$=(\sin\lambda_v K)/\lambda_v K, \quad (26)$$

where

$$\begin{aligned}K&=[p^2-(k'p)^2/(k')^2]^{\frac{1}{2}}\\&=k'p/\kappa_v.\end{aligned} \quad (27)$$

These integrals are invariant functions of their arguments and are most easily evaluated by using special Lorentz frames to simplify the calculation; for example, Eq. (24) would be evaluated by using the reference system in which $\mathbf{q}=0$, $q^4=\pm\kappa_u$. In Eq. (27) we have set p^2 equal to zero because of the factor $\delta'(p^2)$ occurring in the remainder of the integrand; this is not strictly correct, but should introduce little error into our later semi-quantitative arguments.[4]

The expressions (24) and (26) are responsible for making $S^{(2)}$ convergent, and accordingly they will be called convergence factors. They are of the form $x^{-1}\sin x$, a function whose properties are well known. To see the nature of these convergence factors, let us suppose in Eqs. (24) and (25) that $\mathbf{q}=0$, $q^4=\pm\kappa_u$; then $Q=|\mathbf{k}|$ and $F=(\lambda_u|\mathbf{k}|)^{-1}\sin(\lambda_u|\mathbf{k}|)$. If $\lambda_u|\mathbf{k}|$ is small compared to unity, F will be close to unity; accordingly, if λ_u is small enough, the factor F will affect the S-matrix only for a very large momentum $|\mathbf{k}|$. For real processes, the effect of the factor F would be detected only at high energies; for virtual processes, F becomes a cut-off factor which helps produce convergence in the integrations over the intermediate momenta. For later work, it will be useful to express Q and K in different forms, so that at least one of the momenta in each expression represents the momentum of a real particle. This is possible because the momenta are always connected by an expression of the form

$$k-q-k'-p=0,$$

from which we can derive

$$\begin{aligned}kq&=-k'p-\tfrac{1}{2}\kappa_u^2-\tfrac{1}{2}p^2\\&=-k'p-\tfrac{1}{2}\kappa_u^2,\end{aligned} \quad (28)$$

where we again have set p^2 equal to zero.[4] Thus Q and K can be expressed in terms of kq or $k'p$ and some constants; at least one of the momenta k, q, or k' is the momentum of a real particle.

Making use of these results, $S^{(2)}$ becomes

$$S^{(2)}=(i\hbar g/4\pi^3c^5)\int\cdots\int\prod_{i=1}^{4}(dk_i)^4\delta(k_i^2+\kappa_v^2)$$
$$\times\prod_{j=1}^{2}(dq_j)^4\delta(q_j^2+\kappa_u^2)(dp)^4\{(p^4/|p^4|)\delta'(p^2)$$
$$\times\delta(k_1-q_1-k_2-p)\delta(k_3+q_2-k_4+p)$$
$$\times v^*(k_1)v(k_2)v^*(k_3)v(k_4)u(q_1)u^*(q_2)$$
$$\times F(k_1q_1)F(k_4q_2)G(k_2p)G(k_3p)\}. \quad (29)$$

Except for the convergence factors F and G and the use of the Fourier transform of $D(x)$, this expression is much like the one which would be obtained from local field theory. There are three different self-energy terms in Eq. (29), one for each type of particle. Owing to the way in which we have chosen the interaction, the self-energy of the V_+-particle differs from that of the V_--particle; in fact, as we shall see, the self-energy of

[4] This implies neglecting the derivatives of F and G with respect to p, which is reasonable since these functions are slowly varying compared with the other functions in the integrand of Eq. (21).

the V_+-particle is finite, while that of the V_--particle is infinite. This arises from the particular choice of L' given by Eq. (5). By splitting the V field into positive and negative frequency parts, and using a slightly different interaction, it is possible to eliminate this asymmetry between V_+- and V_--particles (see below, Eq. (36)).

We shall consider the self-energy calculation for the V_+-particle in detail, and indicate briefly the results for the other two types of particles. There are two possible contributions to this self-energy which are given by[5]

$$v^*(k_1)v(k_4)\langle v(k_2)v^*(k_3)\rangle_0\langle u(q_1)u^*(q_2)\rangle_0, \tag{30}$$

$$v(k_2)v^*(k_3)\langle v^*(k_1)v(k_4)\rangle_0\langle u(q_1)u^*(q_2)\rangle_0. \tag{31}$$

As a matter of fact, the contribution from Eq. (31) vanishes because it is impossible to satisfy simultaneously the relations

$$\begin{gathered}(k_3+q_2-k_4)^2=p^2=0,\\ (q_2)_0>0,\quad (k_3)_0>0,\quad (k_4)_0<0.\end{gathered} \tag{32}$$

This is connected with the fact, mentioned by Yukawa in the preceding paper,[3] that with this S-matrix formalism there is no self-energy of the vacuum.

