Quantum Theory of Non-Local Fields. Part II. Irreducible Fields and their Interaction*

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General properties of non-local operators are considered in connection with the problem of invariance with respect to the group of inhomogeneous Lorentz transformations. It is shown that irreducible fields can be classified by the eigenvalues of four invariant quantities. Three of these quantities can be interpreted, respectively, as the mass, radius, and magnitude of the internal angular momentum of the particles associated with the quantized non-local field in question. Further, space-time displacement operators are introduced as a particular kind of non-local operator. As a tentative method of dealing with the interaction of non-local fields, an invariant matrix is defined by the space-time integral of a certain invariant operator, which is a sum of products of non-local field operators and displacement operators. It is shown that the matrix thus constructed satisfies the requirements that it be unitary and invariant and that the matrix elements are different from zero only if the initial and final states had the same energy and momentum. However, the remaining conditions of correspondence and convergence cannot be fulfilled simultaneously, in general, by the S-matrix for the non-local fields. It is yet to be investigated whether all of these requirements are satisfied by an appropriate change in the definition of the S-matrix.

I. ELEMENTARY NON-LOCAL SYSTEMS

THE notion of an elementary particle has been intimately connected with the procedure of decomposing a quantized field into its irreducible parts. Accordingly, if the concept of the field itself is so extended as to include the non-local field, the definition of the elementary particle will be altered in its turn. In Part I,[1] we confined our attention to certain types of non-local fields which satisfied a set of operator equations and were supposed to represent assemblies of elementary particles with finite radii. Our problem is now to decompose more general non-local fields into irreducible parts. Again we start from an arbitrary unquantized non-local scalar field U, which can be represented by an arbitrary matrix $(x'|U|x'')$, where x' and x'' stand for x_μ' and x_μ'' ($\mu=1, 2, 3, 4$), respectively. The matrix $(x'|U|x'')$ can be regarded as a function $U(X, r)$ of two sets of real variables,

$$X_\mu=\tfrac{1}{2}(x_\mu'+x_\mu''), \quad r_\mu=x_\mu'-x_\mu'' \tag{1}$$

as in Part I. Then an arbitrary function $U(X, r)$ can be expanded in the form

$$U(X, r)=\int\cdots\int u(k, r)\exp(ik_\mu X^\mu)(dk_\mu)^4 \tag{2}$$

and further in the form

$$U(X, r)=\int\cdots\int u(k, l)\exp(ik_\mu X^\mu)\times\prod_\mu\delta(r_\mu-l_\mu)(dk_\mu)^4(dl_\mu)^4, \tag{3}$$

where $u(k, r)$ and $u(k, l)$ are arbitrary functions of parameters k, r and k, l, respectively.

Now, if we perform an arbitrary homogeneous Lorentz transformation,

$$x_\mu'=a_{\mu\nu}x_\nu, \tag{4}$$

where x_μ' ($\mu=1, 2, 3, 4$) denote this time the space-time operators in the new coordinate system. Therewith, two sets of parameters, X and r, are transformed into

$$X_\mu'=a_{\mu\nu}X_\nu, \quad r_\mu'=a_{\mu\nu}r_\nu \tag{5}$$

and $U(X, r)$ becomes

$$U(X', r')=\int\cdots\int u'(k', l')\exp(ik_\mu' X'^\mu)\times\prod_\mu\delta(r_\mu'-l_\mu')(dk_\mu')^4(dl_\mu')^4, \tag{6}$$

where $u'(k', l')=u(k, l)$. k', l' are connected with k, l just as X', r' are connected with X, r. In order that Eq. (6) retain the same form as Eq. (3) for an arbitrary Lorentz transformation (4), either one of the following two requirements must be satisfied:

(i) $u(k, l)$ is a function of k and l, which retains its form under an arbitrary Lorentz transformation;

(ii) $u(k, l)$ is not a mere function of k and l, but is an ensemble of quantities, which are distinguished by the parameters k and l and which are to be subject to second quantization.