Using Eq. (29), we find for the self-energy of V_+-particles

$$\begin{aligned}&(\cdots, N_{k_1}{}^+=1, \cdots|S^{(2)}|\cdots, N_{k_4}{}^+=1, \cdots)\\ &=(i\hbar g^2/4\pi^3c^5)\delta(k_1-k_4)\delta(k_1{}^2+\kappa_v{}^2)\delta(k_4{}^2+\kappa_v{}^2)I_1(k_4),\end{aligned} \tag{33}$$

where

$$\begin{aligned}I_1(k_4)=\int_{k_0>0,\ q_0>0}\cdots\int &(dk)^4(dq)^4(dp)^4\\ &\times\{\delta(k+q-k_4+p)\delta(k^2+\kappa_v{}^2)\delta(q^2+\kappa_u{}^2)\\ &\quad\times(p^4/|p^4|)\delta'(p^2)[F(k_4q)G(kp)]^2\}.\end{aligned} \tag{34}$$

We note that $I_1(k_4)$ is an invariant function of k_4 and can therefore depend only on $k_4{}^2=-\kappa_v{}^2$, accordingly we simplify further considerations by setting $\mathbf{k}_4=0$, $(k_4)_0=\kappa_v$. We also use Eq. (28) to express G as a function of k_4q. The integrals over k and p in Eq. (34) are then easily performed since they are independent of the convergence factors; the details of this calculation will be found in the Appendix. The resulting expression for I_1 is

$$\begin{aligned}I_1(k_4)=\pi\int_{q_0>0}\cdots\int &\delta(q^2+\kappa_u{}^2)A(q_0)\\ &\times\{[1/(l^2+\kappa_v{}^2)]-(1/2l^2)\}\\ &\times[F(\kappa_vq_0)G(k_vq_0-\tfrac{1}{2}\kappa_u{}^2)]^2(dq)^4,\end{aligned} \tag{35}$$

where

$$A(q_0)=\begin{cases}\tfrac{1}{2}\text{ for } q_0>(\kappa_u{}^2+\kappa_v{}^2)/2\kappa_v\\ 1\text{ for } q_0<(\kappa_u{}^2+\kappa_v{}^2)/2\kappa_v,\text{ if } \kappa_u<\kappa_v\\ 0\text{ for } q_0<(\kappa_u{}^2+\kappa_v{}^2)/2\kappa_v,\text{ if } \kappa_u>\kappa_v\end{cases}$$

and

$$l^2=-\kappa_u{}^2-\kappa_v{}^2+2\kappa_vq_0.$$

The integrand of Eq. (35) has a singularity at just the value of q_0 at which $A(q_0)$ has a discontinuity; consequently, the integral is not convergent. This divergence of the integral will be ignored at present, however, because the discussion of Sec. III and the Appendix will show that it is a consequence of the structure of the S-matrix rather than of the particular properties of the fields. A suitable modification of the S-matrix should eliminate divergences occurring at finite intermediate momenta. We are more interested here in the convergence of Eq. (35) at the upper limit of integration. At this limit, the combined convergence factors produce a factor of the order of $|\mathbf{q}|^{-4}$, so convergence of Eq. (35) is secured; we shall accordingly consider Eq. (35) to be convergent.

Having seen that the self-energy of the V_+-particle is finite, we consider briefly the other two types of particles. The self-energy of the V_--particle involves an integral of the form (34) except that the integration is over k_4, while k is a fixed vector. Making use of Eq. (28) then, the convergence factors are found to depend only on kp. Since the integration over p is convergent without the convergence factors, we see that the value of the integral is essentially independent of these factors, and the integral will diverge. The self-energy in this case diverges linearly; this is in contrast to the ordinary field theory, where the divergence is only logarithmic. This difference arises because in the present calculation there is a contribution from Eq. (31) but not from Eq. (30). In ordinary field theory both of these terms contribute and partially cancel in such a way that the order of the divergence is reduced. The self-energy of the U-particle is found to be finite because the convergence factor for that calculation can be written as $[F(k_4q)\times G(-k_4q-\tfrac{1}{2}\kappa_u{}^2)]^2$. Since q is fixed, integration over k_4 will always be convergent.

The asymmetry between the V_+-particle and the V_--particle can be removed by revising the interaction operator L'.[6] We separate the field V into two parts: V_+ and V_-. V_+ corresponds to $k_0>0$, and V_- to $k_0<0$. We may then take L' to be

$$\begin{aligned}L'=g\{&V_+{}^*UV_++V_-UV_-{}^*\\ &+V_+{}^*UV_-+V_-{}^*UV_+\},\end{aligned} \tag{36}$$

or we may symmetrize the last two terms:

$$\begin{aligned}L'=g\{&V_+{}^*UV_++V_-UV_-{}^*+\tfrac{1}{2}(V_+{}^*UV_-\\ &+V_-UV_+{}^*+V_-{}^*UV_++V_+UV_-{}^*)\}.\end{aligned} \tag{36'}$$

Using either one of these interactions, we have the two

[5] Operators referring to different momenta may be freely commuted.

[6] The author is indebted to Prof. Yukawa for suggesting the following scheme.

self-energies given by

$\frac{1}{2}(ig/\hbar c)^2\{V_+{}^*UV_+DV_+{}^*UV_+\}$ for the V_+-particle,
$\frac{1}{2}(ig/\hbar c)^2\{V_-UV_-{}^*DV_-UV_-{}^*\}$ for the V_--particle.

Because of the creation and annihilation properties given previously, it is seen that these two expressions are symmetrical in the two types of particles and both are convergent (provided that we again ignore the divergence occurring at finite intermediate momenta). This procedure seems to be quite arbitrary at present, but it may turn out that this, or something similar, will be necessary in the future development of the non-local field theory.

The calculations of this section seem to indicate that when a satisfactory method of introducing interactions is found, the non-local field theory will be convergent. The present S-matrix seems to be unsatisfactory for various reasons, but the non-local character of the fields introduces convergence factors which eliminate the usual divergences of local field theory. These convergence factors should be carried over into the more correct formulation when it is found.

III. THE LIMIT OF LOCAL FIELD THEORY

If we let λ_u and λ_v approach zero in Eqs. (8) and (10), we find that U and V become local fields $U(x)$ and $V(x)$ with the following non-vanishing commutation relations:

$$[V^*(x'), V(x'')]=(\hbar/ic)\Delta_v(x'-x''), \tag{37}$$

$$[U(x'), U(x'')]=(\hbar/ic)\Delta_u(x'-x''), \tag{38}$$

where Δ_v and Δ_u are the usual invariant functions of local field theory and are given by

$$\Delta=[-1/(2\pi)^3 i]\int\cdots\int\delta(k^2+\kappa^2)(k_0/|k_0|)\times\exp(ikx)(dk)^4. \tag{39}$$

We have now to compare the results of calculations using the S-matrix given by Eq. (3) with the results from the usual field theory. In the usual theory, the S-matrix takes the form[7]

$$S'=1+(i/\hbar c)\{L'\}+(i/\hbar c)^2\{L'\epsilon_+L'\}+(i/\hbar c)^3\{L'\epsilon_+L'\epsilon_+L'\}+\cdots, \tag{40}$$

where ϵ_+ is a non-invariant operator given by[8]

$$(x'|\epsilon_+|x'')=\epsilon_+(x'-x'')=\begin{cases}1 & \text{for } x'^4>x''^4\\ 0 & \text{for } x'^4<x''^4\end{cases}. \tag{41}$$

The relativistic invariance of Eq. (40) is a result of the condition of integrability

$$[L'(x'), L'(x'')]=0 \text{ for } (x'-x'') \text{ space-like}. \tag{42}$$

[7] F. J. Dyson, Phys. Rev. **75**, 486 (1949).

[8] More generally, ϵ_+ is based on the ordering in time of space-like surfaces on which x' and x'' are situated (reference 7), but we may specialize to constant time surfaces for the present application.

We thus see that the relativistic invariance of Eq. (3) has been obtained by purely kinematical means while invariance of Eq. (40) depends on the dynamical properties of the field and the form of the interaction.

The second-order contributions to the S-matrix in the two cases are found to be[9]

$$S^{(2)}=\tfrac{1}{4}(i/\hbar c)^2\int\cdots\int D(x'-x'')\times[L'(x'), L'(x'')](dx')^4(dx'')^4, \tag{43}$$

$$S^{(2)}=\tfrac{1}{4}(i/\hbar c)^2\int\cdots\int \epsilon(x'-x'')\times[L'(x'), L'(x'')](dx')^4(dx'')^4, \tag{44}$$

where

$$\epsilon(x'-x'')=\begin{cases}1 & \text{for } x'^4>x''^4\\ -1 & \text{for } x'^4<x''^4\end{cases}. \tag{45}$$

These two expressions appear to be equivalent for the following reason: the integrands are equivalent for $(x'-x'')$ time-like, and they both vanish for $(x'-x'')$ space-like. This equivalence is only apparent, however, for the commutator $[L'(x'), L'(x'')]$ has a singularity of the δ-function type when $(x'-x'')$ lies on the light cone. (For spinor fields the singularity would be even stronger.) In Eq. (43), $D(x'-x'')$ changes discontinuously as $(x'-x'')$ crosses the light cone; the δ-function from the commutator averages out this discontinuity in D so that the contribution from the light cone is only half as great as in Eq. (44), where ϵ is continuous across the light cone. The effect of this will be seen in more detail with the aid of an example.

Using the previous interaction (5) we have

$$\begin{aligned}[L'(x'), L'(x'')] &= g^2\{V^*(x')V(x')V^*(x'')V(x'')[U(x'), U(x'')]\\ &+U(x'')U(x')V^*(x')V(x'')[V(x'), V^*(x'')]\\ &+U(x'')U(x')V^*(x'')V(x')[V^*(x'), V(x'')]\}.\end{aligned} \tag{46}$$

To compare Eqs. (43) and (44), we may use for example the first term of Eq. (46); this amounts to comparing

$$A(x)=\tfrac{1}{2}D(x)\Delta_u(x) \tag{47}$$

with

$$A'(x)=\tfrac{1}{2}\epsilon(x)\Delta_u(x). \tag{48}$$

Equation (48) is the function $\bar{\Delta}_u$ defined by Schwinger[10] and is known to have the Fourier expansion

$$A'(x)=\bar{\Delta}_u(x)=(1/(2\pi)^4)P\int\cdots\int\exp(ikx)(dk)^4/(k^2+\kappa_u^2). \tag{49}$$

We shall compare the two expressions (47) and (48) by finding the Fourier transform of Eq. (47). This is most easily done by means of a theorem of Fourier trans-

[9] A similar result has been given by F. Koba, Prog. Theor. Phys. **5**, 139 (1950).