In the first case, it is required that

$$u(k', l')=u(k, l) \tag{7}$$

for an arbitrary transformation

$$k_\mu'=a_{\mu\nu}k_\nu, \quad l_\mu'=a_{\mu\nu}l_\nu, \tag{8}$$

so that $u(k, l)$ must be the function of invariant quantities such as $k_\mu k^\mu$, $l_\mu l^\mu$ and $k_\mu l^\mu$ alone. In many cases, however, we can confine our attention to the subgroup of the homogeneous Lorentz group which does not include the reversal of the time, so that $u(k, l)$ may depend also on $k_4/|k_4|$, provided that k_μ is a time-like vector, and similarly for l_μ. Thus $U(X, r)$ can be

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[1] H. Yukawa, Phys. Rev. 77, 219 (1950). See also B. Kwal, J. de phys. et rad. 11, 213 (1950).

written, in general, in the form

$$U(X,r)=\int\cdots\int w(K,L,M)\delta(k_\mu k^\mu-K)\delta(l_\mu l^\mu-L)\times\delta(k_\mu l^\mu-M)\exp(ik_\mu X^\mu)\prod_{\mu=1}^{4}\delta(r_\mu-l_\mu)\times(dk_\mu)^4(dl_\mu)^4dKdLdM, \quad (9)$$

where $w(K, L, M)$ is an arbitrary function of the real parameters K, L, and M, which can be positive, negative, or zero. If we restrict the transformations to those belonging to the subgroup mentioned above, w may depend also on $k_4/|k_4|$ (and on $l_4/|l_4|$). In any case, the operator U of this type has nothing to do with the quantized non-local field, because there is no room for the application of the method of second quantization. However, an important family of space-time displacement operators belongs to this category as will be shown later on.

The case (ii) is the more important, because the field operator U can be quantized as in the usual theory. Namely, the coefficients $u(k, l)$ can be regarded as an ensemble of creation and annihilation operators for the quanta associated with the field U. The requirement of invariance is fulfilled simply by identifying $u(k, l)$ $[=u'(k', l')]$ with the creation or annihilation operator for a particle in the quantum state characterized by k and l according as k_4 is positive or negative. The only effect of a transformation of the type (4) or (8) is to give a new notation $u'(k', l')$ to the operator $u(k, l)$, owing to the change in name for the same quantum states caused by the change of the reference system. Thus, there is the one-to-one correspondence between $u(k, l)$ and $u'(k', l')$ in two representations, (3) and (6), of the same operator U. Since $k_\mu k^\mu$, $l_\mu l^\mu$, and $k_\mu l^\mu$ are invariant with respect to any Lorentz transformation, the one-to-one correspondence remains, even if the domain of integrations on the right-hand side of Eq. (3) is restricted to definite values K, L, M of these invariant quantities $k_\mu k^\mu$, $l_\mu l^\mu$, and $k_\mu l^\mu$, respectively. In such a case, $U(X, r)$ reduces to

$$U(X,r)=\int\cdots\int u(k,l)\exp(ik_\mu X^\mu)\prod_\mu\delta(r_\mu-l_\mu)\times\delta(k_\mu k^\mu-K)\delta(l_\mu l^\mu-L)\delta(k_\mu l^\mu-M)(dk_\mu)^4(dl_\mu)^4. \quad (10)$$

It is now clear that the scalar non-local field, which was dealt with in detail in Part I, is a particular example with

$$K\equiv-\kappa^2, \quad L\equiv+\lambda^2, \quad M\equiv0. \quad (11)$$

More generally, L and M can be either positive or negative including zero, but K can only be negative or zero, because a positive K has no correspondence with the classical model of particles with real mass. Positive values of L correspond to the assembly of elementary particles with a finite dimension which is extended to space-like directions, whereas negative values of L corresponds to that which is extended to time-like directions.