[10] J. Schwinger, Phys. Rev. **75**, 651 (1949).

forms,[11] which gives the Fourier transform of $A(x)$ in terms of the transforms of $D(x)$ and $\Delta_u(x)$:

$$A(x)=\int\cdots\int f(k)\exp(ikx)(dk)^4, \tag{50}$$

where

$$f(k)=(1/16\pi^5)P\int\cdots\int\delta[(k-p)^2+\kappa_u{}^2]\times[(k^4-p^4)/|k^4-p^4|]\delta'(p^2)(p^4/|p^4|)(dp)^4. \tag{51}$$

This is the integral J_2 which is calculated in the Appendix; the result for $f(k)$ is, therefore,

$$f(k)=[1/(2\pi)^4]\{[1/(k^2+\kappa_u{}^2)]-(1/2k^2)\}. \tag{52}$$

In using this result in further integrations, it is implied that a principal value is to be taken whenever this part of the integrand becomes singular.[11] The first term of Eq. (52) obviously agrees with Eq. (49), while the second term is a consequence of the combination of the δ-function with a discontinuity on the light cone.

The result (52) shows that we must modify the S-matrix (3) if that expression is to reduce properly in the limit of local field theory. This can be accomplished by a modification of the D function such that the discontinuity occurs outside the light cone. Since D is an odd function of x^4, this cannot be done in an invariant manner; but we can do the calculations with the modified D function and then let the discontinuity approach the light cone. The result will be that the last term of Eq. (52) will disappear, thus bringing about agreement of the two S-matrices in second order. The calculations are given in the Appendix.

We must now re-examine the self-energy calculations in terms of the limiting process. We now find that both Eq. (30) and Eq. (31) contribute to the self-energy; in terms of the interaction (36), the additional terms to be added to the self-energy are

$$\tfrac{1}{2}(ig/\hbar c)^2\{V_-{}^*UV_+DV_+{}^*UV_-\}$$
for the V_+- or the V_--particle.

In the self-energy calculation of Sec. II, there was a divergence due to a singularity in the integrand at a finite value of the intermediate momentum. The addition of the second type of self-energy process and the use of the limiting process on D serves to eliminate this difficulty; the calculations are discussed in the Appendix. For the V_+-particle, the operator V_+ corresponds to a fixed momentum; from the considerations of Sec. II, it is therefore apparent that this contribution to the self-energy is not convergent. For the V_--particle, the operator V_- corresponds to a fixed momentum, and this contribution to the self-energy is convergent. Using the interaction (36′) makes the self-energy of both types of particles divergent. The self-energies calculated with the S-matrix (3) and with the modified D_+ function are accordingly divergent. For this reason, the S-matrix (3) will have to be modified or replaced by another method of introducing interactions; but when this has been done, it seems likely that a convergent theory can be obtained with the use of non-local fields.

In higher orders it becomes more difficult to make a comparison between the two S-matrices. We can note, however, some further difficulties with Eq. (3); for example, the third-order term is given by

$$S^{(3)}=(i/\hbar c)^3\int\cdots\int L'(x')D_+(x'-x'')L'(x'')\times D_+(x''-x''')L'(x''')(dx')^4(dx'')^4(dx''')^4. \tag{53}$$

The presence of the D_+ functions insures that $(x'-x'')$ and $(x''-x''')$ must be space-like, or time-like in the future. These functions do not directly impose conditions on the character of $(x'-x''')$, however; in fact, it is even possible for this vector to be time-like in the past. No such situation occurs in the case of Eq. (40), where all of the interactions have a definite ordering in time.

The conclusion of this section would seem to be that the proposed method of introducing interactions in non-local field theory is not correct in its present form, but that an approach along this line is not yet ruled out. The essential difficulty is that the commutation relations for two non-local fields depend not only on their properties as creation and annihilation operators, but also on their matrix character with respect to space-time coordinates. It may be that the matrix character of non-local fields has been overemphasized; that is, it may perhaps be better to consider them simply as fields depending on a center-of-mass coordinate (X) and an internal structure coordinate (r).

IV. INTERNAL ANGULAR MOMENTUM OPERATOR

In the preceding paper[3] Yukawa has discussed the process of decomposing general non-local fields into irreducible parts. The reduced fields $U(X,r)$ were eigenfunctions of the operators

$$\alpha=\partial^2/\partial X_\mu\partial X^\mu, \tag{54}$$

$$\beta=r_\mu r^\mu, \tag{55}$$

$$\gamma=r_\mu\partial/\partial X_\mu, \tag{56}$$

with eigenvalues $-K$, L, and iM, respectively. For correspondence to particles with real mass, $-K$ must be positive, but L and M are unrestricted in value (except that they must be real). In addition, the field may be further decomposed into separate parts characterized by the internal angular momentum. It will be the purpose of this section to produce an operator for the

[11] This theorem is briefly:
Given $G(x)=\int g(k)\exp(ikx)dk$; $H(x)=\int h(k)\exp(ikx)dk$, then $G(x)H(x)=\int f(x)\exp(ikx)dk$, where $f(k)=\int g(k-l)h(l)dl$. When this is applied to Eq. (50), it is necessary to use care in handling the limiting processes implied by the principal value in the definition of D. These limiting processes are to be left as the last step of the successive integrations in Eq. (50); the result of this is that we have an ordinary principal value implied for the singularities which occur in $f(k)$.

internal angular momentum. This operator should satisfy the following conditions: (i) it should be an invariant operator; (ii) it should commute with the previous three operators. Except for a multiplicative factor and an additive constant depending only on α, β, and γ, the desired operator is determined by these two requirements. The calculation is a little lengthy, but it is completely straightforward. We shall merely sketch in the procedure here.

Using the operator $(\partial/\partial r_\mu)$ along with r_μ and $(\partial/\partial X_\mu)$, we can form three new invariant operators.

$$\partial^2/\partial r_\mu\partial r^\mu, \quad r_\mu\partial/\partial r_\mu, \quad \partial^2/\partial r_\mu\partial X^\mu. \tag{57}$$

No linear combination of the elements (57) commutes with α, β, and γ; we therefore form from elements (57) an operator δ, which is the most general linear differential operator of second order in r, with arbitrary coefficients depending only on α, β, and γ.

$$\begin{aligned}\delta\alpha = (\partial^2/\partial r_\mu\partial r^\mu)a + (r_\mu\partial/\partial r_\mu)^2 b + (\partial^2/\partial r_\mu\partial X^\mu)c \\ + (r_\mu\partial^3/\partial r_\mu\partial r_\lambda\partial X^\lambda)d \\ + (r_\mu\partial/\partial r_\mu)e + (\partial^2/\partial r_\mu\partial X^\mu)f + g.\end{aligned} \tag{58}$$

Equation (58) satisfies the condition of invariance (i). The commutation conditions (ii) are

$$[\delta,\alpha]=0, \quad [\delta,\beta]=0, \quad [\delta,\gamma]=0. \tag{59}$$

The relations (59) determine the first six coefficients of Eq. (58) except for a common factor; at this stage, the operator takes the form

$$\begin{aligned}\delta\alpha = \{(\partial^2/\partial r_\mu\partial r^\mu)(\alpha\beta-\gamma^2) - (r_\mu\partial/\partial r_\mu)^2\alpha \\ - (\partial^2/\partial r_\mu\partial X^\mu)^2\beta - 2r_\mu(\partial^3/\partial r_\mu\partial r_\lambda\partial X^\lambda)\gamma \\ - 7r_\mu(\partial/\partial r_\mu)\alpha + 8(\partial^2/\partial r_\mu\partial X^\mu)\gamma + g'\alpha\}a'.\end{aligned} \tag{60}$$

If $-K=\kappa^2$; the rest system is characterized by

$$\partial/\partial X_\mu = i\kappa\delta_4{}^\mu. \tag{61}$$

Using Eq. (61), we have the following reductions in the rest system

$$(\beta\alpha-\gamma^2) = \sum_{i=1}^{3} r_i{}^2\alpha, \quad (\partial^2/\partial r_\mu\partial X^\mu)^2 = (\partial^2/\partial r_4\partial r^4)\alpha, \\ (\partial^2/\partial r_\mu\partial X^\mu)\gamma = (\partial/\partial r_4)r_4\alpha, \tag{62}$$

so that δ reduces to

$$\delta = \left\{\frac{1}{2}\sum_{i,j=1}^{3}\left(r_i\frac{\partial}{\partial r_j} - r_j\frac{\partial}{\partial r_i}\right)^2 + 12 + g'\right\}a'. \tag{63}$$

This will reduce to the usual form for an angular momentum operator if we choose the arbitrary constants to be

$$g'=-12, \quad a'=-\hbar^2. \tag{64}$$

This result can be put into simpler and more compact form by introducing a new operator by

$$\alpha r_\mu' = \alpha r_\mu - \gamma\partial/\partial X^\mu. \tag{65}$$

The meaning of this operator is somewhat clearer if we note that in the rest system

$$\alpha r_\mu' = \begin{cases}\alpha r_\mu & \text{if } \mu=1,2,3 \\ 0 & \text{if } \mu=4\end{cases}, \tag{66}$$

so that Eq. (65) represents, in a sense, a translation of coordinates. Using Eq. (65), the internal angular momentum operator becomes

$$\begin{aligned}\alpha^2\delta = -\tfrac{1}{2}\hbar^2[(r_\mu'\partial/\partial r_\nu) - (r'^\nu\partial/\partial r^\mu)] \\ \times[(r'^\mu\partial/\partial r^\nu) - (r_\nu'\partial/\partial r_\mu)]\alpha^2 \\ - \hbar^2(\partial^2/\partial r_\mu\partial X^\mu)^2(\gamma^2-\alpha\beta).\end{aligned} \tag{67}$$

As is well known, the eigenfunctions of δ in the rest system are the spherical harmonics and the corresponding eigenvalues are $\hbar^2 l(l+1)$ (where $l=1,2,3,\cdots$). Thus α, β, and γ have continuous eigenvalues and δ has discrete eigenvalues. These four operators allow us to decompose completely a general non-local field into irreducible parts because an expansion in terms of simultaneous eigenfunctions of these operators is unique and no further decomposition is possible. Each irreducible part is characterized by the four quantities κ^2, L, M, and l. This justifies the procedure of Sec. II, in which the non-local fields had $L=\lambda^2$, $M=0$, and internal angular momentum given by $l=0$.