It should be noticed, however, that the field characterized by a set of values of K, L, and M can be decomposed further into irreducible parts, each of which corresponds to a definite value for the absolute magnitude of the internal angular momentum. Namely, as shown in Part I, the non-local scalar field U with a given set of values K, L, M can be expanded in the form

$$U=\sum_{k_1k_2k_3}\sum_{l,m}(2\pi/L)^3[\lambda/4\kappa(k^2+\kappa^2)^{\frac{1}{2}}]\times[u(k,l,m)U(k,l,m)+v^*(k,l,m)U^*(k,l,m)], \quad (12)$$

provided that K is negative, L is positive, and M is zero, where u, v^*, U, U^* are defined by the expressions (43) and (44) in Part I. The parameter l in (12) is the quantum number which characterizes the magnitude of the internal angular momentum in the coordinate system moving with the particle with a given wave vector k_1, k_2, k_3. Since l thus defined is invariant with respect to Lorentz transformations, each part of U with a definite value of l transforms into itself and constitutes an irreducible representation of the non-local scalar field. Thus, it is possible that the elementary particles with the integer spins are classified by the value of four constants K, L, M, and l, provided that they are represented by irreducible representations of the non-local scalar field. Among these four constants, $-K$, L, and l can be interpreted, apart from the numerical factors depending on $\hbar$ and c alone, as the mass, radius, and magnitude of the internal angular momentum of the particle, whereas M has no immediate physical meaning.[2]

As was pointed out recently by Fierz,[3] each of these irreducible representations of the non-local scalar field finds its counterpart in the usual field theory of elementary particles with arbitrary integer spin, so far as the behavior with respect to Lorentz transformations is concerned. The essential difference between local and non-local fields will be clear only when the interaction between fields is taken into account. In the case of the non-local spinor field, however, the situation is somewhat different. Namely, a non-local spinor operator ψ_i $(i=1, 2, 3, 4)$ is equivalent to a set of four functions $\psi_i(X, r)$, which can be expanded in the form

$$\psi_i(X,r)=\int\cdots\int u_i(k_\mu,l_\mu)\exp(ik_\mu X^\mu)\times\prod_\mu\delta(r_\mu-l_\mu)(dk_\mu)^4(dl^\mu)^4. \quad (13)$$

This can be decomposed into parts in an invariant manner by giving each of $k_\mu k^\mu$, $l_\mu l^\mu$, $k_\mu l^\mu$ a definite value. Each part can further be regarded as a sum of operators, which differ from one another by their behaviors with respect to space rotations in the rest

[2] D. Yennie, Phys. Rev., following paper.
[3] M. Fierz, Phys. Rev. **78**, 184 (1950); Helv. Phys. Acta **23**, 412 (1950).

system. Each of these operators thus obtained is not yet irreducible in general, because it is a mixture of two types of fields belonging to the same resultant (half-integral) spin. For instance, the operator corresponding to the resultant spin 1/2 may have an internal orbital angular momentum of either zero or unity. In the usual local field theory, however, a spinor field with the spin 1/2, for example, is already irreducible. Thus the difference between the non-local spinor field and the local spinor fields with arbitrary half-integral spins is apparent without taking into account the interaction between the fields.[4]

So far we have considered the problem of invariance of non-local operators with respect to homogeneous Lorentz transformations. Now we go over to the more general inhomogeneous Lorentz transformation of the type

$$x_\mu' = a_{\mu\nu}(x_\nu + b_\nu) \tag{14}$$

or

$$x_\mu' = a_{\mu\nu}x_\nu + b_\mu', \tag{15}$$

with $b_\mu' = a_{\mu\nu}b_\nu$, X and r are transformed thereby into

$$X_\mu' = a_{\mu\nu}(X_\nu + b_\nu), \quad r_\mu' = a_{\mu\nu}r_\nu. \tag{16}$$

Accordingly, we have

$$k_\mu' = a_{\mu\nu}k_\nu, \quad l_\mu' = a_{\mu\nu}l_\nu, \tag{17}$$

and

$$u'(k', l') = \exp(-ik_\mu b^\mu)u(k, l), \tag{18}$$

in order that U be invariant with respect to the transformation (14). The implication of the relation (18) must be considered for the cases (i) and (ii) separately.

In case (i), relation (18) is compatible with the assumption that $u(k, l)$ is an invariant function of k and l, only if $u(k, l)$ is zero for all values of k_μ except $k_\mu = 0$ ($\mu = 1, 2, 3, 4$). This is equivalent to the following statement:

(i)′ A non-local operator U which satisfies requirement (i) is invariant with respect to the whole group of inhomogeneous Lorentz transformations only if $U(X, r)$ is an invariant function of r alone.

It will be shown in the next section that some of the invariant operators satisfying the requirement (i)′ will be of importance in constructing the S-matrix for the interacting non-local fields.

In case (ii), relation (18) reflects the situation that the creation or annihilation operator $u(k, l)$ or $u^*(k, l)$ is defined unambiguously except for an arbitrary phase factor. In spite of this ambiguity or complication, the operator $u^*(k, l)u(k, l)/|k_4|$, which is to be identified with the occupation operator for particles in the quantum state characterized by k_μ and l_μ apart from the purely numerical factor, is defined uniquely and is invariant with respect to the whole group of inhomogeneous Lorentz transformations. So are the commutation relations for $u(k, l)$ and $u^*(k, l)$ as given by Eq. (37) of Part I, for example. Of course, it must always be kept in mind that the time-reversal is associated with the interchange of the annihilation operator $u(k, l)$ and the creation operator $u^*(k, l)$.

These arguments can be applied to non-local spinor fields without essential change. In this way we arrive at the following suggestion: according to the non-local field theory it is possible that there are only two kinds of elementary particles, Bose-Einstein particles and Fermi-Dirac particles, which are described by a scalar field and a spinor field, respectively. The customary discrimination of particles with spins 0, 1, 2, etc., among Bose-Einstein particles, for instance, may well be reduced to the difference in the quantum number l for the internal motion of the same kind of particles.

II. S-MATRIX IN NON-LOCAL FIELD THEORY[5]

Now we must undertake the problem of interaction between non-local fields. In the usual field theory we could always start from the Schrödinger equation for the total system. The Hamiltonian in the Schrödinger equation is derived from the Lagrangian which, in turn, is so chosen as to give the correct field equations for unquantized fields, when the classical variation principle was applied to the system consisting of unquantized fields. In the non-local field theory, however, it is difficult to follow the same procedure as in local field theories for two reasons. Firstly, even in the case of the free field, it is difficult to deduce all of the field equations, (4), (5), and (12), for example, for the scalar non-local field from an invariant operator which is supposed to correspond to the Lagrangian in the usual theory. Moreover, the procedure of variation itself is ambiguous.[6] Secondly, it is rather dubious whether the differentiation of the Schrödinger function with respect to time will play an important role in non-local field theory because other operators, in general, are related to two time instants, which differ from each other by a finite amount. Even the existence of the Schrödinger function in the same sense as in the local field theory is not at all certain.

Although it is not yet clear whether these difficulties could be overcome without renouncing the fundamental principles of quantum mechanics, there seems to exist a tentative solution which retains many of the characteristics of the present field theory. Namely, we can start from the so-called interaction representation in the usual theory, laying aside for the moment the question of whether the free field equations in non-local field theory can be deduced from the Lagrangian formalism or not. Furthermore, we can adopt the integral formalism of the usual theory, which has been

[4] Detailed discussions of non-local spinor field will be made elsewhere.

[5] A preliminary account of the subject was published by H. Yukawa, Phys. Rev. **77**, 849 (1950).

[6] Variation principles in the non-local field theory were discussed by C. Bloch, Kgl. Danske Vid. Sel. Math.-Fys. Medd. See also C. Gregory, Phys. Rev. **78**, 67, 479 (1950).