We have yet to show that there are no other independent invariant operators which are functions of elements (57) and which commute with α, β, and γ. We shall prove this for operators which are finite polynomials of the operators (57). The most general operator of this form may be written as

$$d = \sum_{n=0}^{N}\sum_{m=0}^{M}(r_\mu\partial/\partial r_\mu)^n(\partial^2/\partial r_\lambda\partial X^\lambda)^m a_{nm}, \tag{68}$$

where the a_{nm} are functions of α, β, γ, and δ; the operator $(\partial^2/\partial r_\mu\partial r^\mu)$ does not appear explicitly because it can be expressed in terms of $(r_\mu\partial/\partial r_\mu)$, $(\partial^2/\partial r_\mu\partial X^\mu)$, and δ. Commuting Eq. (68) with β and γ, we find terms of the form

$$\begin{aligned}[(r_\mu\partial/\partial r_\mu)^n(\partial^2/\partial r_\lambda\partial X^\lambda)^m, \beta] \\ = 2n(r_\mu\partial/\partial r_\mu)^{n-1}(\partial^2/\partial r_\lambda\partial X^\lambda)^m\beta \\ + 2m(r_\mu\partial/\partial r_\mu)^n(\partial^2/\partial r_\lambda\partial X^\lambda)^{m-1}\gamma + \cdots,\end{aligned} \tag{69a}$$

$$\begin{aligned}[(r_\mu\partial/\partial r_\mu)^n(\partial^2/\partial r_\lambda\partial X^\lambda)^m, \gamma] \\ = n(r_\mu\partial/\partial r_\mu)^{n-1}(\partial^2/\partial r_\lambda\partial X^\lambda)^m\gamma \\ + m(r_\mu\partial/\partial r_\mu)^n(\partial^2/\partial r_\lambda\partial X^\lambda)^{m-1}\alpha + \cdots.\end{aligned} \tag{69b}$$

The dots indicate terms in lower powers of the operators $(r_\mu\partial/\partial r_\mu)$ and $(\partial^2/\partial r_\mu\partial X^\mu)$. Applying Eq. (69) to d and investigating the highest order terms, we find that it is necessary to take $N=M=0$; thus d is a function of α, β, γ, and δ.

We can easily write a tensor operator $J_\mu{}^\nu$ whose components may be considered to be the components of the internal angular momentum; this operator is

$$\alpha J_\mu{}^\nu=(\hbar/i)[(r_\mu'\partial/\partial r_\nu)-(r'^\nu\partial/\partial r^\mu)]\alpha$$
$$-(\hbar/i)[(r_\mu'\partial/\partial X_\nu)-(r'^\nu\partial/\partial X^\mu)]\partial^2/\partial r_\lambda\partial X^\lambda. \quad (70)$$

In the rest system, this reduces to

$$J_\mu{}^\nu=\begin{cases}0 & \text{if } \mu \text{ or } \nu=4\\ (\hbar/i)[(r_\mu\partial/\partial r_\nu)-(r^\nu\partial/\partial r_\mu)] & \text{if } \mu,\ \nu=1,\,2,\,3.\end{cases} \quad (71)$$

The total internal angular momentum is easily seen to be given by

$$\delta=\tfrac{1}{2}J_\mu{}^\nu J_\nu{}^\mu. \quad (72)$$

The components of the internal angular momentum also commute with α, β, and γ:

$$[J_\mu{}^\nu,\alpha]=[J_\mu{}^\nu,\beta]=[J_\mu{}^\nu,\gamma]=0. \quad (73)$$

We also have the following commutation relations between the components of the internal angular momentum and the total internal angular momentum:

$$\alpha[J_\mu{}^\nu,J_\sigma{}^\lambda]=(\hbar/i)\{J_\mu{}^\lambda[g_\sigma{}^\nu\alpha-(\partial^2/\partial X_\nu\partial X^\sigma)]$$
$$-J_\sigma{}^\nu[g_\mu{}^\lambda\alpha-(\partial^2/\partial X_\lambda\partial X^\mu)]$$
$$-g_{\sigma\alpha}J_\mu{}^\alpha[g^{\nu\lambda}\alpha-(\partial^2/\partial X_\lambda\partial X_\nu)]$$
$$+g^{\lambda\alpha}J_\alpha{}^\nu[g_{\sigma\mu}\alpha-(\partial^2/\partial X^\sigma\partial X^\mu)]\}, \quad (74)$$

$$[J_\mu{}^\nu,\delta]=0, \quad (75)$$

where the $g_{\nu\lambda}$ are the components of the metrical tensor:

$$g_{\nu\lambda}=g^{\nu\lambda}=\begin{cases}0 & \nu\neq\lambda\\ +1 & \nu=\lambda=1,\,2,\,3\\ -1 & \nu=\lambda=4\end{cases} \quad (76)$$
$$g_\mu{}^\nu=g_{\mu\lambda}g^{\nu\lambda}.$$

For convenience, we note the following relations which are useful in proving Eqs. (72)–(75):

$$\alpha\partial^2 r_\mu'/\partial X^\lambda\partial r_\lambda=0,\quad \alpha r_\mu' r'^\mu=\alpha\beta-\gamma^2,$$
$$r_\mu'\partial/\partial X_\mu=0,\quad J_\mu{}^\nu\partial/\partial X_\mu=0. \quad (77)$$

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APPENDIX

1. Fourier Expansion of $D(x)$

The desired expansion of D is

$$D(x)=-(i/\pi^2)P\int\cdots\int(p^4/|p^4|)\delta'(p^2)\exp(ipx)(dp)^4. \quad \text{(A-1)}$$

In Eq. (A-1), the symbol P standing before the integral denotes that a principal value must be taken when performing the integral. We define this principal value in the following manner. Delete from the region of integration a small volume about $p_\mu=0$; the volume deleted is to contain the point $p_\mu=0$ and to be invariant under the reflection $p_\mu\to-p_\mu$. As this deleted volume shrinks to zero, the integral approaches a limit which is the desired principal value. For definiteness, we assume that a similar sort of limiting process is implied for $p_\mu\to\infty$.