proved to be equivalent to the differential formalism and in which the S-matrix, instead of the Schrödinger wave function, came in the foreground. Then the S-matrix for local fields can be transformed in the following manner so as to be easily extended to the case of non-local fields. We consider a system of local fields, for which the interaction Hamiltonian density $H'(x, y, z, t)$ is invariant and is equal to $-L'(x, y, z, t)$, where L' is the interaction part in the Lagrangian density for the system. In the usual one-time formalism, the Schrödinger equation has the form

$$ih\partial\Psi(n', t)/\partial t=\sum_{n''}(n'|\bar{H}'(t)|n'')\Psi(n'', t), \quad (19)$$

where each of n' and n'' stands for a set of eigenvalues of occupation operators of various types of particles in the system in various quantum states. $\bar{H}'(t)=-\bar{L}'(t)$ is the space integral of the Hamiltonian density $H'(x, y, z, t)$ or $-L'(x, y, z, t)$. The differential equation (19) can be integrated with respect to time, at least formally, by the method of successive approximation and we obtain

$$\Psi(n', +\infty)=\Psi(n', -\infty)+(i/\hbar)\int_{-\infty}^{+\infty}\sum_{n''}(n'|\bar{L}'(t)|n'')dt\cdot\Psi(n'', -\infty)+(i/\hbar)^2\int\int_{t>t'}\sum_{n'', n'''}(n'|\bar{L}'(t)|n'')(n''|\bar{L}'(t')|n''')\times dtdt'\Psi(n''', -\infty)+\cdots, \quad (20)$$

where $\Psi(n', +\infty)$ and $\Psi(n', -\infty)$ are Schrödinger wave functions in the infinite future and infinite past, respectively. Thus the S-matrix for this case is given by

$$(n'|S|n'')=(n'|1|n'')+(i/\hbar)\int(n'|\bar{L}'(t)|n'')dt+(i/\hbar)^2\int\int_{t>t'}\sum_{n'''}(n'|\bar{L}'(t)|n''')(n'''|\bar{L}'(t')|n'')\times dtdt'+\cdots. \quad (21)$$

In order to generalize this expression for the S-matrix to the case of the system of non-local fields, we introduce an invariant Hermitian non-local operator L' which is represented by a matrix $(n', x'|L'|n'', x'')$ and which reduces to

$$(n', x'|L'|n'', x'')=(n'|L'(x')|n'')\prod_{\mu}\delta(x'^{\mu}-x''^{\mu}) \quad (22)$$

in the limiting case of the system of local fields, where each of x' and x'' stands for a set of eigenvalues of space-time operators $x^1=x$, $x^2=y$, $x^3=z$, $x^4=ct$. With the help of Eq. (22) and of another non-local operator ϵ which is represented by the matrix

$$(n', x'|\epsilon|n'', x'')=\tfrac{1}{2}\{[(x'^4-x''^4)/|x'^4-x''^4|]+1\}(n'|1|n^4). \quad (23)$$

Equation (21) can be written in the form

$$(n'|S|n'')=(n'|1|n'')+(i/\hbar c)\int\int(n', x'|L'|n'', x'')(dx')^4(dx'')^4+(i/\hbar c)^2\int\cdots\int\sum_{n''', n^{\mathrm{IV}}}(n', x'|L'|n''', x''')\times(n''', x'''|\epsilon|n^{\mathrm{IV}}, x^{\mathrm{IV}})(n^{\mathrm{IV}}, x^{\mathrm{IV}}|L'|n'', x'')\times(dx')^4(dx'')^4(dx''')^4(dx^{\mathrm{IV}})^4+\cdots. \quad (24)$$

If we define $\{A\}$ for an arbitrary non-local operator A by

$$(n'|\{A\}|n'')=\int\cdots\int(n', x'|A|n'', x'')(dx')^4(dx'')^4, \quad (25)$$

the S-matrix with matrix elements as given by Eq. (24) can be written symbolically in the form

$$S=1+(i/\hbar c)\{L'\}+(i/\hbar c)^2\{L'\epsilon L'\}+(i/\hbar c)^3\{L'\epsilon L'\epsilon L'\}+\cdots. \quad (26)$$

This could be used as the definition of the S-matrix in non-local field theory as well as in local field theory. Alternatively, we can define S or $S-1$ by

$$S-1=\{R\}, \quad (27)$$

where

$$R=(i/\hbar c)L'+(i/\hbar c)^2L'\epsilon L'+(i/\hbar c)^3L'\epsilon L'\epsilon L'+\cdots. \quad (28)$$

Incidentally, the non-local operator R satisfies a linear operator equation

$$R=(i/\hbar c)L'+(i/\hbar c)L'\epsilon R. \quad (29)$$

The physical interpretation of the S-matrix remains the same as in the usual theory in spite of the fact that the S-matrix is non-local field theory is defined directly by Eq. (26) or by Eqs. (27) and (28) without recourse to the Schrödinger equation of type (19). Thus, $|(n'|S|n'')|^2$ is the probability that the system will be in the state characterized by n' in the infinite future provided that it was in the state characterized by n'' in the infinite past. In fact, the S-matrix as defined by (26) satisfies two conditions:

(i) S is a unitary matrix which satisfies the relation

$$S^*S=SS^*=1. \quad (30)$$

(ii) The matrix element $(n'|S|n'')$ is different from zero only if the states characterized by n' and n'', respectively, have the same total energy and momentum.

Before going into the proof of these statements, we have to take into account the third condition:

(iii) S must be an invariant matrix.

In local field theories, the S-matrix defined above is

invariant, in spite of the fact that the operator ϵ as defined by Eq. (23) is not invariant with respect to Lorentz transformations. This is due to the fact that the Hamiltonian density $H'(x')$ at a point x' is commutative with the density $H'(x'')$ at any other point x'', which is located in a space-like direction with respect to x'. It is not so, in general, in non-local field theory. An obvious way of guaranteeing the invariance of the S-matrix in such a case is to replace the operator ϵ in Eq. (26) by a suitable invariant non-local operator D_+ such that conditions (i) and (ii) are still fulfilled. Thus the S-matrix for the system of non-local fields takes the form

$$S=1+(i/\hbar c)\{L'\}+(i/\hbar c)^2\{L'D_+L'\}+(i/\hbar c)^3\{L'D_+L'D_+L'\}+\cdots. \quad (31)$$

The actual form of the operator D_+ can be determined in the following manner. If we assume that the invariant operator L' is a sum of products of non-local field operators, condition (ii) is satisfied for any displacement operator D_+ whose matrix element $(x'|D_+|x'')\equiv D_+(X, r)$ is an invariant function of r_μ alone. The proof is simple. Any non-local operator A can be represented by a matrix $(x'|A|x'')$ or a function $A(X, r)$ and Eq. (25) can be written alternatively in the form

$$(n'|\{A\}|n'')=\int\int (n'|A(X,r)|n'')(dX)^4(dr)^4. \quad (32)$$

If the operator A consists of a sum of products of L' and D_+, $(n'|A(X, r)|n'')$ can be expanded into a series with the typical term

$$(n'|a(k_\mu^{(i)}, r_\mu)|n'')\exp(iK_\mu X^\mu), \quad (33)$$

where

$$K_\mu=\textstyle\sum_i n''^{(i)}k_\mu^{(i)}-\sum_i n'^{(i)}k_\mu^{(i)}. \quad (34)$$

Evidently $\hbar K$ is the difference in momenta between the initial state n'' and the final state n' and $-\hbar cK_4$ is the difference in energies of the states n'' and n'. If we insert Eq. (33) into Eq. (32) and integrate with respect to X, we find that each term of $(n'|\{A\}|n'')$ contains a factor $\prod_\mu \delta(K_\mu'-K_\mu'')$, so that $(n'|\{A\}|n'')$ is different from zero only if the states n' and n'' have the same energy and momentum. It should be noticed, however, that we mean by the energy and momentum of a particle the energy and momentum of its center of mass. Thus the energy of internal motion is supposed to be included already in the mass $\hbar\kappa/c$. In other words, κ must be, in general, a function of other constants such as λ and l. The problem of determining the form of such a function is still completely open.