To show that Eq. (A-1) is correct, we note first that the quantity on the right is an invariant function of x and therefore depends only on x^2 and $x^4/|x^4|$. (Note that $(p^4/|p^4|)$ is really an invariant quantity because p must lie on the light cone, and also that the volume of integration is invariant in the limit.) In addition, owing to the factor $(p^4/|p^4|)$, the integral is an odd function of x^4. Because of these two conditions, the integral must vanish for space-like distances x. For time-like x, we specialize to that Lorentz frame in which $\mathbf{x}=0$, $x^4=a$. The integral becomes

$$(-4i/\pi)P\int_{-\infty}^{\infty}dp_0\int_0^\infty p^2dp(p_0/|p_0|)\times\exp(-ip_0a)\partial\delta(p^2-p_0^2)/2p\partial p.$$

Integration by parts with respect to p, yields

$$(2i/\pi)P\int_{-\infty}^{\infty}dp_0\int_0^\infty dp(p_0/|p_0|)\delta(p^2-p_0^2)\exp(-ip_0a).$$

The integrated part does not vanish, but is rapidly oscillating in a (before the limit is taken) and would average to zero in any application. Integrating over p makes

$$D=(i/\pi)P\int_{-\infty}^{\infty}dp_0\exp(-ip_0a)/p_0.$$

P now stands for the ordinary principal value, and we have

$$D(0,0,0,a)=\begin{cases}1 & \text{for } a>0\\ -1 & \text{for } a<0\end{cases}$$

This verifies that Eq. (A-1) is the correct Fourier expansion of $D(x)$.

2. Evaluation of Certain Integrals

Consider the integral (34), which arises in the self-energy calculation; after the integration over k is performed, the integration with respect to p becomes

$$J_1=P\underset{(l-p)^4>0}{\int\cdots\int}\delta((l-p)^2+\kappa_v^2)\delta'(p^2)(p^4/|p^4|)(dp)^4, \quad \text{(A-2)}$$

where

$$l=k_4-q.$$

This integral is an invariant function of l and is therefore most easily carried out by using special Lorentz frames to simplify the calculation. For l time-like, we choose $\mathbf{l}=0$, $l^4=l'$; J_1 then becomes

$$J_1=4\pi P\int_{-\infty}^{l'}dp_0\int_0^\infty p^2dp\delta(p^2-p_0^2-l'^2+2l'p_0+\kappa^2)\times(p_0/|p_0|)\delta'(p^2-p_0^2),$$
$$=2\pi P\int_{-\infty}^{l'}dp_0(p_0^2+l'^2-2l'p_0-\kappa_v^2)^{\frac{1}{2}}(p_0/|p_0|)\delta'(l'^2-2l'p_0-\kappa_v^2)$$
$$=(\pi/l')P\int_{-\infty}^{l'}dp_0(1-l'/p_0)\delta(2l'p_0-l'^2+\kappa_v^2)$$
$$=\begin{cases}0 & \text{if } l'<0\\ \pi\{[1/(-l'^2+\kappa_v^2)]+1/2l'^2\} & \text{if } l'>0.\end{cases}$$

For l space-like, we take $l^4=0$, and we take the vector $\mathbf{l}$ of magnitude l'' as the direction of the polar axis for a set of spherical coordinates in the three-dimensional space part of $(dp)^4$. J_1 then becomes

$$J_1=-2\pi\int_{-\infty}^{0}dp_0\int_0^\pi\sin\theta d\theta\int_0^\infty p^2dp\delta(l''^2-2l''p\cos\theta+p^2-p_0^2+\kappa_0^2)\delta'(p^2-p_0^2).$$

Integrating over p_0 and making the substitution $\cos\theta=\mu$ gives

$$J_1=-\pi\int_{-1}^{1} d\mu\int_0^\infty p^2dp\delta'(2l''p\mu-l''^2-\kappa_v^2)/(p^2+l''^2-2l''p\mu+\kappa_v^2)^{\frac{1}{2}}$$
$$=-(\pi/2l'')\int_0^\infty dp\delta(2l''p-l''^2-\kappa_v^2)+(\pi/4l'')\int_{(l''^2+\kappa_0^2/2l'')}^\infty dp/p^2$$
$$=\pi/2\{[1/(l''^2+\kappa_v^2)]-1/2l''^2\}.$$

We can express both of these results in terms of a function

$$J=\pi\{[1/(l^2+\kappa^2)]-1/2l^2\}. \tag{A-3}$$

Then

$$J_1=\begin{cases}\frac{1}{2}J_v & \text{for } l \text{ space-like}\\ 0 & \text{for } l \text{ time-like, } l^4<0\\ J_v & \text{for } l \text{ time-like, } l^4>0.\end{cases} \tag{A-4}$$

The space- or time-like character of l depends on the sign of

$$l^2=-\kappa_u^2-\kappa_v^2+2\kappa_v q_0. \tag{A-5}$$

If l is time-like, it is easy to see that

$$\begin{aligned} l^4>0 \text{ if } \kappa_v>\kappa_u,\\ l^4<0 \text{ if } \kappa_v<\kappa_u.\end{aligned} \tag{A-6}$$

These are the relations which determine the factor $A(q_0)$ occurring in Eq. (35).