The condition (i) is also fulfilled, if we further imply the condition

$$D_+ + D_+^* = E \quad (35)$$

on D_+, where D_+^* is the Hermitian conjugate of D_+ and E is an invariant displacement operator with the matrix element

$$(n', x'|E|n'', x'')=(n'|1|n'') \quad (36)$$

for any values of x' and x''. In order to prove this, we have only to multiply S as given by Eq. (31) by

$$S^*=1-(i/\hbar c)\{L'\}+(i/\hbar c)^2\{L'D_+^*L'\}-(i/\hbar c)^3\{L'D_+^*L'D_+^*L'\}+\cdots. \quad (37)$$

Then the condition of unitarity

$$\sum_{n'''}(n'|S^*|n''')(n'''|S|n'')=\sum_{n'''}(n'|S|n''')(n'''|S^*|n'')=(n'|1|n'') \quad (38)$$

comes out by the help of Eq. (32) and the relation

$$\{AEB\}=\{A\}\{B\}, \quad (39)$$

which holds for any two non-local operators A and B.

The operators D_+^* and D_+ which satisfy all of these conditions are given by matrices

$$(n', x'|D_+|n'', x'')=(n'|1|n''),\ \tfrac{1}{2}(n'|1|n''),\text{ or } 0;$$
$$(n', x'|D_+^*|n'', x'')=0,\ \tfrac{1}{2}(n'|1|n''),\text{ or } (n'|1|n''), \quad (40)$$

according as $x'-x''$ is future-like, space-like, or past-like.

This modification of the definition of S-matrix gives rise to the new question: does it reduce to the usual definition (21) in the limit of local fields? This question is very intimately connected with another, and probably the most important, question: is the S-matrix for non-local fields convergent? In order to answer these questions, we begin with the investigation of the particular matrix element $(0|S|0)$ of $(n'|S|n'')$, where both the initial state n'' and the final state n' are complete vacua; i.e., all eigenvalues n' and n'' are zero. Now $(0|S|0)$ has the general form

$$(0|S|0)=1+(i/\hbar c)(0|\{L'\}|0)+(i/\hbar c)^2(0|\{L'D_+L'\}|0)+\cdots. \quad (41)$$

Let us consider a very simple case of a system consisting of a complex non-local scalar field V, V^* and a real non-local scalar field U with the interaction of the form

$$L'=gV^*UV. \quad (42)$$

We have first

$$(0|\{L'\}|0)=0 \quad (43)$$

because L' is linear in U and hence has no term which connects the state 0 with itself. As for the third term in Eq. (38), we have the relation

$$(0|\{L'D_+L'\}|0)=\tfrac{1}{2}\sum_{n'}(0|\{L'\}|n')(n'|\{L'\}|0)+\tfrac{1}{2}(0|\{L'DL'\}|0) \quad (44)$$

on account of relations (32) and (36), where the operator D is defined by

$$D=D_+-D_+^* \quad (45)$$

with the matrix element

$$(n', x'|D|n'', x'')=(n'|1|n''),\quad 0,\quad -(n'|1|n''), \quad (46)$$

according as $x'-x''$ is future-like, space-like, or past-like.[7] The first term on the right-hand side of (44)

[7] This operator was introduced by Koba independently. See Z. Koba, Prog. Theor. Phys. 5, 139 (1950).

vanishes on account of the fact that $(n'|\{L'\}|0)$ is zero provided that $\kappa_U<2\kappa_V$ and the second term also vanishes for the following reason: first we expand U, V, V^*, and D in Fourier series and integrate each of the terms of $(0|\{L'DL'\}|0)$ with respect to all of the space-time parameters. Actually we have eight sets of such parameters. Then we are left with the expression of the form

$$\int\int\int f(k_\mu^{(1)}, k_\mu^{(2)}, k_\mu^{(3)})\delta'(K_\mu K^\mu) \times (dk_\mu^{(1)})^4(dk_\mu^{(2)})^4(dk_\mu^{(3)})^4, \quad (47)$$