The integral J_2 occurring in Eq. (51) is:

$$J_2=P\int\cdots\int\delta[(k-p)^2+\kappa_u^2]\delta'(p^2)\times[(k^4-p^4)p^4(dp)^4/|k^4-p^4||p^4|]. \tag{A-7}$$

Using the results of the J_1 integration, this integral is readily evaluated. It is an even function of k^4; and for k time-like with $k^4>0$, it is easy to see that $J_2=J_1$. Also, for k space-like, J_2 has an integrand which is an even function of p_0, and, therefore, $J_2=2J_1$. In general, then

$$J_2=J_u. \tag{A-8}$$

3. Limiting Process for D Function

The D function is to be modified so that its discontinuity occurs outside the light cone; such a function is D', defined by

$$D'(x)=-(i/\pi^2)P\int\cdots\int(p^4/|p^4|)\delta'(p^2)\exp(i\mathbf{p}\cdot\mathbf{x}-i\rho p^4x^4)(dp)^4, \tag{A-9}$$

where

$$\rho>1$$

In some Lorentz frame we may take ρ to be a constant; in this system the discontinuity occurs outside the light cone at $|\mathbf{x}|=\rho|x^4|>|x^4|$. With a change of variable Eq. (A-9) can be written as

$$D'(x)=-(i/\pi^2)P\int\cdots\int(p^4/|p^4|)\delta'(\mathbf{p}^2-\sigma p_0^2)\times\exp(ipx)(1/\rho)(dp)^4, \tag{A-10}$$

where

$$\sigma=1/\rho^2<1.$$

We shall illustrate the use of this modified D function by a calculation of the modified J' function when k is time-like. Take $\mathbf{k}=0$, $k^4=k'$; then this integral becomes

$$J'=4\pi P\int_{-\infty}^\infty dp_0\int_0^\infty p^2dp\delta(p^2-p_0^2-k'^2+2k'p_0+\kappa^2)\times[p_0(k'-p_0)/|p_0||k'-p_0|]\delta'(p^2-\sigma p_0^2)(1/\rho)$$
$$=2\pi P\int_{-\infty}^\infty \sigma^{\frac{1}{2}}dp_0(p_0^2+k'^2-2k'p_0-\kappa^2)^{\frac{1}{2}}\times[p_0(k'-p_0)/|p_0||k'-p_0|]\delta'[p_0^2(1-\sigma)+k'^2-2k'p_0-\kappa^2]$$
$$=\pi P\int_{-\infty}^\infty \sigma^{\frac{1}{2}}dp_0\left\{\frac{k'}{[k'-p_0(1-\sigma)]^2}-\frac{k'}{p_0[k'-p_0(1-\sigma)]}\right\}\frac{k'-p_0}{|k'-p_0|}\times\delta'[p_0^2(1-\sigma)+k'^2-2k'p_0-\kappa^2].$$

The argument of the δ-function vanishes at two points:

$$p_0^{(1)}=(k'^2-\kappa^2)/2k';\quad p_0^{(2)}=2k'/(1-\sigma).$$

The contribution to J' from the singularity at $p_0^{(1)}$ is simply J (if we let $\sigma\to1$, $\sigma<1$). The contribution to J' from the singularity at $p_0^{(2)}$ turns out to be $(-\pi/2k'^2)$, so that we have

$$J'=\pi/(k^2+\kappa^2). \tag{A-11}$$

This result is true for both space-like and time-like k. This shows that the use of the modified D-function does eliminate the undesirable term from J.

We wish to discuss finally the effect of the limiting process on the self-energy calculation. This will be done for small intermediate momenta so that the effect of the convergence factors may be neglected. In calculating the self-energy of the V_+-particle, we must now add in the contribution from Eq. (31), which no longer vanishes; after the momenta have been renamed in a suitable manner, this contribution may be written

$$I_2(k_4)=-\int_{k_0<0,\,q_0<0}\cdots\int(dk)^4(dq)^4(dp)^4\{\delta(k+q-k_4+p)\times\delta(k^2+\kappa_v^2)\delta(q^2+\kappa_u^2)(p^4/|p^4|)\delta'(p^2-\sigma p_0^2)\}. \tag{A-12}$$

This may be combined with $I_1(k_4)$, Eq. (34), by inserting under the integral sign a factor $\frac{1}{2}(k^4/|k^4|+q^4/|q^4|)$. The resulting integral can be expressed in terms of the modified J' function just calculated, so that we have for the contribution to the self-energy

$$I(k_4)=(\pi/2)\int\cdots\int\delta(q^2+\kappa_u^2)/[(k_4-q)^2+\kappa_v^2](dq)^4+(\pi/2)\int\cdots\int\delta(k^2+\kappa_v^2)/[(k_4-k)^2+\kappa_u^2](dk)^4. \tag{A-13}$$

In Eq. (A-13), it is to be understood that a principal value is to be taken at all singular points of the integrand.[11] This eliminates the difficulty of Sec. II, in which a singularity in the integrand was coincident with a discontinuity. For higher momenta, this procedure is no longer valid because the convergence factors in I_1 and I_2 may be different. The two processes are then to be considered separately.