where $K_\mu=\sum_i k_\mu^{(i)}$ and $k_\mu^{(1)}$, $k_\mu^{(2)}$, $k_\mu^{(3)}$ are the wave vectors of the three particles created in the intermediate state. The first of them is a particle of U-type and the other two are particles of $V-V^*$-type. $f(\cdots)$ is a function of $k_\mu^{(1)}$, $k_\mu^{(2)}$, $k_\mu^{(3)}$, which could be determined by elementary calculations, but it is not necessary for our purpose to write it explicitly. δ' denotes the derivative of the δ-function with respect to the argument, which comes from the Fourier transform of the operator D, as discussed in detail by Yennie.[2] Thus, $(0|\{L'DL'\}|0)$ must be zero, unless the condition

$$K_\mu K^\mu=0 \quad (48)$$

is fulfilled. The condition (48) can be satisfied by certain sets of $k_\mu^{(1)}$, $k_\mu^{(2)}$, $k_\mu^{(3)}$ only if both types of particles have the rest mass zero.

The above arguments can be applied to local fields as well as to non-local fields. According to the usual theory of local fields, the third term on the right-hand side of Eq. (41) must be the divergent self-energy of the vacuum, whereas it is actually zero according to our formalism, except for the very particular case of particles both with the rest mass zero. The same argument can be applied to the case of charged particles interacting with the electromagnetic field, and according to our formalism the self-energy of the vacuum is zero, at least up to the second order, if we assume that there is no charged particle with the rest mass zero. Thus, the discrepancy between our formalism and the usual theory is already clear; they give different answers to the same problem for local fields.

Next we consider the matrix element $(1|S|1)$ of S, where only one particle of the same type in the same state exists in the initial and final states. The second-order term of $(1|S|1)$ corresponds to the divergent self-energy of the particle in local field theory. As discussed by Yennie in detail,[2] if we start from a system of two non-local scalar fields of U-type and $V-V^*$-type with the interaction operator L' as given by Eq. (42), the self-energy term is again divergent. However, the fields U and $V-V^*$ can be decomposed further into positive frequency and negative frequency parts without destroying the invariance with respect to the subgroup of Lorentz transformation, in which the direction of time is not reversed. Namely, we can write

$$U=U_+ + U_-, \quad V=V_+ + V_-, \quad V^*=V_+^* + V_-^*, \quad (49)$$

where U_+, U_- are positive and negative frequency parts of U, while V_+, V_+^* and V_-, V_-^* are corresponding parts of V and V^*. If we take the new interaction operator

$$L'=g\{V_+^*UV_+ + V_-UV_-^* + V_+^*UV_- + V_-^*UV_+\} \quad (50)$$

instead of Eq. (39), the self-energy terms for the U-type particle as well as the $V-V^*$-type particles are convergent, although there still remains an undesirable feature, as discussed by Yennie.[2]

Now, in order to remove the discrepancy between the present formalism and the usual formalism in the limit of local fields, we may imagine that D-operator above defined is a limit of the operator with the matrix element, which is a function of r_μ and is different from zero in a narrow region outside the light cone in r-space. Then the correspondence between the present formalism and the usual formalism in the limit of local fields is restored up to the second order, but the essential difference between ϵ- and D-operators remains in the third- and higher order terms. Moreover, the divergences reappear in the case of non-local fields. It is very difficult to construct an S-matrix which is convergent and which reduces to the usual S-matrix in the limit of local fields. It is not yet clear whether the S-matrix formalism itself is not adequate for dealing with the problem of interaction of non-local fields. It might be possible that the S-matrix as defined by Eq. (24) is invariant, if the interaction operator L' has an appropriate form, even in the case of non-local fields. However, it is more probable that the clean-cut separation of the free fields from their interaction is justified only if we are dealing with the weak coupling between local fields. If so, we must go back in search of the Lagrangian formalism for the whole system of non-local fields interacting with one another. In any case, the compatibility conditions for the field equations or the integrability conditions for any substitute for the Schrödinger equation will be of fundamental importance.

In this connection it should be noticed that so far we have not been able to find any relation between the mass and other constants. It is clear that a relation which connects the mass of an elementary particle with other constants such as the radius, the internal angular momentum, and the constants of coupling with other particles will be of vital importance in any future theory of elementary particles. Again this is closely related to the problem of finding the Lagrangian operator for the whole system or any substitute for it.

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